CMDA 4604 · INTERMEDIATE TOPICS IN MATHEMATICAL MODELING

Problem Set 1

Posted Friday 28 August 2015. Due Friday 4 September 2014, 5pm.

1. [25 points]
   Consider the temperature function

   $$u(x,t) = e^{-\kappa\theta^2 t/(\rho c)} \sin(\theta x)$$

   for constant κ, ρ, c, and θ.

   (a) Show that this function $u(x,t)$ is a solution of the homogeneous heat equation

   $$\rho c \frac{\partial}{\partial t} u(x,t) = \kappa \frac{\partial^2}{\partial x^2} u(x,t), \quad \text{for } 0 < x < \ell \text{ and all } t.$$

   (b) For which values of θ will u satisfy homogeneous Dirichlet boundary conditions at $x = 0$ and $x = \ell$, i.e., $u(0,t) = u(\ell,t) = 0$?

   (c) Suppose $\kappa = 2.37$ W/(cm K), $\rho = 2.70$ g/cm^3, and $c = 0.897$ J/(g K) (approximate values for aluminum found on Wikipedia), and that the bar has length $\ell = 10$ cm. Let θ be such that $u(x,t)$ satisfies homogeneous Dirichlet boundary conditions as in part (b) and $u(x,t) \geq 0$ for all x and t.

   Use MATLAB to plot the solution $u(x,t)$ for $0 \leq x \leq \ell$ and time $0 \leq t \leq 20$ sec.
   You may choose to do this in one of the following ways: (1) Plot the solution for $0 \leq x \leq \ell$ at times $t = 0, 4, 8, \ldots, 20$ sec., superimposing all six plots on the same axis (helpful commands: `linspace`, `plot`, `hold on`); (2) Create a three-dimensional plot of the data using `surf`, `mesh`, or `waterfall`. In either case, be sure to produce an attractive, well-labeled plot.

2. [15 points]
   Consider the following sets of functions. Demonstrate whether or not each is a vector space, with addition and scalar multiplication defined in the usual way. (Recall that $C[0,1]$ functions are continuous on $[0,1]$; $C^1[0,1]$ functions are continuous and have a continuous first derivative on $[0,1]$; $C^2[0,1]$ functions are continuous and have continuous first and second derivatives on $[0,1]$.)

   (a) $\{\mathbf{x} \in \mathbb{R}^2 : x_2 = x_1^3\}$
   (b) $\{\mathbf{x} \in \mathbb{R}^3 : x_1 + 2x_2 + 3x_3 = 0\}$
   (c) $\{f \in C[0,1] : \max_{x\in[0,1]} f(x) \leq 1\}$
   (d) $\{f \in C^1[0,1] : f'(0) = 0\}$
   (e) $\{f \in C^2[0,1] : f''(x) = 0 \text{ for all } x \in [0,1]\}$

3. [15 points]
   Determine whether each of the following functions $(\cdot,\cdot)$ determines an inner product on the vector space $\mathcal{V}$. If not, show ***all*** the properties ($(u,v) = (v,u)$; $(\alpha u, v) = \alpha(u,v)$; $(u+v,w) = (u,w)+(v,w)$; $(u,u) \geq 0$ with $(u,u) = 0$ if and only if $u = 0$) of the inner product that are violated.

   (a) $\mathcal{V} = C^1[0,1]$, $(u,v) = \displaystyle\int_0^1 u(x)v'(x)\,dx$ (b) $\mathcal{V} = C[0,1]$: $(u,v) = \displaystyle\int_0^1 |u(x)||v(x)|\,dx$

   (c) $\mathcal{V} = C[0,1]$: $(u,v) = \displaystyle\int_0^1 u(x)v(x)e^{-x}\,dx$

4. [15 points]
Suppose $\mathcal{V}$ is a vector space with an associated inner product. The angle $\angle(u,v)$ between u and $v \in \mathcal{V}$ is defined via the equation

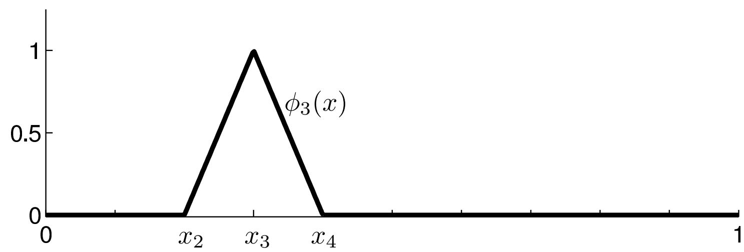
$$(u,v) = \|u\|\|v\| \cos \angle(u,v).$$

Let $\mathcal{V} = C[0,1]$ and $(u,v) = \int_0^1 u(x)v(x)\,dx$. Compute $\cos \angle(x^n, x^m)$ between $u(x) = x^n$ and $v(x) = x^m$ for nonnegative integers m and n. What happens to $\angle(x^n, x^{n+1})$ as $n \to \infty$?

5. [30 points]
Suppose $N \geq 1$ is an integer and define $h = 1/(N+1)$ and $x_k = kh$ for $k = 0, \ldots, N+1$. Consider the $N+2$ hat functions, defined as

$$\phi_k(x) = \begin{cases} (x - x_{k-1})/h, & x \in [x_{k-1}, x_k); \\ (x_{k+1} - x)/h, & x \in [x_k, x_{k+1}); \\ 0, & \text{otherwise.} \end{cases}$$

for $x \in [0,1]$ and $k = 0, \ldots, N+1$. We call these piecewise linear functions *hat functions* because of their shape. They will be important functions later in the course. For example, when $N = 9$ and $k = 3$, this function takes the following form.



(a) Write a MATLAB function called `hat.m` that computes $\phi_k(x)$. It should take in as input x, k, and N. It should return the value $\phi_k(x)$. It should also be able to take in a vector for $\mathbf{x} = (x_1, \ldots, x_m)$ and return the vector $\phi_k(\mathbf{x}) = (\phi_k(x_1), \ldots, \phi_k(x_m))$.

(Alternatively, you may write `hat.m` so that `hat(k,N)` creates a Chebfun representation of ϕ_k for the given N. In this case, you will want to run the command `splitting('on')` before building your Chebfun, to optimally handle the discontinuities in the first derivative of ϕ_k.)

(b) Let $N = 9$. Plot $\phi_0(x), \phi_4(x), \phi_5(x), \phi_6(x), \phi_{10}(x)$ on the same figure. Find some way (e.g., line color, line style, text labels) to label which function is which.

(c) Let $(u,v) = \int_0^1 u(x)v(x)\,dx$. By hand, compute for arbitrary $j = 1, \ldots, N$

(i) (ϕ_j, ϕ_j);

(ii) (ϕ_j, ϕ_{j+1});

(iii) (ϕ_j, ϕ_k) for $|j - k| > 1$.

Hint: your answers might depend on N but will not depend on j.