CMDA 4604 · INTERMEDIATE TOPICS IN MATHEMATICAL MODELING

Problem Set 8

Posted Monday 16 November 2015. Due Monday 30 November 2015, 5pm.
This assignment is worth 75 points (plus 5 bonus points are available).

1. [40 points: (a)=8 pts; (b)=8 pts; (c)=8 pts; (d)=8 pts; (e)=8 pts; (f) = 5 bonus pts]
Our model of the vibrating string predicts that motion induced by an initial pluck will propagate forever with no loss of energy. In practice we know this is not the case: a string eventually slows down due to various types of *damping*. For example, *viscous damping*, a model of air resistance, acts in proportion to the velocity of the string. The partial differential equation becomes

$$u_{tt}(x,t) = u_{xx}(x,t) - 2du_t(x,t),$$

where $d > 0$ controls the strength of the damping. Impose homogeneous Dirichlet boundary conditions,

$$u(0,t) = u(1,t) = 0$$

and suppose we know the initial position and velocity of the pluck:

$$u(x,0) = u_0(x), \qquad u_t(x,0) = v_0(x).$$

In our previous language, we write this PDE in the form

$$u_{tt} = -Lu - 2du_t,$$

where the operator L is defined as $Lu = -u_{xx}$ with boundary conditions $u(0) = u(1) = 0$; as you know well by now, this operator has eigenvalues $\lambda_n = n^2\pi^2$ and eigenfunctions $\psi_n(x) = \sqrt{2}\sin(n\pi x)$. We will look for solutions to the PDE of the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\psi_n(x).$$

For simplicity, assume that $d \in (0,\pi)$.

(a) From the differential equation and this form for $u(x,t)$, show that the coefficients $a_n(t)$ must satisfy the ordinary differential equation

$$a_n''(t) = -\lambda_n a_n(t) - 2da_n'(t).$$

(b) Show that the following function satisfies the differential equation in part (a):

$$a_n(t) = C_1 \exp\big(\big(-d + \sqrt{d^2 - n^2\pi^2}\big)t\big) + C_2 \exp\big(\big(-d - \sqrt{d^2 - n^2\pi^2}\big)t\big)$$

for arbitrary constants C_1 and C_2. (Don't fret about the fact that we have square roots of negative numbers; proceed in the same way you would for an exponential with real argument.)

(c) Now assume that the string starts with zero displacement ($u_0(x) = 0$) but some velocity

$$v_0(x) = \sum_{n=1}^{\infty} b_n(0)\psi_n(x).$$

Determine the values of the constants C_1 and C_2 in part (b) for these initial conditions.

(d) Suppose we have $u_0(x) = 0$ and initial velocity $v_0(x) = x\sin(3\pi x)$, for which

$$b_n(0) = \frac{-6n\sqrt{2}(1 + (-1)^n)}{(n^2 - 9)^2\pi^2} \quad \text{for } n \neq 3, \qquad b_3(0) = \frac{\sqrt{2}}{4}.$$

Take damping parameter $d = 1$, and plot the solution $u(x,t)$ (using 20 terms in the series) at times $t = 0.15, 0.3, 0.6, 1.2, 2.4$. (You may superimpose these on one well-labeled plot; for clarity, set the vertical scale to $[-0.1, 0.1]$.)

(e) Take the same values of u_0 and v_0 used in part (d). Plot the solution at time $t = 2.5$ for $d = 0, .5, 1, 3$ on one well-labeled plot, again using vertical scale $[-0.1, 0.1]$. How does the solution depend on the damping parameter d?

(f) [5 point bonus]
Suppose we wish to damp the string so that the solution becomes small as quickly as possible. What value of $d > 0$ will give the most rapid decay of the solution as $t \to \infty$? Explain your thinking, and provide evidence (plots). If your critical value of d is finite, contract the behavior of smaller and larger values of d. If your value of d is infinite, show plots for increasing values of d to make your case.

2. [35 points: (a)=5 points; (b),(c),(d)=7 points each; (e)=9 points]
In lectures we will see that solutions to the wave equation $u_{tt}(x,t) = u_{xx}(x,t)$ with boundary conditions $u(0,t) = u(1,t) = 0$ and initial conditions $u(x,0) = u_0(x)$ and $u_t(x,0) = v_0(x)$ can be approximated using finite elements in the form

$$u(x,t) = \sum_{j=1}^{N} a_j(t)\phi_j(x),$$

where $\phi_1, \ldots, \phi_N$ denote our usual hat functions, and the coefficients $a_1, \ldots, a_N$ satisfy the differential equation

$$\mathbf{M}\mathbf{a}''(t) = -\mathbf{K}\mathbf{a}(t).$$

This equation can be written in the first order form

$$\begin{bmatrix} \mathbf{a}'(t) \\ \mathbf{a}''(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{a}(t) \\ \mathbf{a}'(t) \end{bmatrix}.$$

In class, we shall see that the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & \mathbf{0} \end{bmatrix}$$

have the form

$$\alpha_{\pm k} = \pm\sqrt{-\lambda_k}, \quad k = 1, \ldots, N$$

where $\lambda_1, \ldots, \lambda_N$ denote the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ that were worked out in class.

(a) For $N = 16$, plot the eigenvalues of $\mathbf{A}$ in the complex plane.

(b) Run the forward Euler method for initial position $u_0(x) = x^2(1-x)$ and $v_0(x) = 0$ with a variety of time steps Δt. What happens to the solution $\mathbf{a}_k$ as $k \to \infty$? Do your results *qualitatively* match the behavior of the exact solution $\mathbf{a}(t)$ as $t \to \infty$? If an instability emerges in $\mathbf{a}_k$ as $k \to \infty$, explain that in terms of the eigenvectors of $\mathbf{A}$. (Which eigenvectors dominate the unstable solution?)

(c) Repeat part (b), but now use the backward Euler method.

(d) Repeat part (b), but now use the trapezoid method introduced in Problem Set 6.

(e) Now suppose we add damping to the wave equation,

$$u_{tt}(x,t) = u_{xx}(x,t) - 2du_t(x,t),$$

for $d > 0$, as in Problem 1. The damping contributes a new term to the differential equation for the coefficients $a_1(t), \ldots, a_N(t)$:

$$\mathbf{M}\mathbf{a}''(t) = -\mathbf{K}\mathbf{a}(t) - 2d\mathbf{M}\mathbf{a}'(t).$$

Show how the new $-2d\mathbf{M}\mathbf{a}'(t)$ term can be incorporated into the first order form

$$\begin{bmatrix} \mathbf{a}'(t) \\ \mathbf{a}''(t) \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathbf{a}(t) \\ \mathbf{a}'(t) \end{bmatrix}.$$

That is, show how the matrix $\mathbf{A}$ changes to include damping.

Compute the eigenvalues of this matrix for $d = \pi$ and $N = 16$.

Based on these eigenvalues, describe the relative merits of solving this system with the forward Euler, backward Euler, and trapezoid method. Recall that with damping, we expect $\mathbf{a}(t) \to \mathbf{0}$ as $t \to \infty$.