CMDA 4604 · INTERMEDIATE TOPICS IN MATHEMATICAL MODELING

Problem Set 7

Posted Monday 9 November 2015. Due Monday 16 November 2015, 5pm.
This assignment is worth 75 points

1. [75 points: (a)=11 pts; (b)=6 pts; (c)=6 pts; (d), (e), (f), (g)=13 pts each]
This problem considers differential equations in two space dimensions. We begin with the steady-state problem. In place of the one dimensional equation, $-u'' = f$, we now have

$$-(u_{xx}(x,y) + u_{yy}(x,y)) = f(x,y), \qquad 0 \le x \le 1, \quad 0 \le y \le 1,$$

with homogeneous Dirichlet boundary conditions $u(x,0) = u(x,1) = u(0,y) = u(1,y) = 0$ for all $0 \le x \le 1$ and $0 \le y \le 1$. The associated operator L is defined as

$$Lu = -(u_{xx} + u_{yy}),$$

acting on the space $C_D^2[0,1] \times C_D^2[0,1]$ consisting of twice continuously differentiable functions on $[0,1] \times [0,1]$ with homogeneous boundary conditions. We can solve the differential equation $Lu = f$ using the spectral method just as we have done in one dimension. This problem will walk you through the process; you may consult Section 8.2 of the text for additional hints.

Important note: Chebfun supports functions of two variables, but uses a different syntax.

define the function f:	`f = chebfun2(@(x,y) x.*(1-y),[0 1 0 1]);`
define the function $\Psi_{j,k}$:	`Psi = chebfun2(@(x,y) 2*sin(j*pi*x).*sin(k*pi*y),[0 1 0 1]);`
compute $(f, \Psi_{j,k})$:	`sum2(f.*Psi);`
surface plot of f:	`plot(f)`

(a) Show that L is symmetric, given the inner product

$$(v,w) = \int_0^1 \int_0^1 v(x,y)w(x,y)\,dx\,dy.$$

(b) Verify that the functions

$$\psi_{j,k}(x,y) = 2\sin(j\pi x)\sin(k\pi y)$$

are eigenfunctions of L for $j,k = 1,2,\ldots$.
(To do this, you simply need to show that $L\psi_{j,k} = \lambda_{j,k}\psi_{j,k}$ for some scalar $\lambda_{j,k}$.)

(c) What is the eigenvalue $\lambda_{j,k}$ associated with $\psi_{j,k}$?

(d) Let $f(x,y) = x(1-y)$. The solution to Laplace's equation is given by the spectral method, but now with a double sum to account for all the eigenvalues:

$$u(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{\lambda_{j,k}} \frac{(f,\psi_{j,k})}{(\psi_{j,k},\psi_{j,k})}\,\psi_{j,k}(x,y).$$

In MATLAB/Chebfun, plot the partial sum

$$u_5(x,y) = \sum_{j=1}^{5}\sum_{k=1}^{5} \frac{1}{\lambda_{j,k}} \frac{(f,\psi_{j,k})}{(\psi_{j,k},\psi_{j,k})}\,\psi_{j,k}(x,y).$$

(e) We seek the solution to the heat equation

$$u_t(x, y, t) = u_{xx}(x, y, t) + u_{yy}(x, y, t)$$

with homogeneous Dirichlet boundary conditions $u(x, 0) = u(x, 1) = u(0, y) = u(1, y) = 0$ and initial condition $u(x, y, 0) = u_0(x, y)$. The solution will have the form

$$u(x, y, t) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{j,k}(t)\psi_{j,k}(x, y).$$

Let $u_0(x, y) = 20xy(1 - x)(1 - y)\exp(\sin(5\pi x))$. Figure out a formula for $a_{j,k}(t)$, and use MATLAB/Chebfun to plot the partial sum

$$u_8(x, y, t) = \sum_{j=1}^{8} \sum_{k=1}^{8} a_{j,k}(t)\psi_{j,k}(x, y)$$

at times $t = 0, 0.01, 0.02, 0.03, 0.04,$ and 0.05. Use `zaxis([-2 2])` for all the plots, to make them easier to compare.

(f) Suppose we wish to add a *forcing function* $f(x, y, t)$ to the heat equation,

$$u_t(x, y, t) = u_{xx}(x, y, t) + u_{yy}(x, y, t) + f(x, y, t)$$

and *inhomogeneous Dirichlet boundary conditions*

$$u(x, 0) = \alpha(x), \quad u(x, 1) = \beta(x), \quad u(0, y) = \gamma(y), \quad u(1, y) = \delta(y).$$

Explain (in detail) how to modify the procedure in part (e) to give a formula for $u(x, y, t)$ with the forcing function and inhomogeneous boundary conditions. (You do not need to do any MATLAB computations for this problem; just describe the procedure in detail.)

(g) Now we seek the solution to the wave equation

$$u_{tt}(x, y, t) = u_{xx}(x, y, t) + u_{yy}(x, y, t)$$

with homogeneous Dirichlet boundary conditions $u(x, 0) = u(x, 1) = u(0, y) = u(1, y) = 0$ and initial condition $u(x, y, 0) = u_0(x, y)$ and $u_t(x, y, 0) = 0$. Write the solution in the form

$$u(x, y, t) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{j,k}(t)\psi_{j,k}(x, y).$$

Again let $u_0(x, y) = 20xy(1 - x)(1 - y)\exp(\sin(5\pi x))$. Figure out a formula for $a_{j,k}(t)$, and use MATLAB/Chebfun to plot the partial sum

$$u_8(x, y, t) = \sum_{j=1}^{8} \sum_{k=1}^{8} a_{j,k}(t)\psi_{j,k}(x, y)$$

at times $t = 0, 0.02, 0.04, 0.06, 0.08,$ and 0.10. Use `zaxis([-2 2])` for all the plots, to make them easier to compare.