CMDA 4604: Intermediate Topics in Mathematical Modeling
Lecture 21: Error Estimates for the Finite Element Method

Finite element methods first rose to prominence in engineering applications, but one great virtue of these methods is the ease with which they yield to rigorous mathematical analysis and the elegance of the resulting bounds. In this lecture we provide a taste of this theory. Proofs of basic error estimates can be found in many mathematical textbooks on the finite element method. This presentation here closely follows that given by Susanne C. Brenner and L. Ridgway Scott, *The Mathematical Theory of Finite Element Methods*, Springer-Verlag, 1994.

Problem Statement.

We wish to solve the differential equation

$$-u''(x) = f(x), \qquad 0 \le x \le 1$$

with $u \in V$, where we are using the shorthand $V = C_D^2[0,1] = \{u \in C^2[0,1] : u(0) = u(1) = 0\}$.

This problem has the equivalent weak form:

$$\text{Find } u \in V \text{ such that} \quad a(u,v) = (f,v) \quad \text{for all } v \in V.$$

Inner Products and Norms.

The inner product (f,v) is defined, as usual, as

$$(f,v) = \int_0^1 f(x)v(x)\,\mathrm{d}x$$

and the energy inner product $a(u,v)$ for this problem takes the form

$$a(u,v) = \int_0^1 u'(x)\,v'(x)\,\mathrm{d}x.$$

With both these inner products we associate norms: the standard norm on $C^2[0,1]$,

$$\|u\| = \sqrt{(u,u)}$$

and the energy norm on $V = C_D^2[0,1]$,

$$\|u\|_E = \sqrt{a(u,u)}.$$

Suppose $u_N \in V_N$ denotes a finite element approximation to u, where V_N is the span of hat functions $\phi_1, \ldots, \phi_N$ on the grid $x_0, x_1, \ldots, x_{N+1}$, where $x_j = jh$ for $h = 1/(N+1)$ and

$$\phi_j(x) = \begin{cases} (x - x_{j-1})/h, & x \in [x_{j-1}, x_j); \\ (x_{j+1} - x)/h, & x \in [x_j, x_{j+1}); \\ 0, & \text{otherwise.} \end{cases}$$

We know the solution u_N of the Galerkin approximation problem satisfies

$$a(u_N, v) = (f, v) \quad \text{for all } v \in V_N.$$

Combining this condition with the weak form of the equation gives *Galerkin orthogonality*:

$$a(u - u_N, v) = 0 \quad \text{for all } v \in V_N.$$

Thus the error $u - u_N$ is orthogonal, in the energy inner product, to all v in the approximation subspace V_N. Recall that this orthogonality of the error is equivalent to least squares approximation. Since the orthogonality is in the energy inner product, the least squares optimization is with respect to the energy norm:

$$\|u - u_N\|_E = \min_{\widehat{u} \in V_N} \|u - \widehat{u}\|_E.$$

As interesting as this description is, it provides little insight into how closely u_N matches the true solution u, and how this error decays as we increase N. Such insight will require more detailed information about V_N. Our first estimate will describe the energy norm of the error, $\|u - u_N\|_E$, as a function of the grid size $h = 1/(N+1)$. Often in practice we would prefer to measure the error in the standard norm, $\|u - u_N\|$. This is the object of our second estimate, whose derivation requires a little more work.

Cauchy–Schwarz Inequality.

Recall from earlier in the semester that any inner product obeys the *Cauchy–Schwarz inequality*, which bounds the inner product of two vectors in terms of the norms of each individual vector. For example, in the standard inner product we have for all $f, g \in C^2[0, 1]$,

$$\boxed{(f, g)| \le \|f\| \, \|g\|.}$$

and in the energy inner product we have for all $u, v \in V = C_D^2[0, 1]$

$$\boxed{|a(u, v)| \le \|u\|_E \|v\|_E.}$$

Interpolants.

A key ingredient in our error bounds will be the function $\widetilde{v} \in V_N$ that *interpolates* the function $v \in V$ at the grid points $0 = x_0, x_1, \ldots, x_N, x_{N+1} = 1$. In the last lecture we constructed high degree polynomials that interpolated a function at set of points. In this context, we will be using *piecewise linear interpolants* – piecing together linear interpolants between adjacent grid points.

Notice that any piecewise linear interpolant $\widetilde{v} \in V_N$ can be written as the sum of the hat functions $phi_1, \ldots, \phi_N$. To see this, write $\widetilde{v}$ as a generic linear combination of these functions:

$$\widetilde{v}(x) = \sum_{j=1}^{N} \gamma_j \phi_j(x).$$

Since $\phi_j(x_k) = 0$ if $j \ne k$ and $\phi_j(x_j) = 1$, we have

$$\widetilde{v}(x_j) = \gamma_j, \qquad j = 1, \ldots, N. \tag{1}$$

Since $\phi_j(0) = 0$ and $\phi_j(1) = 0$ for all $j = 1, \ldots, N$, $\widetilde{v}$ automatically satisfies $\widetilde{v}(0) = v(0) = 0$ and $\widetilde{v}(1) = v(1) = 0$. The condition that $\widetilde{v}$ interpolates v at $x_1, \ldots, x_N$ requires

$$\widetilde{v}(x_j) = v(x_j), \qquad j = 1, \ldots, N. \tag{2}$$

Comparing (1) and (2), we realize that $\gamma_j = v(x_j)$, and thus we have a simple formula for the interpolant:

$$\widetilde{v}(x) = \sum_{j=1}^{N} v(x_j)\phi_j(x).$$

Notice that on each segment $[x_j, x_{j+1}]$, $j = 0, \ldots, N$, the interpolant $\widetilde{v}$ is the straight line that 'connects the dots' between $(x_j, v(x_j))$ and $(x_{j+1}, v(x_{j+1}))$.

How well does the interpolant $\widetilde{v}$ match the function v? There are various ways one can measure this quantity; for us, it will prove most convenient for us to have a bound the energy norm of $v - \widetilde{v}$.

Theorem (Energy Error in Piecewise Linear Interpolants). Suppose $v \in C^2[0, 1]$ and $\widetilde{v}$ is a piecewise linear function that interpolates v at the points $x_0, \ldots, x_{N+1}$ with $x_j = jh = j/(N + 1)$. Then

$$\boxed{\|v - \widetilde{v}\|_E \leq \frac{h}{\sqrt{2}}\|v''\|.}$$

Proof. Denote the error by $e_N(x) = v(x) - \widetilde{v}(x)$. We seek to bound

$$\|v - \widetilde{v}\|_E = \|e_N\|_E = \int_0^1 e_N'(x)^2 \,\mathrm{d}x.$$

Thus we first shall study $e_N'(x)$, for $x \in [x_j, x_{j+1}$ for some $j = 0, \ldots, N$. Notice that $e_N(x_j) = e_N(x_{j+1}) = 0$ since $\widetilde{v}$ interpolates v at $x = x_j$ and $x = x_{j+1}$. Thus by Rolle's Theorem there exists some point $\xi_j \in [x_j, x_{j+1}]$ such that $e_N'(\xi_j) = 0$. The Fundamental Theorem of Calculus then gives

$$e_N'(x) = e_N'(x) - e_N'(\xi_j) = \int_{\xi_j}^{x} e_N''(s) \,\mathrm{d}s.$$

Now we write

$$|e_N'(x)| = \Big|\int_{\xi_j}^{x} e_N''(s) \,\mathrm{d}s\Big| = \Big|\int_{\xi_j}^{x} e_N''(s) \cdot 1 \,\mathrm{d}s\Big|.$$

Use the Cauchy–Schwarz inequality on $C[\xi_j, x]$ (the vector space of functions that are continuous on the intervale $[\xi_j, x]$) to obtain

$$\begin{aligned} |e_N'(x)| &\leq \Big(\Big|\int_{\xi_j}^{x} e_N''(s)^2 \mathrm{d}s\Big|\Big)^{1/2}\Big(\Big|\int_{\xi_j}^{x} 1 \,\mathrm{d}s\Big|\Big)^{1/2} \\ &= \Big|\int_{\xi_j}^{x} e_N''(s)^2 \mathrm{d}s\Big|^{1/2} \;\; |x - \xi_j|^{1/2}. \end{aligned}$$

The absolute values are necessary on the right hand side, because we cannot be sure if $\xi_j \leq x$ or $\xi_j \geq x$: all we know is that both ξ_j and x are in the interval $[x_j, x_{j+1}]$. Now since $e_N''(s)^2 \geq 0$, increasing the domain will only enlarge the integral:

$$\Big|\int_{\xi_j}^{x} e_N''(s)^2 \,\mathrm{d}s\Big| \leq \int_{x_j}^{x_{j+1}} e_N''(s)^2 \,\mathrm{d}s.$$

We do not need the absolute value bars on the right, since we know $x_j < x_{j+1}$. In summary, we can bound

$$|e_N'(x)| \leq \Big(\int_{x_j}^{x_{j+1}} e_N''(s)^2 \,\mathrm{d}s\Big)^{1/2}\ |x-\xi_j|^{1/2}.$$

This seems like a complicated way to bound $|e_N'(x)|$, but now we can put this formula to good use to get our upper bound on the energy norm of the interpolation error. Break the error into pieces over each subinterval $[x_j, x_{j+1}]$:

$$\|e_N\|_E^2 = \int_0^1 |e_N'(s)|^2 \,\mathrm{d}x = \sum_{j=0}^N \int_{x_j}^{x_{j+1}} |e_N'(x)|^2 \,\mathrm{d}x.$$

Now insert the formula for $|e_N'(x)|$ that we have derived for each $x \in [x_j, x_{j+1}]$:

$$\begin{aligned}\|e_N\|_E^2 = \sum_{j=0}^N \int_{x_j}^{x_{j+1}} |e_N'(x)|^2 \,\mathrm{d}x \ &\leq\ \sum_{j=0}^N \int_{x_j}^{x_{j+1}} \int_{x_j}^{x_{j+1}} e_N''(s)^2 \,\mathrm{d}s\ |x-\xi_j| \,\mathrm{d}x \\ &\leq\ \sum_{j=0}^N \int_{x_j}^{x_{j+1}} e_N''(s)^2 \,\mathrm{d}s \int_{x_j}^{x_{j+1}} |x-\xi_j| \,\mathrm{d}x.\end{aligned}$$

Pause for a moment to consider the integral

$$\int_{x_j}^{x_{j+1}} |x-\xi_j| \,\mathrm{d}x,$$

where $x, \xi_j \in [x_j, x_{j+1}]$. Shifting the domain from $[x_j, x_{j+1}]$ to $[0, h]$, without loss of generality we can consider, for $\xi \in [0, h]$,

$$\int_0^h |x-\xi| \,\mathrm{d}x = \int_0^\xi \xi - x \,\mathrm{d}x + \int_\xi^h x - \xi \,\mathrm{d}x = \frac{h^2}{2} + \xi^2 - \xi h.$$

This integral is maximized when ξ takes either extreme value $\xi = 0$ or $\xi = h$, in which case the integral equals $h^2/2$. Thus

$$\int_{x_j}^{x_{j+1}} |x-\xi_j| \,\mathrm{d}x \leq \frac{h^2}{2}, \qquad j = 0, \ldots, N.$$

Substituting this bound into the main bound, we have

$$\begin{aligned}\|e_N\|_E^2 = \sum_{j=0}^N \int_{x_j}^{x_{j+1}} |e_N'(x)|^2 \,\mathrm{d}x \ &\leq\ \sum_{j=0}^N \int_{x_j}^{x_{j+1}} e_N''(s)^2 \,\mathrm{d}s \int_{x_j}^{x_{j+1}} |x-\xi_j| \,\mathrm{d}x \\ &\leq\ \sum_{j=0}^N \int_{x_j}^{x_{j+1}} e_N''(s)^2 \,\mathrm{d}s \Big(\frac{h^2}{2}\Big).\end{aligned}$$

We can factor out the $h^2/2$ term and reassemble the remaining pieces into a single integral:

$$\|e_N\|_E^2 \leq \frac{h^2}{2} \sum_{j=0}^N \int_{x_j}^{x_{j+1}} e_N''(s)^2 \,\mathrm{d}s = \frac{h^2}{2} \int_0^1 e_N''(s)^2 \,\mathrm{d}s = \frac{h^2}{2} \|e_N''\|^2.$$

Since $\widetilde{v}$ is piecewise linear, $\widetilde{v}'' = 0$, so $e_N'' = v'' - \widetilde{v}'' = v''$. This gives

$$\|e_N\|_E \le \frac{h}{\sqrt{2}}\|v''\|$$

as required. ■

Energy Norm Error Bound.

An error estimate in the energy norm will now follow very easily. Recall that from the projection (best approximation) theorem that the approximate solution u_N satisfies

$$\|u - u_N\|_E = \min_{\widehat{u} \in V_N} \|u - \widehat{u}\|_E.$$

Thus $\|u - u_N\|_E$ is at least as small as $\|u - \widehat{u}\|_E$ for any $\widehat{u}$ we pick from V_N. In particular, we have

$$\|u - u_N\|_E \le \|u - \widetilde{u}\|_E, \tag{3}$$

where $\widetilde{u} \in V_N$ is the interpolant to the exact solution u. (You might be troubled by the thought that we cannot construct $\widetilde{u}$ without having access to the exact solution u itself. Regardless, we know that u_N must be at least as good an approximation as this function $\widetilde{u} \in V_N$. We will use this term $u - \widetilde{u}$ as a crutch to get to a more useful description of the error.)

Applying the interpolation error bound for $\|u - \widetilde{u}\|_E$, (3) implies

$$\|u - u_N\|_E \le \frac{h}{\sqrt{2}}\|u''\|.$$

You might find the term involving u'' to be bothersome, since it involves the exact solution u that we are trying to compute. But we actually know that $-u'' = f$ directly from the differential equation, and so we arrive at the beautiful expression

$$\boxed{\|u - u_N\|_E \le \frac{h}{\sqrt{2}}\|f\|.}$$

Note that $\|f\|$, the standard norm of f, is independent of the grid. As we double N, we roughly cut h in half, and so the error must decrease at the same rate.

Figure 1 illustrates this result for the equation $-u''(x) = 1$ with $u(0) = u(1) = 0$. The exact solution is $u(x) = x(1-x)/2$, allowing us to measure the error $\|u - u_N\|_E$, and compare it to the bound we have just proved. The figures uses $N = 2, 4, 8, \ldots, 512$. Notice in particular that the bound has the correct *slope* on this log-log plot: that means that despite all the various inequalities we have made, the bound captures the correct *convergence rate*. We thus can say that the energy norm error $\|u - u_N\|_E$ is $O(h)$ as $h \to 0$.

Standard Norm Error Bound.

To bound $u - u_N$ in the standard norm, we employ a neat idea known as *Nitsche's trick*. First, we imagine that we know the solution u and its approximation u_N, and we use the error $u - u_N$ in place of f in the differential equation. The result is the *dual problem*:

$$-w''(x) = u(x) - u_N(x), \qquad w(0) = w(1) = 0.$$

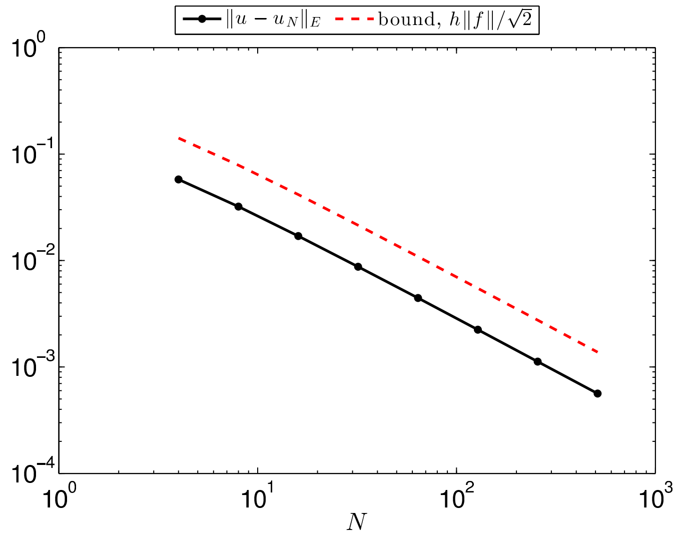

Figure 1: Illustration of the energy norm error bound $\|u-u_N\|_E \leq h\|f\|/\sqrt{2}$ for the equation $-u''(x) = 1$ with $u(0) = u(1) = 0$.

Now we turn to the main quantity of interest:

$$\begin{aligned}\|u-u_N\|^2 &= (u-u_N, u-u_N)\\ &= \int_0^1 (u-u_N)(u-u_N)\,dx\\ &= \int_0^1 (u-u_N)(-w'')\,dx,\end{aligned}$$

where we have substituted the solution to the dual problem for one copy of $u-u_N$ on the last line. Integrate this formula by parts, using the fact that $u(0)-u_N(0) = u(1)-u_N(1) = 0$ to eliminate the boundary term, to obtain

$$\begin{aligned}\|u-u_N\|^2 &= \int_0^1 (u-u_N)'w'\,dx\\ &= a(u-u_N, w).\end{aligned}$$

Since u_N solves the Galerkin problem, we have that $a(u-u_N, v) = 0$ for all $v \in V_N$. In particular, this holds for $\widetilde{w} \in V_N$, the interpolant to w, and so

$$\begin{aligned}\|u-u_N\|^2 &= a(u-u_N, w) - 0\\ &= a(u-u_N, w) - a(u-u_N, \widetilde{w})\\ &= a(u-u_N, w-\widetilde{w}).\end{aligned}$$

Now apply the Cauchy–Schwarz inequality (for the energy inner product) to the last line to obtain

$$\begin{aligned}\|u-u_N\|^2 &= a(u-u_N, w-\widetilde{w})\\ &\leq \|u-u_N\|_E\,\|w-\widetilde{w}\|_E.\end{aligned}$$

We already have a bound on the first quantity: this is just the error in the energy norm that we computed earlier. For the second quantity, we again use our interpolation error bound to obtain

$$\|w - \widetilde{w}\|_E \leq \frac{h}{\sqrt{2}} \|w''\|,$$

but since w solves the dual problem, we have $-w'' = u - u_N$. This is the key step! It will lead to an error that decays like h^2 as $h \to 0$. We arrive at the bound

$$\|u - u_N\|^2 \leq \Big(\frac{h}{\sqrt{2}}\|u''\|\Big)\Big(\frac{h}{\sqrt{2}}\|u - u_N\|\Big),$$

and after canceling $\|u - u_N\|$ from both sides and simplifying, we have

$$\|u - u_N\| \leq \frac{h^2}{2}\|u''\|.$$

Just as before, we can now use the differential equation $-u'' = f$ to simplify the right hand side, giving the elegant bound

$$\boxed{\|u - u_N\| \leq \frac{h^2}{2}\|f\|.}$$

Thus the error in the standard norm *decreases like* h^2, one order of h better than we obtained for the error decrease in the energy norm. When h is *halved*, the error $\|u - u_N\|$ will be *quartered*.

Figure 2 illustrates this bound on $\|u - u_N\|$ for the problem $-u''(x) = 1$ with $u(0) = u(1) = 0$ shown earlier. Again, notice that the bound captures the correct convergence rate: $\|u - u_N\| = O(h^2)$.

Suppose now that you wanted to improve the accuracy from $O(h^2)$ to $O(h^4)$. Review the above analysis to determine what aspect of the finite element method you should modify to obtain such improvement.

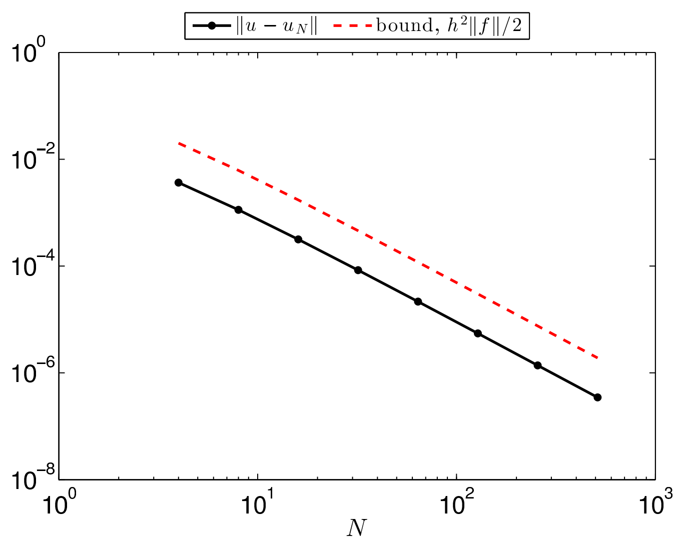

Figure 2: Illustration of the error bound $\|u - u_N\| \leq h^2\|f\|/2$ for the equation $-u''(x) = 1$ with $u(0) = u(1) = 0$.