CMDA 4604 · INTERMEDIATE TOPICS IN MATHEMATICAL MODELING

Problem Set 5

Posted Wednesday 21 October 2015. Du Wednesday 28 October 2015, 5pm.

Solve problems 1 and 2, and either 3 or 4 (to make a total of 100 points).

1. [45 points]
   This problem considers the heat equation with homogeneous Neumann boundary conditions.

   (a) Consider the function $u_0(x) = \begin{cases} 1, & x \in [0, 1/2]; \\ 0, & x \in (1/2, 2/3); \\ 1, & x \in [2/3, 1]. \end{cases}$
   Recall that the eigenvalues of the operator $L : C^2_N[0,1] \to C[0,1]$,

   $$Lu = -u''$$

   are $\lambda_n = n^2\pi^2$ for $n = 0, 1, \ldots$ with associated (normalized) eigenfunctions $\psi_0(x) = 1$ and

   $$\psi_n(x) = \sqrt{2}\cos(n\pi x), \qquad n = 1, 2, \ldots.$$

   We wish to write $u_0(x)$ as a series of the form

   $$u_0(x) = \sum_{n=0}^{\infty} a_n(0)\psi_n(x),$$

   where $a_n(0) = (u_0, \psi_n)$.

   - Compute formulas for these inner products $a_n(0) = (u_0, \psi_n)$ (by hand, Mathematica, etc.; don't leave in integral form).
   - For $m = 0, 1, 2, 4, 80$, plot the partial sums

     $$u_{0,m}(x) = \sum_{n=0}^{m} a_n(0)\psi_n(x).$$

     (You may superimpose these on one single, well-labeled plot if you like.)

   (b) Write down a series solution to the homogeneous heat equation

   $$u_t(x,t) = u_{xx}(x,t), \qquad 0 \le x \le 1, \quad t \ge 0$$

   with Neumann boundary conditions

   $$u_x(0,t) = u_x(1,t) = 0$$

   for the initial condition $u(x, 0) = u_0(x)$ given in part (a).
   Plot the solution at times $t = 0, 0.002, 0.05, 0.1$.
   You will need to truncate your infinite series to show this plot.
   Include enough terms for your plots to appear converged.

   (c) Describe the behavior of your solution as $t \to \infty$.
   (To do so, write down a formula for the solution in the limit $t \to \infty$, if such a limit exists.)

(d) Now consider the inhomogeneous heat equation

$$u_t(x,t) = u_{xx}(x,t) + 1, \qquad 0 \le x \le 1, \quad t \ge 0$$

with Neumann boundary conditions

$$u_x(0,t) = u_x(1,t) = 0.$$

Work out the solution $u(x,t)$ to this equation with forcing function $f(x,t) = 1$.

Plot the solution at times $t = 0, 0.002, 0.05, 0.1, 0.5$ (one time more than in part (b)). Does your solution tend to a steady state?

2. [25 points]

(a) Describe how to solve the heat equation

$$u_t(x,t) = u_{xx}(x,t) + f(x,t), \qquad 0 < x < 1, \quad t \ge 0$$

with *inhomogeneous* Neumann boundary conditions

$$u_x(0,t) = \alpha, \qquad u_x(1,t) = \beta$$

and initial condition $u(x,0) = u_0(x)$.
(Hint: *you are permitted to alter f and u_0 if that is helpful. . . .*)

(b) Consider the inhomogeneous heat equation from Problem 1(d): $u_t(x,t) = u_{xx}(x,t) + f(x,t)$ with $f(x,t) = 1$ and $u(x,0) = u_0(x)$ as specified in Problem 1. In place of the homogeneous Neumann conditions $u_x(0,t) = u_x(1,t) = 0$, we now want to use

$$u_x(0,t) = \alpha = -1, \qquad u_x(1,t) = \beta = 2.$$

Using your solution to Problem 2(a), repeat Problem 1(d) with these inhomogeneous Neumann boundary conditions. (You may use Chebfun to compute all inner products you need.)

3. [30 points: do either Problem 3 or 4]
Consider the *fourth order* partial differential equation

$$u_t(x,t) = u_{xx}(x,t) - u_{xxxx}(x,t)$$

with *hinged* boundary conditions

$$u(0,t) = u_{xx}(0,t) = u(1,t) = u_{xx}(1,t) = 0$$

and initial condition (that should satisfy the boundary conditions)

$$u(x,0) = u_0(x).$$

(This equation is related to a model that arises in the study of thin films.)
To solve this PDE, we introduce the linear operator $L : C_H^4[0,1] \to C[0,1]$, where

$$Lu = -u'' + u''''$$

and

$$C_H^4[0,1] = \{u \in C^4[0,1], u(0) = u''(0) = u(1) = u''(1) = 0\}$$

is the set of C^4 functions that satisfy the hinged boundary conditions.

(a) The operator L has eigenfunctions

$$\psi_n(x) = \sqrt{2}\sin(n\pi x), \qquad n = 1, 2, \ldots.$$

Use this fact to compute a formula for the eigenvalues λ_n, $n = 1, 2, \ldots$.

(b) Suppose the initial condition $u_0(x)$ is expanded in the form

$$u_0(x) = \sum_{n=1}^{\infty} a_n(0)\psi_n(x).$$

Briefly describe how one can write the solution to the PDE $u_t = u_{xx} - u_{xxxx}$ as an infinite sum.

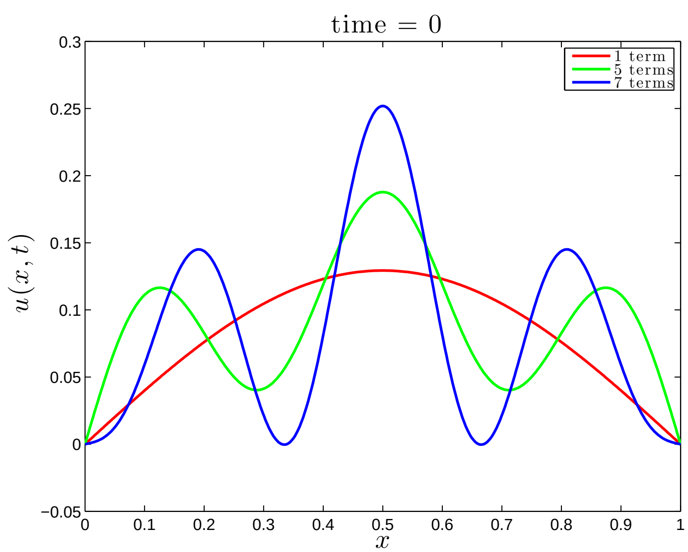
(c) Suppose the initial data is given by

$$u_0(x) = (x - x^2)\sin(3\pi x)^2,$$

with associated coefficients

$$a_n(0) = \begin{cases} \dfrac{432\sqrt{2}(n^4 - 18n^2 + 216)}{(36n - n^3)^3\pi^3}, & n \text{ odd;} \\ 0, & n \text{ even.} \end{cases}$$

Use Chebfun to compute the solution you describe in part (b) up to seven terms in the infinite sum. At each time $t = 0$; 10^{-5}; 2×10^{-5}; 4×10^{-5}, produce a plot comparing the sum of the first 1, 5, and 7 terms of the series. For example, at time $t = 0$, your plot should appear as shown below. (Alternatively, you can produce attractive 3-dimensional plots over the time interval $t \in [0, 4 \times 10^{-5}]$ using 1, 5, and 7 terms in the series.)



4. [30 points: do either Problem 3 or 4]
Recall the *advection–diffusion* equation from Problem Set 3. We now wish to solve the time-dependent version of this important equation in physics and engineering:

$$u_t(x,t) = u_{xx}(x,t) - cu_x(x,t)$$

for $x \in [0,1]$ with $u(0,t) = u(1,t) = 0$ and initial condition $u(x,0) = u_0(x)$. (The $-u_{xx}$ term describes diffusion of a fluid; the constant c describes the strength with which the fluid advects across the domain through the cu_x term.) You might review the solution to question 5 on Problem Set 3 before proceeding.

Define the linear operator $L : C^2_D[0,1] \to C[0,1]$ by $Lu = -u'' + cu'$. The eigenvalues of L are

$$\lambda_n = n^2\pi^2 + \frac{c^2}{4}, \qquad n = 1, 2, \ldots$$

with corresponding eigenfunctions

$$\psi_n(x) = e^{cx/2}\sin(n\pi x), \qquad n = 1, 2, \ldots.$$

Recall that L is not symmetric, but has an adjoint $L^* : C^2_D[0,1] \to C[0,1]$ given by $L^*u = -u'' - cu'$. This adjoint has the same eigenvalues as L, but its eigenfunctions are instead

$$\widehat{\psi}_n(x) = e^{-cx/2}\sin(n\pi x), \quad n = 1, 2, \ldots.$$

Notice that the eigenfunctions of L and L^* are *biorthogonal*:

$$(\psi_n, \widehat{\psi}_m) = 0, \qquad n \neq m.$$

In particular, when expanding a function f in the eigenfunctions $\{\psi_n\}$, the coefficients will involve eigenfunctions of both L and L^*:

$$f(x) = \sum_{n=1}^{\infty} \frac{(f, \widehat{\psi}_n)}{(\psi_n, \widehat{\psi}_n)} \psi_n(x).$$

(a) Adapt our strategy for solving the heat equation to now solve

$$u_t(x,t) = u_{xx}(x,t) - cu_x(x,t)$$

with $u(0,t) = u(1,t) = 0$ with some generic initial condition $u(x,0) = u_0(x)$. More specifically, write down a series solution for $u(x,t)$ in the eigenfunctions $\{\psi_n\}$.

(b) Use Chebfun to plot solutions to the advection diffusion equation with initial condition

$$u_0(x) = x(1-x)^2.$$

and advection parameter $c = 1$. (You can use Chebfun to compute the inner products needed to expand u_0.) Show your solution at times $t = 0, 0.01, 0.02, \ldots, 0.10$. Take enough terms in your series so that the solution appears converged at all these times.

(c) Repeat part (b), but now with parameter $c = 5$. You should now see your initial condition clearly *advect* across the domain, from left to right, as time increases.

(d) In part (c): at what time (roughly) does $\max_{x\in[0,1]} |u(x,t)|$ take its maximum value? How does this contrast to what you are accustomed to seeing for the heat equation?