Matrix Methods for Computational Modeling and Data Analytics

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Chapter 3
Simple Structures at Equilibrium

Simple mechanical structures may seem a far cry from the circuits discussed in the last lecture, but the underlying mathematical models are strikingly similar. Here we illustrate how this different physical scenario gives rise to the same $\mathbf{A}^T\mathbf{K}\mathbf{A}\mathbf{x} = \mathbf{b}$ equation we studied earlier. This observation shows the great merit of developing a mathematical understanding of general systems having this form: in mastering the underlying theory, we develop tools for handling a diversity of applications.

These notes draw heavily in spirit, details, and examples from the texts of Gilbert Strang[1] and Steve Cox[2].

[1] Gilbert Strang. *Introduction to Applied Mathematics*. Wellesley-Cambridge Press, Wellesley, MA, 1986

[2] Steven J. Cox. *Matrix Analysis* in Situ. Rice University, 2013

3.1 *A springy column*

Consider the arrangement of four springs in a vertical column shown in Figure 3.1, with three masses separating the springs. The springs are fixed at the top and bottom of this arrangement, and when forces are applied to these masses, the springs will compress or extend. (For now we do not allow the springs to move left or right out of this vertical arrangement.)

Our goal is to determine how the forces f_1, f_2, and f_3, applied to the masses m_1, m_2, and m_3, affect the displacements x_1, x_2, and x_3. The solution will depend on the material properties of the springs (i.e., the spring constants k_1, k_2, and k_3): we expect stiff springs will allow smaller deformations than more flexible springs. In any case, we presume that our springs behave according to Hooke's Law; when dealing with real springs, this will not generally be the case if the forces are too small or too great — with the extreme case being the fracture of the spring under an excessive load. (Moreover, while we speak of "springs," we might instead envision a "truss," perhaps a steel girder, a timber beam, or a concrete pier that we assume to behave in a roughly Hookean fashion.)

You can consider the forces coming from gravity acting on the masses, with the springs being essentially massless.

The procedure for determining the displacements x_1, x_2, and x_3 will closely resemble our methodology for modelings circuits. Again, we break the process into four steps. Let us agree to measure positive quantities in the down direction; e.g., if the top spring gets longer under the applied load, the displacement x_1 of mass m_1 will be positive.

STEP 1 **Compute the extension of each spring.**

We first measure the elongation of the four different springs. As seen in Figure 3.1, the loads will cause some springs to stretch while others compress; a positive "elongation" means the spring is stretched, while a negative value indicates compression. The first elongation is easy to compute: it is simply x_1, the amount the first mass has descends under the load. We thus set

The springs will change length, but the sum of the changes must be zero, since the top and bottom are fixed.

$$e_1 = x_1.$$

The amount the second spring stretches equals the amount by which the drop of mass m_2 exceeds the drop of mass m_1. (In the cartoon in Figure 3.1, the top two masses have dropped by the same amount, so the second spring has zero elongation.) Thus, the extension of the second spring is

$$e_2 = x_2 - x_1.$$

The third spring stretches similarly:

$$e_3 = x_3 - x_2.$$

Finally, the last spring get shorter by the amount that mass m_3 descends, so

$$e_4 = -x_3.$$

As usual, we arrange these four equations in matrix-vector form:

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad (3.1)$$

which we write as

$$\mathbf{e} = \mathbf{A}\mathbf{x}. \qquad (3.2)$$

STEP 2 **Apply Hooke's Law.**

Next we seek to relate the elongation of spring j to the restoring force y_j that the spring exerts. Hooke's Law does the trick: the force is proportional to the elongation, with the proportionality given by the spring constant:

$$y_j = k_j e_j, \qquad j = 1, \ldots, 4.$$

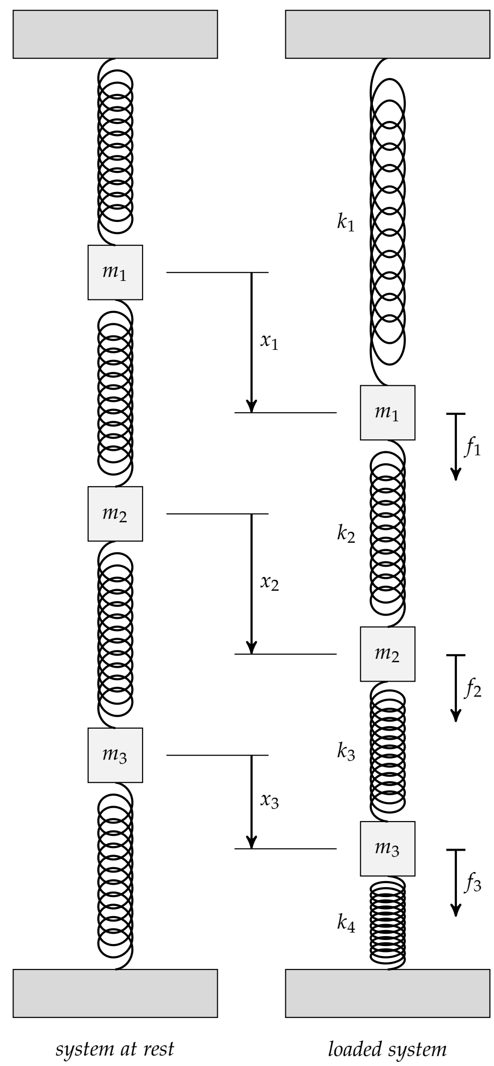


Figure 3.1: Four springs stacked vertically, separated by three masses and fixed at the top and bottom. The figure on the left shows the network with no load applied. When forces f_1, f_2, and f_3 are applied to the masses m_1, m_2, and m_3, the springs deform, as sketched in the schematic on the right. These forces cause the jth mass to drop by x_j units, as controlled by the spring constants $k_1, \ldots, k_4$.

In matrix-vector form,

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix},$$

which we write as

$$\mathbf{y} = \mathbf{K}\mathbf{e}. \tag{3.3}$$

STEP 3 **Balance forces at each mass.**

We aim to figure out how much the applied forces f_1, f_2, and f_3 cause the masses to descend at equilibrium. The key step is to balance these known forces acting on each mass against the restoring force of each spring. Since the system is at rest (static), the forces balance at each mass. Getting the signs of the restoring forces correct can be tricky. At mass m_1, the force exerted by the top spring acts to restore to the spring to its original length, hence it pulls m_1 *up*, which, by our convention, is the negative direction. Meanwhile, as the second spring seeks to be restored to its rest length, it tugs mass 1 downward, the positive direction. Hence, the applied force f_1 balances the restoring force $-y_1 + y_2$:

You can imagine that $f_j = m_j g$, where g denotes the gravitational constant, but other external forces are possible too.

$$f_1 - y_1 + y_2 = 0.$$

The same argument applied to masses m_2 and m_3 to give

$$f_2 - y_2 + y_3 = 0$$

$$f_3 - y_3 + y_4 = 0.$$

We rearrange to get the matrix-vector form,

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}. \tag{3.4}$$

The matrix here encodes the connectivity of the spring network, mapping spring forces to masses; in perfect parallel to the circuit model, it is the transpose of the matrix in (3.1) from step 1, which mapped mass displacements to spring extensions: hence we write (3.4) as

$$\mathbf{A}^T\mathbf{y} = \mathbf{f}. \tag{3.5}$$

STEP 4 **Assembly.**

Now we simply put the pieces together to arrive at an equation for the unknown $\mathbf{x}$. Inserting equation (3.3) into (3.5) gives

$$\begin{aligned} \mathbf{f} &= \mathbf{A}^T\mathbf{y} \\ &= \mathbf{A}^T\mathbf{K}\mathbf{e}. \end{aligned}$$

Now insert equation (3.2) for $\mathbf{e}$ to give the fundamental relation

$$\boxed{\mathbf{A}^T\mathbf{K}\mathbf{A}\,\mathbf{x} = \mathbf{f},} \tag{3.6}$$

the same equation we arrived at for our circuit model.

In a different context, Richard Feynman mused on the question of why so many entirely different physical phenomena give rise to the same equations. See Lecture 12 (and especially Section 12.7, "The 'underlying unity' of nature") in:

Richard Feynman, Robert B. Leighton, and Matthew L. Sands. *The Feynman Lectures on Physics*, volume 2. Addison-Wesley, Reading, MA, 1964

STUDENT EXPERIMENTS

3.7. Suppose all the springs are identical, $k_1 = k_2 = k_3 = k_4 = k$, as are the masses, $m_1 = m_2 = m_3 = m$, and the applied forces come from gravity: $f_j = mg$. Set up and solve (3.6) for x_1, x_2, and x_3.

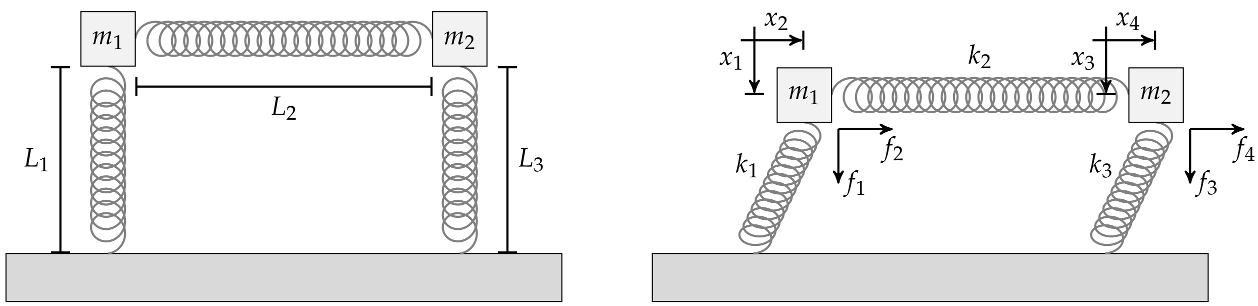
3.8. Generalize the configuration in Figure 3.1 to have N equal masses and $N+1$ identical springs. Fix the total mass in the system, independent of N, and divide it evenly across the masses. How do the displacements $x_1, \ldots, x_N$ behave as N gets large? How should the spring constant k scale with N so that your solution tends to a clean limit as $N \to \infty$?

ULIMATELY WE SEEK to understand when equations like $\mathbf{A}^T\mathbf{K}\mathbf{A}\mathbf{x} = \mathbf{f}$ have a solution, and when that solution is unique. The one-dimensional truss is very clean: you can see that indeed a solution always exists, and it is unique, for the matrix $\mathbf{A}^T\mathbf{K}\mathbf{A}$ is invertible. Yet we ultimately we will extend this modeling procedure to handle more interesting two-dimensional trusses, and the solvability question gets much more interesting.

3.2 *Two dimensions and linearization*

Circuits do not notice the angles at which we arrange the wires relative to the nodes, but mechanical structures certainly do. We avoided these concerns in the last lecture by forcing all the displacements to occur in the same direction. However, we mostly care about structures in two or three dimensions, where the springs can attach at odd angles. This geometry makes it more difficult to compute the elongation of the springs. To set the stage, consider the tipsy two-dimensional table shown in Figure 3.2. We presume that $L_1 = L_3$, so that at rest, the horizontal and vertical springs join at right angles.

Suppose, following Step 1 of the procedure outlined in the last lecture, we want to compute the elongation of the spring on the left, which at rest has length L_1. When forces are applied, mass m_1 moves



system at rest *loaded system*

Figure 3.2: A tipsy table. Three springs with rest lengths L_j and spring constants k_j are connected via two masses, m_1 and m_2. Forces vertical (f_1 and f_3) and horizontal (f_2 and f_4) are applied to each mass, causing vertical (x_1 and x_3) and horizontal (x_2 and x_4) displacements of each mass.

x_1 units down and x_2 units to the right. Use the Pythagorean Theorem to compute the length of the deformed spring,

$$\text{loaded length of spring 1} = \sqrt{(L_1 - x_1)^2 + x_2^2},$$

from which we deduce the formula

$$\text{elongation of spring 1} = \sqrt{(L_1 - x_1)^2 + x_2^2} - L_1. \tag{3.7}$$

Similar formulas hold for the other two springs:

$$\text{elongation of spring 2} = \sqrt{((x_1 - x_3)^2 + (L_2 + x_4 - x_2)^2} - L_2 \tag{3.8}$$

$$\text{elongation of spring 3} = \sqrt{(L_3 - x_3)^2 + x_4^2} - L_3. \tag{3.9}$$

Following the pattern of the last lecture, we should now try to write these elongation equations in the matrix-vector form

$$\mathbf{e} = \mathbf{A}\mathbf{x}.$$

However, we run into a fundamental obstacle: equations (3.7)–(3.9) are *nonlinear* functions of x_1, x_2, x_3, and x_4; they involve squares, square roots, and products like $x_1 x_3$. There is no way to write these elongations as *linear* functions of the x_j variables, as implied by the equation $\mathbf{e} = \mathbf{A}\mathbf{x}$.

Despite the exaggerated elongations shown in Figure 3.2, we are generally envisioning deformations that are quite modest, compared to the rest lengths of the springs. In this parameter regime, the elongations are *approximately linear* functions of the deformations. To see this, take a closer look at the elongation of spring 1, which we can rewrite as

In a substantial structure, trusses with lengths measured in meters or tens of meters might deform on the order of millimeters or centimeters.

$$\begin{aligned}
\text{elongation of spring 1} &= \sqrt{(L_1 - x_1)^2 + x_2^2} - L_1 \\
&= \sqrt{L_1^2\left(\left(1 - \frac{x_1}{L_1}\right)^2 + \left(\frac{x_2}{L_1}\right)^2\right)} - L_1 \\
&= L_1\sqrt{1 - \frac{2x_1}{L_1} + \frac{x_1^2 + x_2^2}{L_1^2}} - L_1.
\end{aligned} \tag{3.10}$$

If the rest length L_1 is much larger in magnitude than the deformations x_1 and x_2, then the term $-2x_1/L_1 + (x_1^2 + x_2^2)/L_1^2$ under the radical will be quite small. This calls to mind the Taylor series

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \cdots$$

whose leading terms give an excellent approximation when $|x|$ is small. Substituting the square root from (3.10) into the Taylor series, we get

$$\begin{aligned}\sqrt{1 - \frac{2x_1}{L_1} + \frac{x_1^2 + x_2^2}{L_1^2}} &= 1 + \frac{1}{2}\left(-\frac{2x_1}{L_1} + \frac{x_1^2 + x_2^2}{L_1^2}\right) + \frac{1}{8}\left(-\frac{2x_1}{L_1} + \frac{x_1^2 + x_2^2}{L_1^2}\right)^2 + \cdots \\ &= 1 - \frac{x_1}{L_1} + O\left(\frac{x_1^2 + x_2^2}{L_1^2}\right) \\ &\approx 1 - \frac{x_1}{L_1}.\end{aligned}$$

This is the Taylor (Maclaurin) series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots,$$

with the function $f(x) = \sqrt{1+x}$. The series converges for $|x| < 1$.

The "big-oh" notation $O((x_1^2 + x_2^2)/L_1^2)$ takes the place of terms that are smaller than some constant times $(x_1^2 + x_2^2)/L_1^2$ as $(x_1^2 + x_2^2)/L_1^2 \to 0$. This is notation is a convenient tool for tracking the size of neglected terms, without writing out their formulas in detail.

Now insert this approximation into the elongation formula (3.10):

$$\text{elongation of spring 1} \approx L_1\left(1 - \frac{x_1}{L_1}\right) - L_1 = -x_1. \qquad (3.11)$$

This approximation seems quite reasonable, as it matches the formula we would have obtained if spring 1 were constrained to only deform in the vertical direction (like the springy column in the last lecture). Next, apply the same approximation strategy to the elongation formulas (3.8) and (3.9) to obtain

$$\text{elongation of spring 2} \approx x_4 - x_2 \qquad (3.12)$$

$$\text{elongation of spring 3} \approx -x_3 \qquad (3.13)$$

Notice a key property of the approximations (3.11)–(3.13):

> The dominant deformation occurs
> in the direction of the spring's orientation.

That is, the approximate elongation formula is the same thing we would obtain if the spring were constrained to only deform in the direction of its main axis. This general rule makes it easy to approximate the elongation of springs at arbitrary orientations, when we approach structures with more interesting geometry than the one shown in Figure 3.2.

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3.9. Use geometry to derive the extension formulas (3.8) and (3.9).

3.10. Test the quality of the approximation. Suppose $L_1 = 10$ m, and let $x_1 = x_2$. Produce a plot that compares the true elongation (3.7) of spring 1 to the approximation (3.11) for small displacements starting at $x_1 = x_2 = 0\ m$ and increasing. (Plot $x_1 = x_2$ on the horizontal axis, and the values of the elongation and its approximation on the vertical axis.) How large can $x_1 = x_2$ be before the approximation (3.11) noticeably loses its accuracy?

3.11. Work out the approximations for the elongations of springs 2 and 3 given in (3.12) and (3.13).

3.3 *Four–steps for the linearized model*

With the approximate elongations at hand, we can proceed with the four steps of the modeling process described in the last lecture.

STEP 1 **Compute the (approximate) extension of each spring.**

Since we are after a *linear* relationship between the displacements and elongations, we use the approximations (3.11)–(3.13):

$$e_1 = -x_1$$

$$e_2 = x_4 - x_2$$

$$e_3 = -x_3.$$

These approximations can be written in the matrix–vector form

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \qquad (3.14)$$

which, of course, we write as

$$\mathbf{e} = \mathbf{A}\mathbf{x}.$$

STEP 2 **Apply Hooke's Law.**

Hooke's Law pays no heed to the way the springs are connected, so there is no need to make a linearizing approximation here. We proceed as before, computing the restoring force in each spring as

$$y_j = k_j e_j, \qquad j = 1, 2, 3,$$

which takes the matrix form

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},$$

or

$$\mathbf{y} = \mathbf{K}\mathbf{e}.$$

STEP 3 **Balance Forces.**

The truss can move within the plane, with forces acting in the vertical and horizontal directions. Hence we must balance forces in two directions at each node. Since (in our linear approximation) the spring elongation occurs in the direction in which the spring is oriented, the force balance step inherits the same assumption. For this simple case, the restoring forces of springs 1 and 3 act in the vertical direction, while spring 2 acts horizontally. So the force balance equations become:

$$\begin{aligned} \text{MASS 1, VERTICAL:} \quad & 0 = f_1 + y_1; \\ \text{MASS 1, HORIZONTAL:} \quad & 0 = f_2 + y_2; \\ \text{MASS 2, VERTICAL:} \quad & 0 = f_3 + y_3; \\ \text{MASS 2, HORITZONTAL:} \quad & 0 = f_4 - y_2. \end{aligned}$$

Collect these four equations as

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}.$$

Yes indeed, the matrix in this last equation once again is $\mathbf{A}^T$, so

$$\mathbf{A}^T\mathbf{y} = \mathbf{f}.$$

STEP 4 **Assembly.**

This step proceeds just as before:

$$\begin{aligned} \mathbf{f} &= \mathbf{A}^T\mathbf{y} \\ &= \mathbf{A}^T\mathbf{K}\mathbf{e} \\ &= \mathbf{A}^T\mathbf{K}\mathbf{A}\mathbf{x}, \end{aligned}$$

resulting in the equation

$$\boxed{\mathbf{A}^T\mathbf{K}\mathbf{A}\,\mathbf{x} = \mathbf{f}.} \tag{3.15}$$

Before proceeding to solve this equation, pause for a moment to consider the matrix dimensions that arise when modeling general planar (2d) trusses. Suppose we have m masses connected by n springs. This gives the following dimensions.

$\mathbf{x}$	$2m \times 1$ vector
$\mathbf{e}$	$n \times 1$ vector
$\mathbf{y}$	$n \times 1$ vector
$\mathbf{f}$	$2m \times 1$ vector
$\mathbf{A}$	$n \times 2m$ matrix
$\mathbf{K}$	$n \times n$ matrix
$\mathbf{A}^T\mathbf{K}\mathbf{A}$	$2m \times 2m$ matrix

These dimensions play a crucial role when it comes to solving for the displacements $\mathbf{x}$ corresponding to a known load $\mathbf{f}$.

3.4 *When Gaussian elimination fails*

Return now to the specific scenario sketched in Figure 3.2. Work out

$$\mathbf{A}^T\mathbf{K}\mathbf{A} = \begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & -k_2 \\ 0 & 0 & k_3 & 0 \\ 0 & -k_2 & 0 & k_2 \end{bmatrix},$$

and, with known loads f_1, f_2, f_3, and f_4, proceed to solve $\mathbf{A}^T\mathbf{K}\mathbf{A}\mathbf{x} = \mathbf{f}$,

$$\begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & -k_2 \\ 0 & 0 & k_3 & 0 \\ 0 & -k_2 & 0 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}.$$

Suppose we try to do so with Gaussian elimination, forming the augmented matrix

$$\left[\begin{array}{cccc|c} k_1 & 0 & 0 & 0 & f_1 \\ 0 & k_2 & 0 & -k_2 & f_2 \\ 0 & 0 & k_3 & 0 & f_3 \\ 0 & -k_2 & 0 & k_2 & f_4 \end{array}\right].$$

Given the large number of zeros in this matrix, elimination looks to be an easy task. First we target the (4,2) entry, which can be eliminated by replacing row 4 by the sum of rows 2 and 4:

$$\left[\begin{array}{cccc|c} k_1 & 0 & 0 & 0 & f_1 \\ 0 & k_2 & 0 & -k_2 & f_2 \\ 0 & 0 & k_3 & 0 & f_3 \\ 0 & 0 & 0 & 0 & f_4 + f_2 \end{array}\right]. \tag{3.16}$$

What just happened? We have an upper triangular matrix on the left, but a strange one. Perhaps it helps to write out the equations:

$$k_1 x_1 = f_1 \tag{3.17}$$

$$k_2x_2 - k_2x_4 = f_2 \quad (3.18)$$

$$k_3x_3 = f_3 \quad (3.19)$$

$$0 = f_2 + f_4. \quad (3.20)$$

The last of these equations imposes a *consistency condition*. Remember that we seek x_1, x_2, x_3, and x_4 that satisfy a *static equilibrium*: if the condition $f_2 + f_4 = 0$ is violated, then the horizontal load is imbalanced ($f_2 \neq -f_4$), and the structure will not stand: no static equilibrium exists. You knew this instinctively from Figure 3.2, and now you see it confirmed by the linear algebra.

. . . aside perhaps from the pile of debris that occurs when the structure collapses!

But what if the forces *do* balance, $f_2 + f_4 = 0$? Then equations (3.17) and (3.19) immediately give

$$x_1 = f_1/k_1, \qquad x_3 = f_3/k_3.$$

Regarding x_2 and x_4, we only know

$$x_2 = x_4 + f_2/k_2.$$

We call x_4 a *free variable*: for any value it takes, we can construct a solution to (3.17)–(3.20). Thus, any vector of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} f_1/k_1 \\ \gamma + f_2/k_2 \\ f_3/k_3 \\ \gamma \end{bmatrix}, \qquad f_2 = -f_4 \quad (3.21)$$

solves $\mathbf{A}^T\mathbf{K}\mathbf{A}\mathbf{x} = \mathbf{f}$. That is, there are *infinitely many* static configurations of the structure for any single consistent load.

One particular choice of consistent forces illuminates the key problem. Consider the trivial load

$$f_1 = f_2 = f_3 = f_4 = 0,$$

in which case the solution space (3.21) becomes

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ \gamma \\ 0 \\ \gamma \end{bmatrix}, \qquad \mathbf{f} = \mathbf{0} \quad (3.22)$$

for any value of γ. Think about the physical significance of this solution. When *no load is applied*, the structure is free to displace in any of the solutions (3.22), which correspond to both masses shifting to the right (or left) by the same amount. A structure that permits such unforced shifts cannot stand: it is *unstable*, and the linear algebra reveals the nature of this instability.

Student experiments

3.12. How would you change the tipsy table in Figure 3.2 to stabilize it? How would your modification affect the linear algebra leading to the equation $\mathbf{A}^T\mathbf{K}\mathbf{A}\mathbf{x} = \mathbf{f}$?

Now, in the simple arrangement of Figure 3.2 the trouble is easy to diagnose. In structures with hundreds or thousands of struts, instabilities can be much more difficult to eyeball. We will thus develop tools for diagnosing these instabilities, and, later, for understanding near-instabilities too. Before doing so, we should fix that instability.

3.5 *Trusses with oblique supports*

Our analysis of the truss in Figure 3.2 was simplified by the fact that the springs meet at right angles. More interesting structures inevitably present more complicated geometry. How do we resolve springs at oblique angles?

Suppose mass m_j is connected to mass m_k by spring ℓ, forming an angle θ measured clockwise from the horizontal, as illustrated in Figure 3.3 Then our approximation rule (that springs are mainly deformed in their direction of rest orientation) gives

The angle is measured this way because of our convention that downward vertical displacements correspond to positive x_j values.

$$\text{elongation} \approx \sin(\theta_\ell)\Big(\text{net vertical displacement}\Big) + \cos(\theta_\ell)\Big(\text{net horizontal displacement}\Big)$$

and so

$$e_\ell = \sin(\theta_\ell)(x_{2k-1} - x_{2j-1}) + \cos(\theta_\ell)(x_{2k} - x_{2j}). \tag{3.23}$$

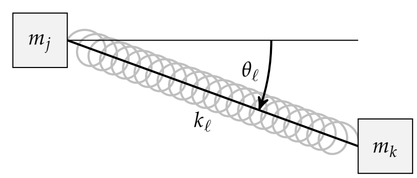

Figure 3.3: A spring's angle is measured clockwise, from the vertical.

Try checking this formula by applying it to the horizontal and vertical springs that make up the truss in Figure 3.2.

Now put equation (3.23) into action in a more interesting scenario, by introducing a diagonal brace to support the truss in Figure 3.2. Suppose for simplicity that the three springs in the original table have equal length, $L_1 = L_2 = L_3$, with a new diagonal spring connecting m_1 to the floor, forming an angle of $\pi/4$ with the second spring, as shown in Figure 3.4.

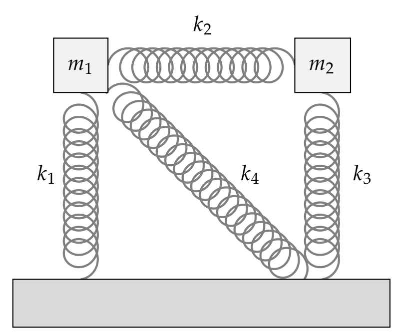

Figure 3.4: Take the truss in Figure 3.2 (with $L_1 = L_2 = L_3$) and add a diagonal brace that meets the second spring at an angle of $\pi/4$.

We quickly recapitulate the steps of our modeling procedure. The first three springs elongate as before:

$$e_1 = -x_1$$
$$e_2 = x_4 - x_2$$
$$e_3 = -x_3$$

The fourth spring is obviously the interesting one. Appealing to (3.23) with $\theta = \pi/4$ gives

$$e_4 = \sin(\pi/4)(0 - x_1) + \cos(\pi/4)(0 - x_2)$$

$$= -\frac{\sqrt{2}}{2}(x_1 + x_2).$$

Now we set up the elongation equations in the form $\mathbf{e} = \mathbf{A}\mathbf{x}$:

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ -\sqrt{2}/2 & -\sqrt{2}/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Here we have taken $j = 1$ in (3.23), and since the spring anchored into the floor at the far end, the displacements x_{2k-1} and x_{2k} are both zero.

Step 2 of our modeling procedure proceeds as expected:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}.$$

Step 3 requires that the restoring force of each spring be resolved into its horizontal and vertical components. For the scenario sketched in Figure 3.3, spring ℓ connects to masses m_j and m_k, and so exterts forces on both bodies. Those forces must be resolved into horizontal and vertical components.

mass	direction	contribution
m_j	vertical	$\sin(\theta_\ell)y_\ell$
m_j	horizontal	$\cos(\theta_\ell)y_\ell$
m_k	vertical	$-\sin(\theta_\ell)y_\ell$
m_k	horizontal	$-\cos(\theta_\ell)y_\ell$

Thus, in our scenario we still have four force balance equations, but with new terms for the diagonal springs:

MASS 1, VERTICAL: $0 = f_1 + y_1 + (\sqrt{2}/2)y_4$;

MASS 1, HORIZONTAL: $0 = f_2 + y_2 + (\sqrt{2}/2)y_4$;

MASS 2, VERTICAL: $0 = f_3 + y_3$;

MASS 2, HORITZONTAL: $0 = f_4 - y_2$.

In general, at mass m_j the horizontal and vertical force balances give the equations

$$0 = f_{2j-1} + \sum_\ell \sin(\theta_\ell)y_\ell$$

$$0 = f_{2j} + \sum_\ell \cos(\theta_\ell)y_\ell,$$

where the sum over ℓ includes all springs that attach to mass m_j, and the angles θ_j are measured from the positive horizontal emanating from m_j.

Thus we arrive at the balance equation

$$\begin{bmatrix} -1 & 0 & 0 & -\sqrt{2}/2 \\ 0 & -1 & 0 & -\sqrt{2}/2 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}.$$

Indeed, this is our usual equation

$$\mathbf{A}^T\mathbf{y} = \mathbf{f},$$

leading once more to

$$\mathbf{A}^T\mathbf{K}\mathbf{A}\mathbf{x} = \mathbf{f}$$

In light of the instability we diagnosed for the original tipsy table, we now ask the critical question:

Will the brace stabilize the structure,
i.e., will $\mathbf{A}^T\mathbf{K}\mathbf{A}\mathbf{x} = \mathbf{0}$ only have the trivial solution $\mathbf{x} = \mathbf{0}$?

Work out

$$\mathbf{A}^T\mathbf{K}\mathbf{A} = \begin{bmatrix} k_1 + k_4/2 & k_4/2 & 0 & 0 \\ k_4/2 & k_2 + k_4/2 & 0 & -k_2 \\ 0 & 0 & k_3 & 0 \\ 0 & -k_2 & 0 & k_2 \end{bmatrix}.$$

To simplify the arithmetic, suppose $k_1 = k_2 = k_3 = k_4 = 1$. Then to explore solutions to $\mathbf{A}^T\mathbf{K}\mathbf{A}\mathbf{x} = \mathbf{f}$, set up the augmented matrix

$$\left[\begin{array}{cccc|c} 3/2 & 1/2 & 0 & 0 & f_1 \\ 1/2 & 3/2 & 0 & -1 & f_2 \\ 0 & 0 & 1 & 0 & f_3 \\ 0 & -1 & 0 & 1 & f_4 \end{array}\right].$$

Eliminate the $(2,1)$ entry:

$$\left[\begin{array}{cccc|c} 3/2 & 1/2 & 0 & 0 & f_1 \\ 0 & 4/3 & 0 & -1 & f_2 - f_1/3 \\ 0 & 0 & 1 & 0 & f_3 \\ 0 & -1 & 0 & 1 & f_4 \end{array}\right].$$

Now eliminate the $(4,2)$ entry:

$$\left[\begin{array}{cccc|c} 3/2 & 1/2 & 0 & 0 & f_1 \\ 0 & 4/3 & 0 & -1 & f_2 - f_1/3 \\ 0 & 0 & 1 & 0 & f_3 \\ 0 & 0 & 0 & 1/4 & f_4 + 3f_2/4 - f_1/4 \end{array}\right],$$

reducing the matrix on the left to upper-triangular form. Unlike the previous augmented form (3.16), all then entries on the main diagonal of this upper triangular matrix are *nonzero*. This means that for any choice of $\mathbf{f}$, we can find a unique corresponding $\mathbf{x}$ vector. In this case, the augmented triangular form implies

$$\begin{aligned} \tfrac{3}{2}x_1 + \tfrac{1}{2}x_2 &= f_1 \\ \tfrac{4}{3}x_2 - x_4 &= f_2 - \tfrac{1}{3}f_1 \\ x_3 &= f_3 \\ \tfrac{1}{4}x_4 &= f_4 - \tfrac{3}{4}f_2 - \tfrac{1}{4}f_1, \end{aligned}$$

giving, for each $\mathbf{f}$, the *unique* solution

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} f_1 - f_2 - f_4 \\ -f_1 + 3f_2 + 3f_4 \\ f_3 \\ -f_1 + 3f_2 + 4f_4 \end{bmatrix}.$$

The only way for $\mathbf{A}^T\mathbf{KAx} = \mathbf{f}$ to equal zero is for $\mathbf{f} = \mathbf{0}$, so we conclude that the table with the diagonal brace is *stable* The same result holds any choice of (nonzero) spring constants, k_1, k_2, k_3, and k_4. With the diagonal spring added, the instability has been removed, and the structure sits in its static equilibrium.

In the next lecture, we shall delve more deeply into the scenarios where $\mathbf{A}^T\mathbf{KAx} = \mathbf{f}$ has a unique solution.