CMDA 3606 · MATHEMATICAL MODELING II

Problem Set 5

Posted 1 March 2019. Due at 5pm on Thursday, 7 March 2019.

Basic guidelines: Students may discuss the problems on this assignment, but each student must submit his or her individual writeup and code. (In particular, you *must write up your own individual MATLAB code.*) Students may consult class notes and other online resources for general information; cite all your sources and list those with whom you have discussed the problems.

1. [35 points: 14 points for (a); 7 points each for (b),(c),(d)]

   Consider the vectors

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 4 \\ 2 \\ 5 \\ 2 \end{bmatrix}.$$

   (a) Perform (by hand) the Gram–Schmidt orthogonalization procedure on $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ to produce orthonormal vectors $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$ such that

$$\operatorname{span}\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\} = \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}.$$

   (b) You can arrange the quantities you have computed in the Gram–Schmidt process into a **QR** factorization of the matrix $\mathbf{A} = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3]$: Define $\mathbf{Q} = [\mathbf{q}_1\ \mathbf{q}_2\ \mathbf{q}_3]$ and define $\mathbf{R}$ entrywise as

$$r_{j,k} = \begin{cases} \mathbf{q}_j^T \mathbf{a}_k, & j < k; \\ \|\mathbf{a}_k - \sum_{\ell=1}^{k-1} \mathbf{q}_\ell^T \mathbf{a}_k \mathbf{q}_\ell\|, & j = k; \\ 0, & j > k. \end{cases}$$

   Write out $\mathbf{Q}$ and $\mathbf{R}$, and confirm that $\mathbf{QR} = \mathbf{A}$.

   (c) Compute $\mathbf{Q}\mathbf{Q}^T$ (by hand) and $\mathbf{A}\mathbf{A}^+ = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$ (in MATLAB) and confirm that they are equal. (Both formulas give the orthogonal projector onto $\mathcal{R}(\mathbf{A})$.)

   (d) Consider the least squares problem

$$\min_{\mathbf{x} \in \mathbb{R}^3} \|\mathbf{b} - \mathbf{A}\mathbf{x}\|, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ -5 \\ -1 \\ 1 \end{bmatrix}.$$

   In MATLAB, compute the least squares solution in two ways:

$$\mathbf{x} = \mathbf{R}^{-1}\mathbf{Q}^T\mathbf{b}, \qquad \mathbf{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b},$$

   and confirm that you get the same answer.

2. [30 points: 10 points each for (a), (d); 5 points each for (b), (c)]

   Matrices with orthonormal columns play an important role in this course – and in many applications. Special *rotation matrices* arise often in computer graphics and dynamics.

   (a) Given an angle θ, define the *rotation matrix*

$$\mathbf{Q}_\theta := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

   Prove that $\mathbf{Q}_\theta$ has orthonormal columns for any value of θ, that is, $\mathbf{Q}_\theta^T\mathbf{Q}_\theta = \mathbf{I}$.

(b) The class website contains the file `hw5.mat`. Get this file and `load hw5` to have access to the variable $\mathbf{X} \in \mathbb{R}^{2\times 48}$, which contains data points in the (x, y) plane, stored in the form

$$\mathtt{X} = \begin{array}{|c|c|c|c|} \hline x_1 & x_2 & \cdots & x_{48} \\ \hline y_1 & y_2 & \cdots & y_{48} \\ \hline \end{array}.$$

Visualize this data set via

```
plot(X(1,:),X(2,:),'k.','markersize',14)
```

Be sure to use `axis equal` to optimize viewing.

(c) For the 8 values $\theta = \pi/4,\ \pi/2,\ 3\pi/4,\ \ldots,\ 2\pi$:

(i) construct $\mathbf{Q}_\theta$ in MATLAB

(ii) using a command similar to the one in part (b), plot $\mathbf{Q}_\theta\mathbf{X}$ in the (x, y) plane.

Use `axis equal` and `hold on` to superimpose all 8 plots into one.

(d) The file `hw5.mat` also contains another variable, $\mathbf{Y} \in \mathbb{R}^{3\times 7190}$, that contains a point cloud in three dimensions (derived from the Stanford 3D Scanning Repository):

$$\mathtt{Y} = \begin{array}{|c|c|c|c|} \hline x_1 & x_2 & \cdots & x_{7190} \\ \hline y_1 & y_2 & \cdots & y_{7190} \\ \hline z_1 & z_2 & \cdots & z_{7190} \\ \hline \end{array}.$$

Given angles θ and ϕ, define the two rotation matrices

$$\mathbf{Q}_{xy} := \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{Q}_{yz} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{bmatrix}.$$

Set $\theta = 3\pi/4$ and $\phi = \pi/2$.

Create one plot showing four versions of this data set:

$$\mathbf{Y}, \qquad \mathbf{Q}_{xy}\mathbf{Y}, \qquad \mathbf{Q}_{yz}\mathbf{Y}, \qquad \mathbf{Q}_{xy}\mathbf{Q}_{yz}\mathbf{Y}.$$

To show this three-dimensional data, use the `plot3` command, e.g.,

```
plot3(Y(1,:),Y(2,:),Y(3,:),'k.')
```

(Plot each data set as a different color; use `axis equal`.)

Describe the effect of each of the matrices $\mathbf{Q}_{xy}$ and $\mathbf{Q}_{yz}$ on the data set.

3. [35 points: 14 points for (a); 7 points each for (b), (c), (d)]

Arguably the most fundamental matrix in applied mathematics has the simple form

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{bmatrix},$$

a symmetric matrix with ones on the first super-diagonal and sub-diagonal, with zeros everywhere else.

(a) Let $\mathbf{A}$ be an $N \times N$ matrix of this form. The eigenvalues λ_j and eigenvectors $\mathbf{v}_j$ have the elegant form

$$\lambda_j = 2\cos\left(\frac{j\pi}{N+1}\right), \qquad \mathbf{v}_j = \begin{bmatrix} \sin\left(\frac{j\pi}{N+1}\right) \\ \sin\left(\frac{2j\pi}{N+1}\right) \\ \vdots \\ \sin\left(\frac{Nj\pi}{N+1}\right) \end{bmatrix}$$

for each $j = 1, \ldots, N$.

Verify that these formulas do indeed give an eigenvalue–eigenvector pair for $\mathbf{A}$, for $j = 1, \ldots, N$. (To do so, compute $\mathbf{A}\mathbf{v}_j$ and $\lambda_j \mathbf{v}_j$, and show that all entries of these two vectors are the same.)

Hint: Recall that $2\cos(\phi)\sin(\theta) = \sin(\theta + \phi) + \sin(\theta - \phi)$.

(b) For $N = 32$, plot the eigenvectors of $j = 1,\ 2,\ 31$, and 32.

(Plot the entry number $k = 1, \ldots, 32$ along the horizontal axis, and the entries $(\mathbf{v}_j)_k$ of the eigenvector along the vertical axis. If `vj` is a vector of length 32 containing the entries of $\mathbf{v}_j$, then

```
plot([1:32],vj,'k.-','markersize',24,'linewidth',1)
```

will produce a nice plot. Produce a distinct plot for each of $j = 1, 2, 31, 32$.)

(c) Arrange the 32 vectors $\mathbf{v}_j$ into a matrix $\mathbf{V} = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_N]$.

Check the orthogonality of the eigenvectors by computing all their dot products:

```
imagesc(V'*V), colorbar
```

Explain how this plot reveals the orthogonality of the eigenvectors.

(d) The matrix

$$\mathbf{D} = \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}$$

is an important variation on $\mathbf{A}$ that arises in many applications. Notice that $\mathbf{D} = -2\mathbf{I} + \mathbf{A}$. Show that $\mathbf{D}$ has the same eigenvectors as $\mathbf{A}$. What are the eigenvalues of $\mathbf{D}$?