CMDA 4604 · INTERMEDIATE TOPICS IN MATHEMATICAL MODELING

Problem Set 2

Posted Thursday 10 September 2015. Due Friday 18 September 2015, 5pm.

1. [24 points]
   Consider the polynomials $\phi_1(x) = 1$, $\phi_2(x) = x$, and $\phi_3(x) = 3x^2 - 1$, which form a basis for the set of all quadratic polynomials. These polynomials are orthogonal in $C[-1,1]$ with the usual inner product

$$(u, v) = \int_{-1}^{1} u(x)v(x)\,dx.$$

   (You may use this fact without proving it, i.e., you do not need to calculate (ϕ_j, ϕ_k) for $j \neq k$.)

   In the parts below, "best approximation" is defined with respect to this inner product and its norm. For this problem, please compute formulas for the various inner products you need (either "by hand" or with a symbolic algebra system like Mathematica or Wolfram Alpha).

   Let $v(x) = e^x$.

   (a) Construct the best approximation $w_1(x) = c_1\phi_1(x)$ to $v(x)$ from span$\{\phi_1\}$ (i.e., determine c_1 to minimize $\|f - w_1\|$ in $C[-1,1]$).

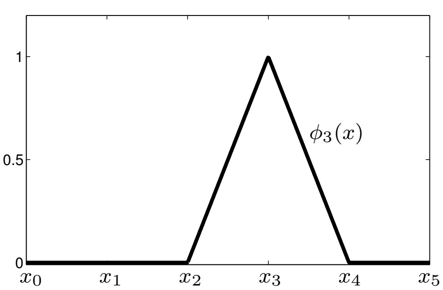
   (b) Construct the best approximation $w_2(x) = c_1\phi_1(x) + c_2\phi_2(x)$ to $v(x)$ from span$\{\phi_1, \phi_2\}$.

   (c) Construct the best approximation $w_3(x) = c_1\phi_1(x)+c_2\phi_2(x)+c_3\phi_3(x)$ to $v(x)$ from span$\{\phi_1, \phi_2, \phi_3\}$.

   (d) Produce a plot that superimposes your best approximation from parts (a), (b), and (c) on top of a plot of $v(x)$. (You may use Chebfun to plot these functions.)

2. [34 points]
   Suppose $N \geq 1$ is an integer and define $h = 1/(N+1)$ and $x_k = kh$ for $k = 0, \ldots, N+1$. Consider the N *hat functions*, defined as

$$\phi_k(x) = \begin{cases} (x - x_{k-1})/h, & x \in [x_{k-1}, x_k); \\ (x_{k+1} - x)/h, & x \in [x_k, x_{k+1}); \\ 0, & \text{otherwise.} \end{cases}$$



   The plot to the right shows $\phi_3(x)$ for $N = 4$.

   Consider the standard inner product on $C[0,1]$,

$$(u, v) = \int_{0}^{1} u(x)v(x)\,dx.$$

   In Problem Set 1, you showed that

$$(\phi_j, \phi_k) = \begin{cases} 2h/3, & j = k; \\ h/6, & |j-k| = 1; \\ 0, & |j-k| > 1. \end{cases}$$

   (a) Write MATLAB code to create the Gram matrix $\mathbf{G}$ (whose (j,k) entry is $g_{j,k} = (\phi_j, \phi_k)$) using these formulas for any specified value of N, and print out the Gram matrix for $N = 4$.

The rest of this problem involves computing the best approximation to $v(x) = x\sin(\pi x)$ for $x \in [0, 1]$ using the hat functions $\phi_1, \ldots, \phi_N$. To help with this problem, we have supplied a code called `hat.m` on the course website; `phi_j = hat(j,N)` creates a Chebfun representation of ϕ_j for the specified values of j and N. Recall that you can create a Chebfun representation of v with the command: `v = chebfun('x.*sin(pi*x)',[0 1])`. Then the inner product can be computed as $(\phi_j, v) =$ `phi_j'*v`.

(b) To compute the best approximation, we need values for (ϕ_j, v) for $j = 1, \ldots, N$ and $v(x) = x\sin(\pi x)$. Write the MATLAB code to compute the vector $\mathbf{b}$ whose jth entry is $b_j = (\phi_j, v)$, and display the resulting vector for $N = 4$.

(You may do this using Chebfun, which makes this very easy but gets slow for larger values of N. If you prefer, you can compute a formula for (ϕ_j, v) by hand (or using Mathematica etc.). The result will depend on j and N.)

(c) Use `c = G\b` in MATLAB to compute the coefficients $c_1, \ldots, c_N$ of the best approximation in MATLAB, and construct the best approximation

$$\widehat{w}(x) = \sum_{j=1}^{N} c_j \phi_j(x).$$

For each of $N = 2, 4, 8$, produce two plots:
(i) Plot $v(x) = x\sin(\pi x)$ and $\widehat{w}(x)$ for $x \in [0, 1]$. (Use `hold on` to superimpose two plots.)
(ii) Plot the error $v(x) - \widehat{w}(x)$ for $x \in [0, 1]$.

(d) For $N = 2, 4, 8, 16, 32$, construct the best approximation and compute the error

$$e_N = \max_{x\in[0,1]} |v(x) - \widehat{w}(x)|,$$

which can be accomplished in Chebfun simply as `max(abs(v-w_hat))`. Produce a `loglog` plot in MATLAB with the N values on the horizontal axis, and the e_N values on the vertical axis.

(You need only turn in your code and the plot for this part; you don't need to show your best approximations for all these N values.)

3. [24 points]
Recall that a function $f : \mathcal{V} \to \mathcal{W}$ that maps a vector space $\mathcal{V}$ to a vector space $\mathcal{W}$ is a *linear operator* provided (1) $f(u + v) = f(u) + f(v)$ for all u, v in $\mathcal{V}$, and (2) $f(\alpha v) = \alpha f(v)$ for all $\alpha \in \mathbb{R}$ and $v \in \mathcal{V}$.

Demonstrate whether each of the following functions is a linear operator.
(Show that both properties hold, or give an example showing that one of the properties must fail.)

(a) $f : \mathbb{R}^n \to \mathbb{R}^m$, $f(\mathbf{u}) = \mathbf{A}\mathbf{u}$ for a fixed matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$.

(b) $f : \mathbb{R}^n \to \mathbb{R}^m$, $f(\mathbf{u}) = \mathbf{A}\mathbf{u} + \mathbf{b}$ for a fixed matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ and fixed *nonzero* vector $\mathbf{b} \in \mathbb{R}^m$.

(c) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(\mathbf{x}) = \mathbf{x}^T\mathbf{x}$.

(d) $f : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, $f(\mathbf{X}) = \mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{B}$ for fixed matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$.

(e) $L : C^1[0, 1] \to C[0, 1]$, $Lu = u\dfrac{\mathrm{d}u}{\mathrm{d}x}$.

(f) $L : C^2[0, 1] \to C[0, 1]$, $Lu = \dfrac{\mathrm{d}^2u}{\mathrm{d}x^2} - \sin(x)\dfrac{\mathrm{d}u}{\mathrm{d}x} + \cos(x)u$.

please see the next page

4. [18 points]
Suppose that

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

(a) Compute *by hand* the eigenvalues and eigenvectors of this symmetric matrix.

(b) Verify *by hand* that these eigenvectors are orthogonal.

(c) Solve the linear system $\mathbf{Ax} = \mathbf{b}$ *using the spectral method*, where

$$\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

5. [optional, for those who want to think more deeply about projectors]
Recall that a linear operator P is a projection from the vector space $\mathcal{V}$ to the vector space $\mathcal{V}$ provided $P^2 = P$, that is, $P(Pf) = Pf$ for all $f \in \mathcal{V}$. Consider $\mathcal{V} = C[-1, 1]$ with the usual inner product

$$(u, v) = \int_{-1}^{1} u(x)v(x)\,dx,$$

and the two linear operators P_e and P_o that project a function onto their even and odd parts. That is,

$$(P_e f)(x) = \frac{f(x) + f(-x)}{2}, \qquad (P_o f)(x) = \frac{f(x) - f(-x)}{2}.$$

(a) Show that P_e and P_o are projections.

(b) Verify that $P_e f$ and $P_o f$ are orthogonal for any $f \in C[-1, 1]$.

(c) Is $P_e + P_o$ a projection? Explain.

6. [optional]
This problem invites you to make a deeper exploration of the convergence behavior of the best interpolant with hat functions determined in Problem 2.

(a) Repeat Problem 2(d), but now measure the error in the norm induced by the inner product:

$$e_N = \|v - \widehat{w}\| = \sqrt{(v - \widehat{w}, v - \widehat{w})}.$$

(Use the `norm` command in Chebfun.)

(b) Like the error in 2(d), you should find that e_N behaves, for sufficiently large N, according to

$$e_N \le Ch^p,$$

where $C > 0$ is some constant, $h = 1/(N + 1)$, and the constant $p > 0$ is *rate of convergence*. Use the numerical evidence from the last subproblem to estimate a value for p.

(c) [challenge problem]
Assume that $v \in C[0, 1]$ satisfies $v(0) = v(1) = 0$ (but it could differ from the function in Problem 2). Prove that

$$\|v - \widehat{w}\| \le Ch^p,$$

where you should specify values for C and p. You may assume that $|v(x)| \le M$ for all $x \in [0, 1]$, or alternatively that $v \in C^1[0, 1]$ and $|v'(x)| \le M$, where in each case $M > 0$ is some number that depends on v but not on N.

7. [optional]

The hat functions studied in Problem 2 are *piecewise linear*, and so the resulting approximation $\widehat{w}$ is in $C[0,1]$ but not $C^1[0,1]$. Some applications require a greater degree of smoothness; one appealing option is to use *cubic Hermite basis functions*; cf. [Burkhardt].

These basis functions are a bit more complicated to define than the usual hat functions. As before, let $N > 0$ be a parameter determining the uniform spacing $h = 1/(N+1)$ between the grid points $x_k = kh = k/(N+1)$ for $k = 0, \ldots, N+1$, so that $x_0 = 0$ and $x_{N+1} = 1$.

We define two classes of basis function. For $k = 0, \ldots, N+1$, let

$$\phi_k(x) = \begin{cases} \left(\dfrac{x - x_{k-1}}{h}\right)^2 \left(1 - \dfrac{2(x - x_k)}{h}\right), & x \in [x_{k-1}, x_k); \\ \left(\dfrac{x - x_{k+1}}{h}\right)^2 \left(1 + \dfrac{2(x - x_k)}{h}\right), & x \in [x_k, x_{k+1}); \\ 0, & \text{otherwise.} \end{cases}$$

and let

$$\psi_k(x) = \begin{cases} \left(\dfrac{x - x_{k-1}}{h}\right)^2 \left(\dfrac{x - x_k}{h}\right), & x \in [x_{k-1}, x_k); \\ \left(\dfrac{x - x_{k+1}}{h}\right)^2 \left(\dfrac{x - x_k}{h}\right), & x \in [x_k, x_{k+1}); \\ 0, & \text{otherwise.} \end{cases}$$

Notice that ϕ_k and ψ_k both have double roots at x_{k-1} and x_{k+1}, which makes these functions transition to zero at x_{k-1} and x_{k+1} with a continuous derivative.

(a) Write Chebfun code to construct these basis functions.

(b) Plot several of these basis functions for $N = 5$, to get an impression for their shape. (Be sure to plot both ϕ_k and ψ_k, as they look quite a bit different from one another.)

(c) Now we wish to approximate $v(x) = x\sin(\pi x)$ from the space $\mathcal{W} = \text{span}\{\phi_0, \ldots, \phi_{N+1}, \psi_0, \ldots, \psi_{N+1},$ a space of dimension $2N+4$. Write code to construct the appropriate Gram matrix $\mathbf{G}$ (using Chebfun is fine) and the right-hand side $\mathbf{b}$.

(d) For $N = 16$, produce a `spy` plot of $\mathbf{G}$. Comment on the zero structure of this matrix.

(e) Repeat Problem 2(c) for these basis functions (with $N = 2, 4, 8$).

(f) Repeat Problem 2(d) for these basis functions. (On the horizontal axis of your `loglog` plot, now show $2N+4$ rather than N, so that the horizontal axis shows the dimension of the approximating subspace $\mathcal{W}$ for each N.) How does the error compare to what you obtained in Problem 2(d)?