CMDA 4604 · INTERMEDIATE TOPICS IN MATHEMATICAL MODELING

Problem Set 4

Posted Monday 5 October 2015. Due Monday 12 October 2015, 5pm.

All of the problems on this set use the inner product

$$(u, v) = \int_0^1 u(x)v(x)\,dx.$$

1. [20 points]
   On Problem Set 3, Question 1(d), we sought the solution of the equation $-u''(x) = f(x)$, $u(0) = u(1) = 0$, where

   $$f(x) = \begin{cases} 1, & 0 \le x < 1/2; \\ 0, & 1/2 < x \le 1. \end{cases}$$

   The exact solution to this problem is:

   $$u(x) = \begin{cases} 3x/8 - x^2/2, & 0 \le x < 1/2; \\ (1-x)/8, & 1/2 < x \le 1. \end{cases}$$

   You can implement u in Chebfun via the commands:

```
x = chebfun('x',[0 1]);
u = chebfun({3*x/8-(x.^2)/2,(1-x)/8},[0 .5 1],'splitting','on');
```

   Throughout this problem, let $\phi_1, \ldots, \phi_N$ denote the usual hat functions for a grid with uniform spacing equal to $h = 1/(N+1)$.

   (a) Suppose that N is odd. Compute *by hand* the inner products (f, ϕ_j) for $j = 1, \ldots, N$.
   (b) Modify the `fem1.m` code on the class website (or write your own) to construct approximate solutions u_N to this equation via the finite element method on the subspace $V_N = \text{span}\{\phi_1, \ldots, \phi_N\}$. Produce plots comparing the exact solution $u(x)$ to the approximate solution $u_N(x)$ for $0 \le x \le 1$ for $N = 3, 7, 15, 31, 63$.
   (c) Produce plots of the error $u(x) - u_N(x)$ for $N = 3, 7, 15, 31, 63$. (These may all be on one plot.)

2. [30 points]
   On Problem Set 3, Question 3, you sought solutions to the equation $-u''(x) + 7u(x) = 1$ with $u(0) = u(1) = 0$. The problem considers the finite element method for this equation.

   (a) Suppose that u satisfies the differential equation. Show that for all $v \in C_D^2[0,1]$, the weak form holds:

   $$a(u, v) = (f, v),$$

   where $a(u, v) = \int_0^1 \big(u'(x)v'(x) + 7u(x)v(x)\big)\,\mathrm{d}x$ and $(f, v) = \int_0^1 f(x)v(x)\,\mathrm{d}x$.

   (b) We now wish to approximate the solution to this equation using the finite element method with standard hat functions $\phi_1, \ldots, \phi_N$ on a grid with uniform spacing, $h = 1/(N+1)$.
   Calculate (by hand) the entries in the stiffness matrix $\mathbf{K}$, i.e., $\mathbf{K}_{j,k} = a(\phi_j, \phi_k)$, for the energy inner product in part (a).
   Hint: you can write your answer in terms of the stiffness matrix we obtained for $-u''(x) = f(x)$, which we shall call $\mathbf{K}_0$ here, and the Gram matrix for hat functions, which we shall call $\mathbf{M}$ here:

   $$\mathbf{K}_0 = \frac{1}{h}\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}, \quad \mathbf{M} = h\begin{bmatrix} 2/3 & 1/6 & & \\ 1/6 & 2/3 & \ddots & \\ & \ddots & \ddots & 1/6 \\ & & 1/6 & 2/3 \end{bmatrix},$$

   where unspecified entries are zero.

(c) Use your new stiffness matrix to construct approximate solutions u_N to $-u''(x) + 7u(x) = f(x)$ with $u(0) = u(1) = 0$ for $N = 2, 4, 8, 16$. Produce a plot showing your approximation solutions u_N and the exact solution, which can be implemented in Chebfun via the commands:

```
x = chebfun('x',[0 1]);
u = (1+exp(sqrt(7))-exp(-sqrt(7)*(x-1))-exp(sqrt(7)*x))/(7*(1+exp(sqrt(7))));
```

(d) Produce plots showing the error $u(x) - u_N(x)$ for $N = 2, 4, 8, 16$. (These may all be on one plot.)

3. [30 points]
In class we have derived the weak form of the differential equation $-u''(x) = f(x)$ with (a) homogeneous Dirichlet boundary conditions, $u(0) = u(1) = 0$, and (b) homogeneous Neumann boundary conditions, $u'(0) = u'(1) = 0$. Recall that the Dirichlet boundary conditions were *essential*, meaning that we had to use test functions that satisfy this boundary conditions, $v \in C^2_D[0, 1]$, while the Neumann bounday conditions were *natural*, meaning that the test functions did not need to include the boundary conditions, $v \in C^2[0, 1]$. (They key issue was making sure the boundary term that arose during integration by parts is zero.)

(a) Derive the weak form of the differential equation $-u''(x) = f(x)$ for the *mixed* boundary conditions $u'(0) = u(1) = 0$. That is, show that if $-u''(x) = f(x)$, then $a(u, v) = (f, v)$. Specify the energy inner product $a(u, v)$, and be very clear about what space the test functions v should come from.

(b) We wish to construct finite element approximate solutions to $-u''(x) = x$ with $u'(0) = u(1) = 0$, using hat functions on a grid with uniform spacing $h = 1/(N + 1)$. In particular, we will use the approximation space

$$V_N = \text{span}\{\phi_0, \phi_1, \dots, \phi_N\}.$$

Write out the entries of the stiffness matrix $\mathbf{K} \in \mathbb{R}^{(N+1)\times(N+1)}$.

(c) Write MATLAB code to compute the finite element approximations u_N from V_N, and produce a plot comparing the solutions $u_N(x)$ to the exact solution $u(x) = (1 - x^3)/6$, for $N = 2, 4, 8, 16$. (You may plot the errors $u(x) - u_N(x)$, if the approximate solutions are difficult to tell apart.)

(d) Do the approximate solutions u_N satisfy the boundary conditions exactly?
That is, does $u_N'(0) = 0$? Does $u_N(1) = 0$?

4. [20 points]
We wish to solve the differential equation

$$-(\kappa(x)u'(x))' = f(x)$$

with $0 \le x \le 1$ and $\kappa(x) = \mathrm{e}^x$, subject to homogeneous Dirichlet boundary conditions:

$$u(0) = u(1) = 0.$$

With this simple coefficient $\kappa(x) = \mathrm{e}^x$ the eigenfunctions would take an intricate form, thus making the spectral method difficult to implement. Thus this problem is a prime candidate for solution with the finite element method. The weak form of the problem is:

$$a(u, v) = (f, v)$$

for all $v \in C^2_D[0, 1]$, where $a(u, v) = \int_0^1 \kappa(x)u'(x)v'(x)\,dx$.
For this problem, we shall use

$$f(x) = x,$$

giving the exact solution

$$u(x) = \frac{(1 - \mathrm{e})x^2 + (2 - 2\mathrm{e})x + 3(\mathrm{e}^x - 1)}{2(1 - \mathrm{e})\mathrm{e}^x}.$$

As usual, we shall look for finite element solutions from the span of hat functions,

$$V_N = \text{span}\{\phi_1, \dots, \phi_N\}.$$

(a) Write down a formula for $a(\phi_j, \phi_\ell)$, the (j, ℓ) entry of the stiffness matrix $\mathbf{K}$. (Hint: your answer can depend on the grid points $x_j = jh$.)

(b) Write MATLAB code to compute the finite element approximation u_N from V_N, and produce a plot showing the solutions $u_N(x)$ for $N = 4, 8, 16, 32$.

(c) Write MATLAB code to plot the error $u(x) - u_N(x)$ for $N = 4, 8, 16, 32$.