## Two Examples of Solving $-u''(x) = f(x)$

We wish to solve $-u''(x) = f(x)$ with homogeneous Dirichlet boundary conditions $u(0) = u(1) = 0$ for three different choices of $f$. The key idea here is that the smoothness of $f$ will vary in these three examples, which will be reflected in the decay rates of the inner products $(\psi_n, f)$.

Let $L : C_D^2[0,1] \to C[0,1]$ be given by $Lu = -u''$.

In what follows, denote the $n$th eigenvalue and eigenfunction of $L$ by

$$\lambda_n = n^2\pi^2, \qquad \psi_n(x) = \sqrt{2}\sin(n\pi x).$$

Notice that the leading $\sqrt{2}$ factor in $\psi_n$ ensures that $(\psi_n, \psi_n) = 1$, i.e., it makes $\psi_n$ a unit vector.

1. Solve $-u''(x) = 1$, $u(0) = u(1) = 0$.

   We can find the exact solution by just integrating twice and using the boundary parameters to determine the constants of integration. This gives $u(x) = (x - x^2)/2$.

   However (anticipating time dependent problems to come, like $u_t = u_{xx}$), we will solve this by eigenfunction expansion. First we will consider expansions of $f$ (i.e., the limits of best approximations):

$$1 = f(x) = \sum_{n=1}^{\infty} \frac{(\psi_n, f)}{(\psi_n, \psi_n)}\psi_n(x)$$

   and the solution

$$u(x) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n}\frac{(\psi_n, f)}{(\psi_n, \psi_n)}\psi_n(x).$$

   For both we must compute

$$(\psi_n, f) = (\sqrt{2}\sin(n\pi x), 1) = \int_0^1 \sqrt{2}\sin(n\pi x)\cdot 1\,dx = \frac{\sqrt{2}(1-(-1)^n)}{n\pi}.$$

   The first few values are:

$$(\psi_1, f) = \frac{2\sqrt{2}}{\pi}, \quad (\psi_2, f) = 0, \quad (\psi_3, f) = \frac{2\sqrt{2}}{3\pi}, \quad (\psi_4, f) = 0, \quad (\psi_5, f) = \frac{2\sqrt{2}}{5\pi}.$$

   Thus we might want to write

$$\begin{aligned} 1 = f(x) = \sum_{n=1}^{\infty} \frac{(\psi_n, f)}{(\psi_n, \psi_n)}\psi_n(x) &= \sum_{n=1}^{\infty} \frac{\sqrt{2}(1-(-1)^n)}{n\pi}(\sqrt{2}\sin(n\pi x)) \\ &= \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n\pi}\sin(n\pi x). \end{aligned}$$

The right-hand side looks like an absurd way to write $f(x) = 1$, but you can see that it seems to work. Run `lapex1.m`.

We can also write the solution as

$$\frac{x - x^2}{2} = u(x) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \frac{(\psi_n, f)}{(\psi_n, \psi_n)} \psi_n(x) = \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} \frac{\sqrt{2}(1 - (-1)^n)}{n\pi} (\sqrt{2}\sin(n\pi x))$$

$$= \sum_{n=1}^{\infty} \frac{2(1 - (-1)^n)}{n^3\pi^3} \sin(n\pi x).$$

Notice in `lapex1.m` that the series for $u$ converges so much faster than the series for $f$ ! Why? First, note that $u$ satisfies the boundary conditions, but $f$ doesn't (and there is no need for it to do so) – hence the eigenfunctions (which must obey the boundary conditions) can approximate $u$ better than $f$. Secondly, notice that dividing by $\lambda_n = n^2\pi^2$ makes the coefficients in the series decay like $1/n^3$ in the series for $u$, instead of $1/n$ as in the series for $f$. Faster decay of the coefficients means faster convergence.

2. Solve $-u''(x) = f(x)$, $u(0) = u(1) = 0$, where

$$f(x) = \begin{cases} x, & x \in [0, 1/2]; \\ 1 - x, & x \in [1/2, 1]. \end{cases}$$

Notice that this function $f$ is in $C[0, 1]$ but not $C^1[0, 1]$, because the derivative of $f$ is discontinuous at $1/2$. However, $f(0) = f(1) = 0$, so you might have some hope that the eigenfunctions will do a better job of approximating $f$, since $\psi_n(0) = \psi_n(1) = 0$.

Can you find the solution exactly? (Integrate twice on each half of the domain; you will have four constants (two on each domain). Use the boundary conditions to set two of the integration constants; enforce continuity of $u(1/2)$ and $u'(1/2)$ to find the other two.

One can now compute

$$(\psi_n, f) = \int_0^1 \psi_n(x) f(x)\, dx = \frac{8\sqrt{2}\,\cos(n\pi/4)\sin^3(n\pi/4)}{n^2\pi^2}.$$

The first few values are:

$$(\psi_1, f) = \frac{2\sqrt{2}}{\pi^2}, \quad (\psi_2, f) = 0, \quad (\psi_3, f) = -\frac{2\sqrt{2}}{9\pi^2}, \quad (\psi_4, f) = 0, \quad (\psi_5, f) = \frac{2\sqrt{2}}{25\pi^2}.$$

Now the solution is

$$u(x) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \frac{(\psi_n, f)}{(\psi_n, \psi_n)} \psi_n(x) = \sum_{n=1}^{\infty} \frac{16\,\cos(n\pi/4)\sin^3(n\pi/4)}{n^4\pi^4} \sin(n\pi x),$$

which converges even faster than in the last case, because the series for $f$ converged even faster. Notice (by running `lapex2.m`) that $f'$ is not continuous as $x = 1/2$, but the solution $u$ indeed appears smooth.