CMDA 4604 · INTERMEDIATE TOPICS IN MATHEMATICAL MODELING

Problem Set 6

Posted Sunday 1 November 2015. Due Monday 9 November 2015, 5pm.

1. [30 points: (a)=4 points; (b)=5 points; (c)=7 points; (d)=7 points; (e)=7 points]
   Consider the following three matrices:

$$\text{(i)} \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{(ii)} \quad \mathbf{A} = \begin{bmatrix} -50 & 49 \\ 49 & -50 \end{bmatrix} \qquad \text{(iii)} \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

   (a) For each of the matrices (i)–(iii), compute the matrix exponential $e^{t\mathbf{A}}$.

   You may use `eig` for the eigenvalues and eigenvectors, but please construct the matrix exponential "by hand" (not with `expm`). For diagonalizable $\mathbf{A} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}$, recall the formula $e^{t\mathbf{A}} = \mathbf{V}e^{t\boldsymbol{\Lambda}}\mathbf{V}^{-1}$. If you encounter a complex eigenvalue $\lambda = \alpha + i\beta$, you may use the formula

$$e^{t\lambda} = e^{t(\alpha+i\beta)} = e^{t\alpha}(\cos(t\beta) + i\sin(t\beta)).$$

   (b) Use your answers in part (a) to explain the behavior of solutions $\mathbf{y}(t)$ to the differential equation $\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t)$ as $t \to \infty$, given that $\mathbf{y}(0) = [2,\ 0]^T$ (e.g., specify and explain exponential growth, exponential decay, or neither) for each of the three matrices (i)–(iii).

   (c) For the matrix (ii), describe how large one can choose the time step Δt so that the forward Euler method applied to $\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t)$,

$$\mathbf{y}_{k+1} = \mathbf{y}_k + \Delta t\mathbf{A}\mathbf{y}_k,$$

   will produce a solution that qualitatively matches the behavior of the true solution (i.e., the approximations $\mathbf{y}_k$ should grow, decay, or remain of the same size as the true solution does).

   Answer the same question for the backward Euler method

$$\mathbf{y}_{k+1} = \mathbf{y}_k + \Delta t\mathbf{A}\mathbf{y}_{k+1}.$$

   (d) For the matrix in (iii), describe how the forward Euler method behaves *for all* Δt as $k \to \infty$ for $\mathbf{y}(0) = [1, 1]^T$. Now describe how the backward Euler method must behave as $k \to \infty$ for the same matrix and initial condition.

2. [30 points: (a)=4 points; (b)=5 points; (c)=7 points; (d)=7 points; (e)=7 points]
   There exist a host of alternatives to the forward and backward Euler methods for approximating the solution of the differential equation $\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t)$. This problem and the next consider more sophisticated techniques.

   The *trapezoid method* has the form

$$\mathbf{y}_{k+1} = \mathbf{y}_k + \tfrac{1}{2}\Delta t\mathbf{A}(\mathbf{y}_k + \mathbf{y}_{k+1}),$$

   where $\Delta t > 0$ is the time-step, and $\mathbf{y}_k \approx \mathbf{y}(t_k)$ for $t_k = k\Delta t$.

   (a) Like backward Euler, the trapezoid method is an *implicit* technique: $\mathbf{y}_{k+1}$ appears on both the right and left hand side of the formula above that defines it. Describe how to find $\mathbf{y}_{k+1}$ given $\mathbf{y}_k$. In particular, what linear system of algebraic equations needs to be solved at each step? (For comparison, the backward Euler method requires the solution of the system $(\mathbf{I} - \Delta t\mathbf{A})\mathbf{y}_{k+1} = \mathbf{y}_k$ for the unknown $\mathbf{y}_{k+1}$ at each step.)

(b) Consider the matrix and initial condition

$$\mathbf{A} = \begin{bmatrix} -1 & 10 \\ 0 & -2 \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Approximate the solution to $\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t)$ on the interval $t \in [0,5]$ for time step $\Delta t = .05$. Produce a `semilogy` plot showing $t_k = k\Delta t$ versus $\|\mathbf{y}_k\|$ for $k = 0,\ldots,100$. (Use the `norm` command in MATLAB to compute $\|\mathbf{y}_k\|$.)

(c) We wish to understand how the error in our approximation at time $t = 1$ improves as we run the simulation with smaller and smaller Δt values. Produce a `loglog` plot showing Δt versus the error in the trapezoid rule and backward Euler approximations for the matrix and initial condition in part (b) at time $t = 1$. To compute the error, first find the exact solution $\mathbf{y}(1) = \mathrm{e}^{\mathbf{A}}\mathbf{y}(0)$ using the `expm` command, then compute the norms $\|\widehat{\mathbf{y}} - \mathbf{y}(1)\|$, where $\widehat{\mathbf{y}}$ denotes your approximation to $\mathbf{y}(1)$ from the trapezoid or backward Euler methods. Start your plot with $\Delta t = 1/2$ and use sufficiently many smaller values of Δt to make the trend in your plot clear. For which method does the error decay more rapidly as $\Delta t \to 0$?

(d) To get some insight into the accuracy you observed in part (c), apply the trapezoid rule to your the scalar equation $y'(t) = \alpha y(t)$ with $y(0) = y_0 = 1$ for just one step. Write y_1 as a Taylor series, and compare this to the Taylor series for the true solution $y(\Delta t) = e^{\alpha\Delta t}$. How many terms in these Taylor series agree?

Hint: For the backward Euler method,

$$y_1 = (1 - \alpha\Delta t)^{-1} = 1 + \alpha(\Delta t) + \alpha^2(\Delta t)^2 + \cdots,$$

using the expansion $(1-z)^{-1} = 1 + z + z^2 + \cdots$ for small $|z|$.

(e) Forward Euler iterates can be written as $\mathbf{y}_k = (\mathbf{I} + \Delta t\mathbf{A})^k\mathbf{y}_0$, while backward Euler iterates can be written as $\mathbf{y}_k = (\mathbf{I} - \Delta t\mathbf{A})^{-k}\mathbf{y}_0$.

Work out a similar formula for the iterates $\mathbf{y}_k$ generated by the trapezoid method.

Suppose $\mathbf{A}$ is a diagonalizable matrix and all its eigenvalues α_j, $j = 1,\ldots,n$, are negative real numbers. How must you choose the time step Δt to ensure that the iterates $\mathbf{y}_k$ generated by the trapezoid method converge to zero, $\|\mathbf{y}_k\| \to 0$, as $k \to \infty$?

3. [22 points: (a)=5 points; (b)=5 points; (c)=5 points; (d)=7 points]
The *trapezoid method* in Problem 1 is an implicit method. Here we consider an explicit alternative, a member of the class of *Runge–Kutta methods* called *Heun's method*. When applied to $\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t)$, this technique gives

$$\mathbf{y}_{k+1} = \mathbf{y}_k + \tfrac{1}{2}\Delta t\Big(\mathbf{A}\mathbf{y}_k + \mathbf{A}\big(\mathbf{y}_k + \Delta t\mathbf{A}\mathbf{y}_k\big)\Big).$$

(a) In what way can you view Heun's method as an explicit approximation of the trapezoid method?

(b) Repeat Problem 1(d) for Heun's method: apply the method to $y'(t) = \alpha y(t)$ with $y(0) = 1$. How many terms in the Taylor series for $y(\Delta t) = e^{\alpha\Delta t}$ does the Taylor series for y_1 match?

(c) Repeat Problem 1(c) for Heun's method.

(d) Again suppose $\mathbf{A}$ is a diagonalizable matrix with eigenvalues α_j that are negative real numbers. How must you choose the time step Δt to ensure that the Heun approximations $\mathbf{y}_k$ behave qualitatively correctly ($\mathbf{y}_k \to \mathbf{0}$) as $k \to \infty$? (Simplify as much as possible.) How does this stability restriction compare to the one we obtained in class for forward Euler?

4. [18 points: (a)=9 points; (b)=9 points]
Consider the differential equation $u_t(x,t) = -u_{xxxx}(x,t)$ for $x \in [0,1]$ with the hinged boundary conditions $u(0,t) = u_{xx}(0,t) = u(1,t) = u_{xx}(1,t) = 0$ and initial condition $u(x,0) = u_0(x)$.

As you saw on the practice midterm, with the fourth order derivative in space this problem does not lend itself to approximation with hat functions. Instead we use piecewise cubic Hermite polynomials.

(See the optional Problem 7 of Problem Set 2 for details.) As before, we end up with a differential equation of the form

$$\mathbf{M}\mathbf{a}'(t) = -\mathbf{K}\mathbf{a}(t),$$

where $\mathbf{M}$ is the mass matrix for the piecewise cubic Hermite basis functions, and $\mathbf{K}$ is the stiffness matrix associated with the fourth derivative appleid to the same basis functions.

The code `beam_mats.m` on the class website will produce the mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$ for the beam problem, given a grid size N.

(a) For $N = 16$, use numerically computed eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ to determine the upper limit on Δt values that will give stable solution (i.e., $\mathbf{a}_k \to \mathbf{0}$ as $k \to \infty$) for forward Euler.

(b) Now successively double N, and study how this critical maximum Δt value changes. For the heat equation, we proved that doubling N required you to quarter Δt. Based on your numerical calculations, can you deduce a similar rule for the fourth-order equation?