CMDA 3606 · MATHEMATICAL MODELING II

Problem Set 3

Posted 7 February 2019. Due at 5pm on Thursday, 14 February 2019.

MW Section: This week, please write "MW" clearly on the top of your paper, and turn it in directly to Professor Beattie; his office is 552 McBryde. (TR Section: Turn in your work as normal.)

Basic guidelines: Students may discuss the problems on this assignment, but each student must submit his or her individual writeup and code. (In particular, you *must write up your own individual MATLAB code.*) Students may consult class notes and other online resources for general information; cite all your sources and list those with whom you have discussed the problems.

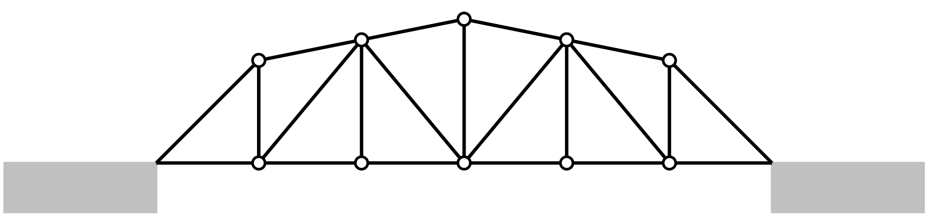
1. [30 points: 2 points for (a); 4 points each for (b)–(h)]

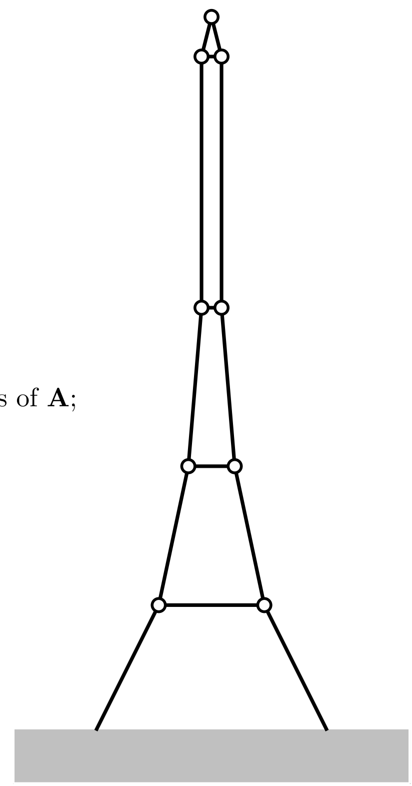
   Consider the structures pictured below: a truss bridge and a pathetic model of the Eiffel Tower. In each graphic, a solid black line represents a *strut* (i.e., a member that behaves like a spring with respect to axial compression or extension); $\circ$ represents a *node* (i.e., a pinned joint where struts are fastened together but can freely rotate). Nodes are allowed to move in both horizontal and vertical directions; both structures are attached to the ground with pinned joints that are not allowed move, although the attached struts are allowed to rotate in these joints. Note that the tower has four horizontal struts; the top two might be a bit tough to see in the diagram.

   You should answer the questions below *without* writing down the entries in the matrix $\mathbf{A}$ for these structures; do not fret about the odd angles between the struts.

   Recall from the lectures, $\mathcal{R}(\mathbf{A})$ denotes the *column space* (or *range*) of $\mathbf{A}$, while $\mathcal{N}(\mathbf{A})$ denotes the *null space* of $\mathbf{A}$:

   $\mathcal{R}(\mathbf{A}) = \{\mathbf{A}\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\} =$ the set of all weighted sums $\mathbf{A}\mathbf{x}$ of the columns of $\mathbf{A}$;
   $\mathcal{N}(\mathbf{A}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{0}\} =$ the set of all $\mathbf{x}$ for which $\mathbf{A}\mathbf{x} = \mathbf{0}$.





   (a) For both the bridge and tower, count the numbers of nodes and struts.

   (b) As described in class, we can model collectively the horizontal and vertical displacements, $\mathbf{x}$, of the nodes induced by a loading of forces, $\mathbf{f}$, acting on the nodes, by solving an equation having the form, $(\mathbf{A}^T\mathbf{K}\mathbf{A})\mathbf{x} = \mathbf{f}$. The first step of the process that leads to this system of equations determines the strut elongations, $\mathbf{e}$, in terms of the node displacements: $\mathbf{e} = \mathbf{A}\mathbf{x}$ for $\mathbf{A} \in \mathbb{R}^{m\times n}$.

   For both the bridge and tower, specify the dimensions m and n of the corresponding $\mathbf{A}$.

   (c) Recall that rank($\mathbf{A}$) equals the number of pivot columns (and pivot rows) in the reduced echelon form of $\mathbf{A}$, and hence equals the dimension of the subspaces $\mathcal{R}(\mathbf{A})$ and $\mathcal{R}(\mathbf{A}^T)$.

   For both the bridge and tower, what is the largest possible value for the dimension of $\mathcal{R}(\mathbf{A})$? (Use only the matrix dimensions: do not try to write down the entries of $\mathbf{A}$.)

(d) Based on your answer to part (c), what is the *smallest* possible dimension of the subspace $\mathcal{N}(\mathbf{A})$ for each structure?

(e) Show that if $\mathbf{x} \in \mathcal{N}(\mathbf{A})$, then $\mathbf{x} \in \mathcal{N}(\mathbf{A}^T\mathbf{K}\mathbf{A})$.
(This means that $\mathcal{N}(\mathbf{A}) \subseteq \mathcal{N}(\mathbf{A}^T\mathbf{K}\mathbf{A})$, so $\mathcal{N}(\mathbf{A})$ cannot be smaller than $\mathcal{N}(\mathbf{A}^T\mathbf{K}\mathbf{A})$.)

(f) We say that a structure is *unstable* if $\mathcal{N}(\mathbf{A}^T\mathbf{K}\mathbf{A})$ contains nonzero vectors.
Based on the previous parts of this problem, one of the structures above is *guaranteed to be unstable* regardless of the stiffness values in $\mathbf{K}$. Which structure is unstable, and why?

(g) By assessing the dimensions computed above, what is the minimum number of struts that you must *add* to the *unstable* structure so it *potentially* can be made stable?

(h) By assessing the dimensions computed above, what is the minimum number of struts that you must *remove* from the *stable* structure to *ensure* it will be unstable?

2. [20 points: 10 points per part]

Let $\mathcal{U}$ and $\mathcal{V}$ be subspaces of $\mathbb{R}^n$.

(a) The *intersection* of $\mathcal{U}$ and $\mathcal{V}$ is defined as:

$$\mathcal{U} \cap \mathcal{V} := \{\mathbf{x} : \mathbf{x} \in \mathcal{U} \textit{ and } \mathbf{x} \in \mathcal{V}\}.$$

(In words: $\mathcal{U} \cap \mathcal{V}$ is the set of all vectors that are in *both* subspaces $\mathcal{U}$ *and* $\mathcal{V}$.)
Prove that $\mathcal{U} \cap \mathcal{V}$ satisfies the two basic properties that are required in order to be a subspace.

(b) The *union* of $\mathcal{U}$ and $\mathcal{V}$ is defined as:

$$\mathcal{U} \cup \mathcal{V} := \{\mathbf{x} : \mathbf{x} \in \mathcal{U} \textit{ or } \mathbf{x} \in \mathcal{V}\}.$$

(In words: $\mathcal{U} \cup \mathcal{V}$ is the set of all vectors that are in *at least one of* the subspaces $\mathcal{U}$ *or* $\mathcal{V}$.)
Give a specific example that shows that $\mathcal{U} \cup \mathcal{V}$ is *not* always a subspace.
(Take $\mathcal{U}$ and $\mathcal{V}$ to both be one-dimensional subspaces of $\mathbb{R}^2$.)

3. [21 points: 7 points per part for (a), (b), and (c)]

Write down a matrix with the required property or explain why no such matrix exists.

(a) The column space contains

$$\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

while the row space contains

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

(b) The column space equals

$$\text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\},$$

while the null space equals

$$\text{span}\left\{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}\right\}.$$

(c) The column space is all of $\mathbb{R}^4$, while the row space is all of $\mathbb{R}^3$.

[This problem is adapted from Steve Cox's *Matrix Analysis in Situ*.]

4. [29 points: (a) = 14 points; (b) and (d) = 4 points; (c) = 7 points]

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 0 & 0 \\ 1 & 3 \end{bmatrix}$$

(a) Construct bases (by hand) for the four fundamental subspaces associated with $\mathbf{A}$: $\mathcal{R}(\mathbf{A})$, $\mathcal{N}(\mathbf{A}^T)$, $\mathcal{R}(\mathbf{A}^T)$, and $\mathcal{N}(\mathbf{A})$.

(b) Characterize each of these four subspaces as a point, line, plane, etc.
Specify in each case whether it is a subspace of $\mathbb{R}^2$ or $\mathbb{R}^3$.

(c) Use the MATLAB scripts `plotline2`, `plotplane2`, `plotline3`, and `plotplane3` (available on the class web site) to create two illustrations: in the first, show both $\mathcal{R}(\mathbf{A})$ and $\mathcal{N}(\mathbf{A}^T)$; in the second, show both $\mathcal{R}(\mathbf{A}^T)$ and $\mathcal{N}(\mathbf{A})$.

You *do not* need to include your MATLAB commands for this problem, but please ***label the subspaces on your plots.*** It is fine to print the plots and add the labels by hand. Use the `axis equal` command to scale all the axes with the same units.

(d) Indicate the orthogonality of the subspaces in your plots.