CMDA 4604 · INTERMEDIATE TOPICS IN MATHEMATICAL MODELING

Problem Set 3

Posted Wednesday 23 September 2015. Due Monday 5 October 2015, 5pm.

All of the problems on this set use the inner product

$$(u, v) = \int_0^1 u(x)v(x)\,dx.$$

Complete any four of these five problems. (Only four will be counted for credit.)

1. [25 points]
   Consider the function $f(x) = \begin{cases} 1, & 0 \le x < 1/2; \\ 0, & 1/2 < x \le 1. \end{cases}$

   (a) Let $\psi_n(x) = \sqrt{2}\sin(n\pi x)$. Work out a formula for (ψ_n, f) (as a function of n).

   (b) Write a MATLAB/Chebfun code to plot the best approximation to f from $\operatorname{span}\{\psi_1, \ldots, \psi_n\}$:

   $$f_N = \sum_{n=1}^{N} \frac{(\psi_n, f)}{(\psi_n, \psi_n)} \psi_n.$$

   Plot $f_N(x)$ for $x \in [0, 1]$ for $N = 4, 16, 64$.
   N.B. You can implement f in Chebfun via the command
   ```
   f = chebfun({1,0},[0 .5 1],'splitting','on')
   ```

   (c) Since $f_N(0) = 0$ but $f(0) = 1$, we cannot have $f_N(x) \to f(x)$ for all $x \in [0, 1]$ as $N \to \infty$. We wish to study another kind of convergence here. Produce a `loglog` plot showing $N = 2, 4, 8, 16, 32, 64$ (or larger, if you like) on the horizontal axis, and $\|f - f_N\|$ on the vertical axis. (Note that you can compute $\|f - f_N\|$ in Chebfun using the `norm` command.)

   (d) Now we seek to solve $-u''(x) = f(x)$ with $u(0) = u(1) = 0$. Use the spectral method discussed in class to construct best approximations u_N to u from $\operatorname{span}\{\psi_1, \ldots, \psi_N\}$. (Do not worry about the fact that f is not continuous, and hence not in $C[0, 1]$. Forge ahead and see if you get a reasonable solution.) Plot the solutions u_N for a few values of N to show convergence.

2. [25 points]
   Consider the linear operator $L : C_b^2[0, 1] \to C[0, 1]$ defined by

   $$Lu = -\frac{d^2u}{dx^2},$$

   where

   $$C_b^2[0, 1] = \Big\{u \in C^2[0, 1] : \frac{du}{dx}(0) = u(1) = 0\Big\}.$$

   (a) Is L symmetric?

   (b) Find the eigenvalues λ_n and eigenfunctions ψ_n of L.

   For each of the following two subproblems, write down the solution u_N, which will involve a sum of N terms involving a best approximation from $\operatorname{span}\{\psi_1, \ldots, \psi_N\}$ (possibly with a correction term to account for boundary conditions). Plot the approximate solutions u_N for several values of N (e.g., $N = 2, 4, 8$). (You may use Chebfun.)

(c) $-\dfrac{d^2u}{dx^2}(x) = 10x, \quad \dfrac{du}{dx}(0) = u(1) = 0.$

(d) $-\dfrac{d^2u}{dx^2}(x) = 10x, \quad \dfrac{du}{dx}(0) = u(1) = 1.$

(e) $-\dfrac{d^2u}{dx^2}(x) = x\,\mathrm{e}^x \sin(20x), \quad \dfrac{du}{dx}(0) = u(1) = 0.$ (Take N large enough to show convergence.)

3. [25 points]

(a) Suppose $\mathbf{A} \in \mathbb{R}^{N\times N}$ is a symmetric matrix with eigenvalues λ_j and associated eigenvectors $\mathbf{v}_j \neq \mathbf{0}$:

$$\mathbf{A}\mathbf{v}_j = \lambda_j \mathbf{v}_j, \qquad j = 1, \ldots, N.$$

Show that the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_N$ are also eigenvectors of the matrix $\mathbf{B} = \mathbf{A} + \beta\mathbf{I}$ for any fixed $\beta \in \mathbb{R}$. What are the corresponding eigenvalues $\mu_1, \ldots, \mu_N$ of $\mathbf{B}$?

For the rest of this problem, consider the linear operator $L : C^2_D[0,1] \to C[0,1]$, where

$$Lu = -\frac{d^2}{dx^2}u + 7u$$

and $C^2_D[0,1] = \{u \in C^2[0,1] : u(0) = u(1) = 0\}$ as usual.

(b) Prove that L is symmetric.

(c) Using part (a) or otherwise, determine the eigenvalues and eigenfunctions of L.

(d) Use the spectral method to determine the solution to the differential equation

$$-\frac{d^2}{dx^2}u + 7u = 1, \qquad u(0) = u(1) = 0.$$

That is, write down the formula for u_N, the best approximation to u from $\mathrm{span}\{\psi_1, \ldots, \psi_N\}$.

(e) Using MATLAB/Chebfun, plot your solutions $u_N(x)$ for several values of N to show convergence.

4. [25 points]

For the problems we have considered thus far, the eigenvalues have always satisfied nice formulas that are fairly easy to compute. This problem illustrates that this is not always the case.

Consider the equation

$$-u''(x) = f(x), \quad x \in [0,1]$$

with a homogeneous *Robin condition* on the left,

$$u(0) - u'(0) = 0$$

and a homogeneous Dirichlet boundary condition on the right,

$$u(1) = 0.$$

Define the linear operator $L : \mathcal{V} \to C[0,1]$ via $Lu = -u''$ with

$$\mathcal{V} = \{u \in C^2[0,1] : u(0) - u'(0) = u(1) = 0\}.$$

(a) Prove that L is symmetric.

(b) Is zero an eigenvalue of L? That is, does there exist a nontrivial solution to $-u''(x) = 0$ with these boundary conditions?

(c) Compute the eigenfunctions of L associated with nonzero eigenvalues, and show that these eigenvalues λ must satisfy the equation

$$\sqrt{\lambda} = -\tan(\sqrt{\lambda}).$$

(d) In MATLAB, create a plot of $g(x) = -\tan(x)$ for $x \in [0, 5\pi]$ and superimpose (`hold on`) a plot of $h(x) = x$. By hand, mark the points where these two functions intersect on your plot.

(e) Use your plot in (d) to argue that L has infinitely many eigenvalues, with $(n - \frac{1}{2})^2\pi^2 < \lambda_n < (n + \frac{1}{2})^2\pi^2$. To what value does λ_n tend as n becomes large?

(f) Estimate the first four eigenvalues to at least six digits of accuracy.
You will need to find the points of intersection you marked in part (d).
Please *don't* just try to 'eyeball' these by zooming in on your plot!
Instead, either use MATLAB's `fzero` function, or write your own implementation of a root-finding algorithm (Newton's method, bisection, etc.). Alternatively, you can use the `roots` command with Chebfun, but you need to be delicate with the domain over which you define your chebfuns....

5. [25 points]

Many fluid dynamics problems lead to *advection–diffusion* equations, the simplest example of which is

$$-u''(x) + cu'(x) = f(x),$$

for $x \in [0, 1]$ with $u(0) = u(1) = 0$. (The $-u''$ term describes diffusion of a fluid; the constant c describes the strength with which the fluid advects across the domain through the cu' term. Think of a drop of dye on the surface of water, with wind blowing across: the dye spreads out slowly in all directions due to diffusion; at the same time the dye advects across the water in the wind's direction.)

Define the linear operator $L : C_D^2[0, 1] \to C[0, 1]$ by $Lu = -u'' + cu'$.

(a) Show that the functions

$$\psi_n(x) = e^{cx/2}\sin(n\pi x), \qquad n = 1, 2, \ldots$$

are eigenfunctions of L. What are the corresponding eigenvalues, λ_n?
Are the eigenfunctions orthogonal? Explain.

(b) This operator is *not* symmetric. To construct solutions for this case, we must introduce the notion of the *adjoint*, which generalizes the transpose of a matrix. An operator $L^* : \mathcal{V} \to \mathcal{W}$ is the *adjoint* of $L : \mathcal{V} \to \mathcal{W}$ with $\mathcal{V} \subseteq \mathcal{W}$ provided $(Lu, v) = (u, L^*v)$ for all $u, v \in \mathcal{V}$.
Show that the adjoint $L^* : C_D^2[0, 1] \to C[0, 1]$ of L is given by

$$L^*u = -u'' - cu'.$$

That is, show that this definition of L^* gives $(Lu, v) = (u, L^*v)$ for all $u, v \in C_D^2[0, 1]$.

(c) Show that L^* has the same eigenvalues λ_n as L. What are the corresponding eigenfunctions of L^*? Denote the eigenfunctions of L^* by $\widehat{\psi}_n$.

(d) Show that $(\psi_m, \widehat{\psi}_n) = 0$ if $m \neq n$.
(This means that the eigenvectors of L and L^* are *biorthogonal*.)

(e) Suppose we seek an approximate solution of the form

$$u_N(x) = \sum_{n=1}^{N} \gamma_n \psi_n(x).$$

The best approximation approach we used to find u_N when L was symmetric no longer holds. Instead, impose the orthogonality condition on the eigenvectors of L^*:

$$(u - u_N, w) = 0 \quad \text{for all } w \in \text{span}\{\widehat{\psi}_1, \ldots, \widehat{\psi}_N\}.$$

Use this condition to derive a formula for each γ_n, $n = 1, \ldots, N$, and hence for the approximate solution u_N.

(f) Plot the approximate solution u_N to $-u''(x)+u'(x) = 1$ with $u(0) = u(1) = 0$ using the technique you designed in part (e) for several values of N. Do you obtain convergence? Does the performance of your method change if you increase c from $c = 1$ to $c = 10$ or $c = 100$?