ON THE ORDER SEQUENCE OF AN EMBEDDING OF THE REE CURVE

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Abstract. In this paper we compute the Weierstrass order-sequence associated with a certain linear series on the Deligne-Lusztig curve of Ree type. As a result, we determine that the set of Weierstrass points of this linear series consists entirely of $\mathbb{F}_q$-rational points.

1. Introduction

Let X be a smooth, geometrically irreducible, projective algebraic curve defined over a finite field $\mathbb{F}_q$ of characteristic p, and let

$$m(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0 \in \mathbb{Z}[t]$$

be the square-free part of the characteristic polynomial of the Frobenius endomorphism Fr_q on the Jacobian of X. Then for any $P, P_0 \in X$ with P_0 an $\mathbb{F}_q$-rational point, we have the fundamental linear equivalence [9]

$$m(\mathrm{Fr}_q)(P) = \mathrm{Fr}_q^n(P) + \cdots + a_1 \mathrm{Fr}_q(P) + a_0 P \sim m(1)P_0. \tag{1}$$

Thus for $m = |m(1)|$, the linear series $\mathcal{D}_X := |mP_0|$, sometimes called the *Frobenius linear series*, is independent of the choice of rational point P_0, and is completely determined by the zeta function $Z_X(t)$.

The linear series $\mathcal{D}_X$ is a useful tool for studying curves with many rational points. It has been used to study $\mathbb{F}_q$-*maximal curves*, that is, curves defined over $\mathbb{F}_q$ whose number of $\mathbb{F}_q$-rational points attains the Hasse-Weil bound [6], [1], [4], [5], as well as $\mathbb{F}_q$-*optimal curves*, which have the greatest number of $\mathbb{F}_q$-rational points among curves of their genus [7].

Notable among the latter group are the Hermitian, Suzuki, and Ree curves, which are the Deligne–Lusztig curves associated to the simple groups ${}^2A_2$, ${}^2B_2$, and ${}^2G_2$. These curves satisfy strong uniqueness properties. Each is characterized among curves over $\mathbb{F}_q$ by its genus, number of rational points, and automorphism group [8]. Moreover, it can be shown that the genus and number of rational points alone are sufficient to characterize the Hermitian and Suzuki curves [12], [7]. Whether this is also the case for the Ree curve remains an open question—one which was the initial motivation for the work in the current paper.

For fixed $s \geq 1$, let $q_0 = 3^s$ and $q = 3^{2s+1}$. The Ree curve $X = X(q)$ over $\mathbb{F}_q$ has a singular affine model given by the two equations

$$y^q - y = x^{q_0}(x^q - x), \qquad z^q - z = x^{q_0}(y^q - y), \tag{2}$$

and has genus $g = \frac{3}{2}q_0(q-1)(q+q_0+1)$ and $N = q^3 + 1$ points defined over $\mathbb{F}_q$. Weil–Serre's explicit formulas can be used to show that X is $\mathbb{F}_q$-optimal, and that any curve defined over $\mathbb{F}_q$ with this g and N has L-polynomial

$$L_X(t) = (1 + 3q_0t + qt^2)^{q_0(q^2-1)}(1 + qt^2)^{\frac{1}{2}q_0(q-1)(q+3q_0+1)}. \tag{3}$$

Since the characteristic polynomial of Fr_q is $t^{2g}L_X(1/t)$, we obtain

$$\begin{aligned} m(t) &= (t^2 + 3q_0t + q)(t^2 + q) \\ &= t^4 + 3q_0t^3 + 2qt^2 + 3qq_0t + q^2 \end{aligned}$$

and $\mathcal{D}_X = |mP_0|$ with $m = m(1) = 1 + 3q_0 + 2q + 3qq_0 + q^2$.

There is a subseries $\mathcal{D} \subset \mathcal{D}_X$ of projective dimension 13 which is invariant under $\mathrm{Aut}(X)$. In [3], Duursma and Eid show that $\mathcal{D}$ is very ample, giving a smooth embedding of X in $\mathbb{P}^{13}$. They also find 105 equations describing the image of this embedding, and use these to compute the Weierstrass semigroup at a rational point when $s = 1$. In this case, it follows from their work that $\mathcal{D} = \mathcal{D}_X$ is a complete linear series. Whether or not $\mathcal{D}$ is complete for $s \geq 2$ is unknown at present. In [10], Kane also gives an embedding of X in $\mathbb{P}^{13}$ that arises from the abstract theory of Deligne–Lusztig varieties. The exact relationship between the two embeddings is not immediately clear.

In this paper we determine the order sequence of $\mathcal{D}$, that is, the orders of vanishing of sections of $\mathcal{D}$ at a general point. Equivalently, these are the intersection multiplicities of hyperplane sections of X embedded in $\mathbb{P}^{13}$ at a general point. We prove the following theorem.

Theorem 1. *The orders of $\mathcal{D}$ are*

$$0,\ 1,\ q_0,\ 2q_0,\ 3q_0,\ q,\ q+q_0,\ 2q,\ qq_0,\ qq_0+q_0,\ qq_0+q,\ 2qq_0,\ 3qq_0,\ q^2.$$

Since $\mathcal{D} \subset \mathcal{D}_X$, these form a subset of the orders of $\mathcal{D}_X$.

As a consequence, we show in the final section that the Weierstrass points of $\mathcal{D}$ consist of the $\mathbb{F}_q$-rational points of X.

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2. Background

The theory of Weierstrass points in characteristic p was developed first by F.K. Schmidt [13]. We briefly give the necessary definitions and results on the subject following the presentation in the paper of Stöhr and Voloch [15].

Given a base-point-free linear series $\mathcal{D}$ on X of dimension r and degree d, and P a point of X, the $(\mathcal{D}, P)$*-orders* consist of the sequence

$$0 = j_0(P) < j_1(P) < \cdots < j_r(P) \leq d$$

of integers j_i such that there is a hyperplane in $\mathcal{D}$ intersecting P with multiplicity equal to j_i. These are the same for all but finitely many points $P \in X$, called *$\mathcal{D}$-Weierstrass points.* The generic values of the $j_i(P)$ are the *$\mathcal{D}$-orders*

$$0 = \epsilon_0 < \epsilon_1 < \cdots < \epsilon_r.$$

This order sequence may be computed by choosing the ϵ_i lexicographically smallest so that

$$(D_x^{\epsilon_i} f_0 : D_x^{\epsilon_i} f_1 : \cdots : D_x^{\epsilon_i} f_r), \qquad i = 1, \ldots, r,$$

are linearly independent in $\mathbb{P}^r_{\mathbb{F}_q(X)}$, where $f_0, f_1, \ldots, f_r$ is a basis for $\mathcal{D}$, and the D_x^i are Hasse derivatives taken with respect to some fixed separating variable x.

The Hasse derivatives D_x^i are defined on $\mathbb{F}_q(x)$ by

$$D_x^i x^j = \binom{j}{i} x^{j-i},$$

and extend to derivations on $\mathbb{F}_q(X)$ satisfying the properties

$$D_x^k(fg) = \sum_{i+j=k} (D_x^i f)(D_x^j g) \qquad \text{and} \qquad D_x^k f^p = \begin{cases} (D_x^{k/p} f)^p & \text{if } p \mid k \\ 0 & \text{otherwise} \end{cases}$$

for any $f, g \in \mathbb{F}_q(X)$. In view of this second property, it will be often be convenient to write $D_x^{k/q}$ for k/q is a rational number with denominator a power of p, adopting the convention that $D_x^{k/q} = 0$ when k/q is not an integer. Furthermore, when the choice of separating variable x is clear from context, we omit the subscript and write simply D^i.

The following "p-adic criterion" for $\mathcal{D}$-orders is quite useful.

Lemma 2 ([15], Corollary 1.9)**.** *If ϵ is a $\mathcal{D}$-order and $\binom{\epsilon}{\mu} \not\equiv 0 \bmod p$, then μ is also a $\mathcal{D}$-order.*

By Lucas's Theorem, the condition $\binom{\epsilon}{\mu} \not\equiv 0 \bmod p$ in the lemma is equivalent to saying that the coefficients in the p-adic expansion of ϵ are greater than or equal to those in the expansion of μ. When this is the case we write $\mu \leq_p \epsilon$. This defines a partial order on the nonnegative integers.

The (q-)*Frobenius orders* $0 = \nu_0 < \nu_1 < \cdots < \nu_{r-1}$ of $\mathcal{D}$ form a subsequence of the order sequence $\{\epsilon_i\}$, and are defined lexicographically smallest so that

$$\begin{aligned}
&(f_0^q : f_1^q : \cdots : f_r^q), \\
&(D_x^{\nu_0} f_0 : D_x^{\nu_0} f_1 : \cdots : D_x^{\nu_0} f_r), \\
&\qquad \vdots \\
&(D_x^{\nu_{r-1}} f_0 : D_x^{\nu_{r-1}} f_1 : \cdots : D_x^{\nu_{r-1}} f_r)
\end{aligned}$$

are linearly independent in $\mathbb{P}^r_{\mathbb{F}_q(X)}$. There is exactly one $\mathcal{D}$-order ϵ_I which is omitted by the sequence $\{\nu_i\}$. The geometric significance of the index I is as follows: it is the smallest $i \geq 0$ such that, for general P, the image of P under the Frobenius endomorphism lies in the ith osculating space at P. The Frobenius orders are closely connected with the $\mathbb{F}_q$-rational points of X, and are used in Stöhr and Voloch's proof of the Riemann Hypothesis for curves over finite fields.

Lemma 3 ([15], discussion preceding Proposition 2.3)**.** *Let $(1 : f_1 : \cdots : f_r)$ be the morphism associated to $\mathcal{D}$. Then the Frobenius orders of $\mathcal{D}$ which are less than q are the first several orders of the morphism $(f_1 - f_1^q : \cdots : f_r - f_r^q)$.*

3. Derivatives on the Ree Curve

The function field of the Ree curve X is $\mathbb{F}_q(x, y, z)$, where y and z satisfy (2). The linear series $\mathcal{D}$ we wish to study corresponds to the $\overline{\mathbb{F}}_q$-vector space $V_{\mathcal{D}}$ spanned by the 14 functions

$$\mathcal{B} = \{1,\ x,\ y,\ z,\ w_1,\ w_2,\ \ldots,\ w_{10}\},$$

where w_i are defined by

$$
\begin{aligned}
w_1 &= x^{3q_0+1} - y^{3q_0} & w_6 &= v^{3q_0} - w_2^{3q_0} + xw_4^{3q_0} \\
w_2 &= xy^{3q_0} - z^{3q_0} & w_7 &= w_2 + v \\
w_3 &= xz^{3q_0} - w_1^{3q_0} & w_8 &= w_5^{3q_0} + xw_7^{3q_0} \\
w_4 &= xw_2^{q_0} - yw_1^{q_0} & w_9 &= w_4 w_2^{q_0} - yw_6^{q_0} \\
v &= xw_3^{q_0} - zw_1^{q_0} & w_{10} &= zw_6^{q_0} - w_3^{q_0} w_4 \\
w_5 &= yw_3^{q_0} - zw_1^{q_0} & &
\end{aligned}
\tag{4}
$$

as in the appendix of [11]. The functions in $\mathcal{B}$ have distinct orders at the pole P_∞ of x, hence are linearly independent. We will use the separating variable x for computing all Hasse derivatives on the Ree curve.

To compute the orders of $\mathcal{D}$, we will need to obtain closed form expressions for the derivatives of the functions $f \in \mathcal{B}$. In addition to the relations in (2), the following equations derived by Pedersen will be useful for computing the derivatives of the w_i.

$$
\begin{aligned}
w_1^q - w_1 &= x^{3q_0}(x^q - x) & w_4^q - w_4 &= w_2^{q_0}(x^q - x) - w_1^{q_0}(y^q - y) \\
w_2^q - w_2 &= y^{3q_0}(x^q - x) & w_5^q - w_5 &= w_3^{q_0}(y^q - y) - w_2^{q_0}(z^q - z) \\
w_3^q - w_3 &= z^{3q_0}(x^q - x) & w_7^q - w_7 &= w_2^{q_0}(y^q - y) - w_3^{q_0}(x^q - x) \\
w_6^q - w_6 &= w_4^{3q_0}(x^q - x) & w_9^q - w_9 &= w_2^{q_0}(w_4^q - w_4) - w_6^{q_0}(y^q - y) \\
w_8^q - w_8 &= w_7^{3q_0}(x^q - x) & w_{10}^q - w_{10} &= w_6^{q_0}(z^q - z) - w_3^{q_0}(w_4^q - w_4)
\end{aligned}
\tag{5}
$$

We have separated these equations into groups of similar form. We call the w_i which appear on the left hand side of (5) of type 1 and the w_i on the right hand side of type 2.

We give an example to show how the expressions in (5) are useful for computing derivatives. To compute the derivatives of y, we let $h = x^{q_0}(x^q - x)$. Since $y^q - y = h$, we may expand y as a series in h whose tail is contained in the kernel of any of the derivations we wish to apply. To compute $D^i y$ for $i < q^2$, we consider

$$y = -h - h^q + y^{q^2} \equiv -h - h^q \mod \overline{\mathbb{F}}_q(X)^{q^2},$$

since $\overline{\mathbb{F}}_q(X)^{q^2} = \bigcap_{i=1}^{q^2-1} \ker D^i$. Then

$$D^i y = -D^i h - (D^{i/q} h)^q.$$

In this manner, the derivatives of y are written in terms of derivatives of h, which can be determined using the basic properties of Hasse derivatives.

Each equation in (5) is of a similar form, with each new function written in terms of previous ones. Therefore, one may in principle write down any derivative $D^i f$ with $f \in \mathcal{B}$ as an element of $\mathbb{F}_q[x, y, z]$ using this method.

For each $f \in \mathcal{B}$, we construct a set S_f containing all indices i in $\{0, 1, \ldots, q^2\}$ such that $D^i f \neq 0$, which we refer to as the *support* of f. We make no claims that

$D^i f \neq 0$ for all i in S_f. By direct calculation as in the example above, the sets

$$\begin{aligned} S_{x^q-x} &= \{0,1,q\} \\ S_{y^q-y} &= \{0,1,q_0,q_0+1,q,q+q_0\} \\ S_y &= \{0,1,q_0,q_0+1,q,q+q_0,qq_0,qq_0+q,q^2\} \\ S_{z^q-z} &= \{0,1,q_0,q_0+1,2q_0,2q_0+1,q,q+q_0,q+2q_0\} \\ S_z &= \{0,1,q_0,q_0+1,2q_0,2q_0+1,q, \\ &\qquad q+q_0,q+2q_0,qq_0,qq_0+q,2qq_0,2qq_0+q,q^2\} \end{aligned}$$

satisfy the desired conditions.

Now we construct S_{w_i} for $i=1,\dots,10$. For $n \geq 1$ and $A, B \subset \{0,\dots,q^2\}$ we use the notation

$$\begin{aligned} nA &= \{na : a \in A\} \cap [0,q^2], \\ A+B &= \{a+b : a \in A, b \in B\} \cap [0,q^2]. \end{aligned}$$

Since $w_1^q - w_1 = x^{3q_0}(x^q - x)$, we define

$$\begin{aligned} S_{w_1^q-w_1} &= 3q_0 S_x + S_{x^q-x} \\ &= \{0,3q_0\} + \{0,1,q\} = \{0,1,3q_0,3q_0+1,q,q+3q_0\} \end{aligned}$$

and

$$\begin{aligned} S_{w_1} &= S_{w_1^q-w_1} \cup q S_{w_1^q-w_1} \\ &= \{0,1,3q_0,3q_0+1,q,q+3q_0,3qq_0,3qq_0+q,q^2\}. \end{aligned}$$

Similarly, we define $S_{w_2^q-w_2} = 3q_0 S_y + S_{x^q-x}$ and $S_{w_2} = S_{w_2^q-w_2} \cup q S_{w_2^q-w_2}$, and so on, using the equations in (5) as a guide.

Let S denote the union of the S_f with $f \in \mathcal{B}$. Since we will use this information later, we include a table of the S_f in an appendix, and note in particular that

$$S_x \subset S_{w_1} \subset S_{w_2} \subset S_{w_3} \subset S_{w_6} = S_{w_8}$$

and

$$S_y \subset S_z \subset S_{w_4} \subset S_{w_7} \subset S_{w_5} \subset S_{w_9} \subset S_{w_{10}}.$$

For $s=1$, the indices appearing in S as represented in the appendix are not all distinct, since for example $3q = qq_0$. To avoid any complications this may cause, we assume going forward that $s \geq 2$. Computations performed in Magma [2] have verified the statements of all our results for $s=1$.

4. Computation of Orders

In this section we compute the orders of $\mathcal{D}$.

Theorem 1. *The orders of $\mathcal{D}$ are*

$$0,\ 1,\ q_0,\ 2q_0,\ 3q_0,\ q,\ q+q_0,\ 2q,\ qq_0,\ qq_0+q_0,\ qq_0+q,\ 2qq_0,\ 3qq_0,\ q^2.$$

Since $\mathcal{D} \subset \mathcal{D}_X$, these form a subset of the orders of $\mathcal{D}_X$.

Lemma 4. *The orders of $\mathcal{D}$ which are less than q are* 0, 1, q_0, $2q_0$, *and* $3q_0$.

Proof. That $\epsilon_0(\mathcal{D}) = 0$ and $\epsilon_1(\mathcal{D}) = 1$ is clear. The rest follows from Lemma 3. Indeed, the orders of the morphism

$$(x^q - x : y^q - y : z^q - z : w_1^q - w_1 : w_2^q - w_2 : \cdots) = (1 : x^{q_0} : x^{2q_0} : x^{3q_0} : y^{3q_0} : \cdots)$$

which are less than q are 0, q_0, $2q_0$, and $3q_0$, so these are the Frobenius orders of $\mathcal{D}$ which are less than q. There is only one order of $\mathcal{D}$ which is not a Frobenius order, and this is $\epsilon_1(\mathcal{D})$. □

Remark. In light of Lemma 3, the fact that $\nu_1(\mathcal{D}) = \epsilon_2(\mathcal{D}) > 1$ means that the matrix

$$\begin{pmatrix} x^q - x & y^q - y & z^q - z & \cdots & w_{10}^q - w_{10} \\ 1 & D^1 y & D^1 z & \cdots & D^1 w_{10} \end{pmatrix}$$

has rank 1, so that

$$f^q - f = (x^q - x) D^1 f \tag{6}$$

holds for all $f \in \mathcal{B}$. Applying the derivatives D^q and D^{kq_0} for $k = 1, 2, 3$ to the previous equation gives the identities

$$D^{kq_0} f = -(x^q - x) D^{kq_0+1} f, \qquad k = 1, 2, 3 \tag{7}$$

$$D^q(f^q - f) = D^1 f + (x^q - x) D^{q+1} f \tag{8}$$

which hold for all $f \in \mathcal{B}$. These will be used extensively in what follows.

From (1), we have for any $P \in X$ a linear equivalence

$$\mathrm{Fr}^4(P) + 3q_0 \,\mathrm{Fr}^3(P) + 2q \,\mathrm{Fr}^2(P) + 3qq_0 \,\mathrm{Fr}(P) + q^2 P \sim m P_\infty.$$

If $P \notin X(\mathbb{F}_q)$, then the terms on the left hand side involve distinct points since X has no places of degrees 2, 3, or 4 over $\mathbb{F}_q$. By applying some multiple of the Frobenius to this equivalence, we obtain each of 1, $3q_0$, $2q$, $3qq_0$, and q^2 as orders of $\mathcal{D}_X$, as in Lemma 3.2 of [7]. By the p-adic criterion it follows that q is also an order of $\mathcal{D}_X$. That said, it is not immediately clear that these are orders of the linear series $\mathcal{D}$. We show now that these are in fact the orders of the subseries $\mathcal{E} \subset \mathcal{D}$ corresponding to $V_{\mathcal{E}} = \overline{\mathbb{F}}_q\langle 1, x, w_1, w_2, w_3, w_6, w_8\rangle$, and hence are orders of $\mathcal{D}$.

Theorem 5. *The orders of $\mathcal{E}$ are 0, 1, $3q_0$, q, $2q$, $3qq_0$, and q^2.*

For ease of notation we write $\ell = x^q - x$ in the proof of the next lemma and throughout the rest of the paper.

Lemma 6. *The image in $\mathbb{P}^6$ of the map $\phi_{\mathcal{E}} = (1 : x : w_1 : w_2 : w_3 : w_6 : w_8)$ lies on the hypersurface*

$$\sum_{i+j=6} X_i^{q^2} X_j = 0.$$

Proof. By using (5) one finds that

$$\begin{aligned} w_8^{q^2} + w_8 &= (w_7^{3q_0})^q \ell^q + w_7^{3q_0} \ell - w_8 \\ x w_6^{q^2} + x^{q^2} w_6 &= ((w_4^{3q_0})^q x + w_6)\ell^q + (w_4^{3q_0} x + w_6)\ell - x w_6 \\ w_1 w_3^{q^2} + w_1^{q^2} w_3 &= ((z^{3q_0})^q w_1 + (x^{3q_0})^q w_3)\ell^q + (z^{3q_0} w_1 + x^{3q_0} w_3)\ell - w_1 w_3 \\ w_2^{q^2+1} &= (y^{3q_0})^q w_2 \ell^q + y^{3q_0} w_2 \ell + w_2^2. \end{aligned}$$

Summing these and collecting terms involving common powers of ℓ gives an expression of the form

$$A_{-1} + A_0\ell + A_1\ell^q.$$

That each $A_i = 0$ on X may be verified using (4) and (5), along with some of the 105 equations found in [3]. This calculation is carried out more explicit detail in the proof of Lemma 4.3 of [14]. □

Lemma 7. *The largest order of $\mathcal{E}$ is q^2.*

Proof. Because the coefficients a_i of $m(t) = t^4 + 3q_0t^3 + 2qt^2 + 3qq_0t + q^2$ satisfy $a_0 \geq a_2 \geq \cdots \geq a_4$ and $\#X(\mathbb{F}_q) > q(m - a_0) + 1$, it follows from Proposition 3.4 of [6] that the largest order of $\mathcal{D}_X$ is q^2 (our numbering of the coefficients a_i is opposite that found in the reference). Since each order of $\mathcal{E}$ is an order of $\mathcal{D}_X$, no order of $\mathcal{E}$ is greater than q^2. We exhibit a family of functions g_P in $V_{\mathcal{E}}$ parameterized by P in X which vanish to order at least q^2 at P.

Write $(1, x, w_1, w_2, w_3, w_6, w_8) = (f_0, \ldots, f_6)$. Then by Lemma 6, the function

$$G(P, Q) = \sum_{i+j=6} f_i^{q^2}(P) f_j(Q)$$

vanishes on the diagonal of $X \times X$. Choose any $P \in X \setminus \{P_\infty\}$. Then $g_P = G(P, \cdot)$ and $h_P = G(\cdot, P)$ are functions on X which vanish at P, and g_P is in $V_{\mathcal{E}}$. Since

$$h_P = \sum_{i+j=6} f_j(P) f_i^{q^2} = \left(\sum_{i+j=6} f_j(P)^{1/q^2} f_i \right)^{q^2},$$

the function h_P vanishes at P to order at least q^2. But

$$\begin{aligned} g_P - h_P &= \sum_{i+j=6} f_i^{q^2}(P) f_j - f_j(P) f_i^{q^2} \\ &= \sum_{i+j=6} (f_i^{q^2}(P) - f_i^{q^2})(f_j(P) + f_j) + f_i^{q^2} f_j - f_i^{q^2}(P) f_j(P) \\ &= \sum_{i+j=6} (f_i(P) - f_i)^{q^2} (f_j(P) + f_j) \end{aligned}$$

also vanishes at P to order at least q^2, hence so does g_P. Since P was chosen in an open subset of X, q^2 is an order of $\mathcal{E}$. □

Remark. The proof of the preceding lemma shows that

$$\sum_{i+j=6} f_i^{q^2}(P) X_j = 0$$

is the equation of the osculating hyperplane at P, and that the function $X \to \mathrm{Div}(X)$ taking $P \mapsto \mathrm{div}(g_P)$ assigns to P the corresponding hyperplane section.

Proof of Theorem 5. Let M be the matrix of derivatives $[D^i f_j]$ with $i \in \{0, 1, 3q_0 + 1, q + 3q_0 + 1, 2q + 3q_0 + 1\}$ and $f_j \in \{1, x, w_1, w_2, w_3\}$. By the appendix, the matrix M is upper triangular. Moreover, one may check by hand that each diagonal entry is equal to 1. Thus, there are at least 5 orders ϵ of $\mathcal{E}$ with $\epsilon \leq 2q + 3q_0 + 1$.

Let $I_{\mathcal{E}} = \{0, 1, 3q_0, q, 2q, 3qq_0, q^2\}$ be the proposed set of orders. The subset of $S_{w_8} \setminus I_{\mathcal{E}}$ of elements minimal with respect to the partial order $\leq_3$ is

$$J = \{3q_0 + 1, q + 1, q + 3q_0, 3q, 3qq_0 + 1, 3qq_0 + 3q_0, 3qq_0 + q, 6qq_0\}.$$

By the p-adic criterion, to prove the theorem it will suffice to show that no $j \in J$ is an order of $\mathcal{E}$. In fact, it will be enough to show that no $j \in \{3q_0+1, q+1, q+3q_0, 3q\}$ is an order. For then we will already know that the six elements of $I_{\mathcal{E}} \setminus \{3qq_0\}$ are orders. Then exactly one of the remaining elements of J is the seventh and final order of $\mathcal{E}$. But each of the remaining elements satisfies $j \geq_3 3qq_0$, so this final order is $3qq_0$.

That $3q_0 + 1$ is not an order follows from (7). Let w be one of the functions w_1, w_2, w_3, w_6, w_8. Each of these is of the form

$$w \equiv -h - h^q \mod \overline{\mathbb{F}}_q(X)^{q^2},$$

where $h = f^{3q_0}(x^q - x)$ and $f \in \{x, y, z, w_4, w_7\}$. Then $D^k w = -D^k h - (D^{k/q} h)^q$ and

$$\begin{aligned} D^k h &= \sum_{3q_0 i + j = k} (D^i f)^{3q_0} D^j (x^q - x) \\ &= (x^q - x)(D^{\frac{k}{3q_0}} f)^{3q_0} - (D^{\frac{k-1}{3q_0}} f)^{3q_0} + (D^{\frac{k-q}{3q_0}} f)^{3q_0}. \end{aligned}$$

We calculate

$$\begin{aligned} D^{3q_0+1} w &= (D^1 f)^{3q_0} & D^{q+3q_0} w &= -(D^1 f)^{3q_0} - \ell (D^{q_0+1} f)^{3q_0} \\ D^q w &= (f^q - f)^{3q_0} - \ell (D^{q_0} f)^{3q_0} & D^{2q} w &= -(D^{q_0} f)^{3q_0} - \ell (D^{2q_0} f)^{3q_0} \\ D^{q+1} w &= (D^{q_0} f)^{3q_0} & D^{3q} w &= -(D^{2q_0} f)^{3q_0}. \end{aligned} \tag{9}$$

Then by using these along with (6) and (7), one immediately verifies that

$$\begin{aligned} \ell D^{q+1} w + D^q w &= \ell^{3q_0} D^{3q_0+1} w \\ \ell^{3q_0} D^{q+3q_0} w + D^q w &= 0 \\ \ell D^{3q} w &= D^{2q} w + D^{q+1} w, \end{aligned}$$

and so $q + 1$, $q + 3q_0$, and $3q$ are not orders of $\mathcal{D}$. This completes the proof. □

Up to this point we have shown that the nine numbers

$$0,\ 1,\ q_0,\ 2q_0,\ 3q_0,\ q,\ 2q,\ 3qq_0,\ q^2$$

are orders of $\mathcal{D}$, and it remains to show that $q + q_0$, qq_0, $qq_0 + q_0$, $qq_0 + q$, and $2qq_0$ are orders.

Proof of Theorem 1. Let $I_{\mathcal{D}}$ be the list of orders of $\mathcal{D}$ proposed in the statement of the theorem. Let M be the 12 by 12 matrix of derivatives $[D^i f_j]$ with i in

$$\begin{aligned} \{0, 1, q_0 + 1, 2q_0 + 1, 3q_0 + 1, q + 3q_0 + 1, 2q + 3q_0 + 1, qq_0 + 2q + q_0, \\ qq_0 + q + 2q_0 + 1, qq_0 + 2q + 3q_0, qq_0 + 3q + 3q_0, 2qq_0 + 3q_0 + 1\} \end{aligned}$$

and f_j in $\{1, x, y, z, w_1, w_2, w_3, w_4, w_7, w_5, w_9, w_{10}\}$. The appendix assures that M is upper triangular. Moreover, one may check by hand that each diagonal entry is equal to 1, except the last, which is x^{2q}. Thus there are at least 11 orders which are less than $2qq_0$, and 12 orders which are at most $2qq_0 + 3q_0 + 1$. Since there are 14 orders in total, and we already know that $3qq_0$ and q^2 are among them, to prove the theorem it will be enough to show that no element of $(S \setminus I_{\mathcal{D}}) \cap [0, 2qq_0 + 3q_0 + 1]$ is an order.

By the p-adic criterion, it suffices to check elements of this set which are minimal with respect to $\leq_3$. These elements comprise the set

$$J = \{q_0+1, 3q_0+1, q+1,\ q+2q_0,\ q+3q_0,\ 2q+q_0,\ 3q,\ qq_0+1,\\ qq_0+2q_0,\ qq_0+3q_0,\ qq_0+q+q_0,\ qq_0+2q,\ 2qq_0+q_0\}. \tag{10}$$

In fact, it will be enough to demonstrate that each element of $J \setminus \{2qq_0+q_0\}$ is not an order, since $2qq_0 \leq_3 2qq_0+q_0$.

That q_0+1 and $3q_0+1$ are not orders follows from Lemma 4. To deal with each remaining ten elements $j \in J$, we give a differential equation

$$c_j D^j f + \sum_{i<j} c_i D^i f = 0, \qquad c_i \in \mathbb{F}_q(X)$$

which is satisfied by all $f \in \mathcal{B}$. These are listed in the following lemma, and proven in the next section. This will complete the proof of the theorem. □

Lemma 8. *The following differential equations are satisfied by each $f \in \mathcal{B}$:*

(A1) $\ell^{q_0} D^{q_0+1} f + \ell^{2q_0} D^{2q_0+1} f + \ell^{3q_0} D^{3q_0+1} f = D^q f + \ell D^{q+1} f$
(A2) $\ell^{q_0}(D^{q+2q_0} f + D^{2q_0+1}) f = D^{q+q_0} f + D^{q_0+1} f$
(A3) $D^q f + \ell^{q_0} D^{q+q_0} f + \ell^{2q_0} D^{q+2q_0} f + \ell^{3q_0} D^{q+3q_0} f = 0$
(A4) $\ell D^{2q+q_0} f = D^{q_0+1} f + D^{q+q_0} f$
(A5) $\ell D^{3q} f = D^{2q} f + D^{q+1} f$
(A6) $\ell D^{qq_0+1} f + D^{qq_0} f = \ell^q(\ell^{q_0} D^{2q_0+1} f - D^{q_0+1} f)$
(A7) $\ell^{2q_0} D^{qq_0+2q_0} f - \ell^{q_0} D^{qq_0+q_0} f = \ell^q(D^{q_0+1} f + D^{q+q_0} f)$
(A8) $\ell^{q_0} D^{qq_0+q_0} f + \ell^{2q_0} D^{qq_0+2q_0} f + \ell^{3q_0} D^{qq_0+3q_0} f = \ell D^{qq_0+1} f$
(A9) $\ell^{q+q_0+1} D^{qq_0+q+q_0} f = (\ell^q - \ell)\ell^{q_0} D^{qq_0+q_0} f$
(A10) $\ell^{2q}(\ell D^{qq_0+2q} f - D^{qq_0+1} f) = (\ell^q - \ell)(\ell^q D^{qq_0+q} f + D^{qq_0} f)$.

Proof. The fact that x, y, and z satisfy these equations may be verified without difficulty by hand. If f is of type 1, then by consulting the appendix we see that the only equations among (A1)–(A10) in which nonzero derivatives of f appear are (A1), (A3), and (A5), and, keeping in mind that some of the terms are zero for f of type 1, these are the equations which were verified in the proof of Theorem 5. The proof for functions of type 2 is contained in the next section. □

5. Verification of Some Differential Equations

Before proceeding to verify equations (A1)–(A10) for functions of type 2, we first list a few identities for w of type 1 which will be useful for this task. For w of type 1, write $w^q - w = f^{3q_0}(b^q - b)$ as in the proof of Theorem 5. Then $D^{2q+1} w = (D^{2q_0} f)^{3q_0}$, and so by (9), we have

$$\ell D^{2q+1} w + D^{2q} w + D^{q+1} w = 0. \tag{11}$$

Also, from the proof of Theorem 5 we recall that

$$D^q w + \ell D^{q+1} w = \ell^{3q_0} D^{3q_0+1} w, \tag{12}$$

$$\ell^{3q_0} D^{q+3q_0} w + D^q w = 0. \tag{13}$$

Now let w be of type 2. From (5), w may be written in the form $w = t_1 - t_2$, where t_i satisfies

$$t_i^q - t_i = h_i = f_i^{q_0}(b_i^q - b_i),$$

for some $f_i \in \{w_1, w_2, w_3, w_6\}$ and $b_i \in \{x, y, z, w_4\}$. Thus, if one of the desired equations holds for all t_i of this form, then it also holds for w. This will be the case for some but not all of the equations we wish to verify.

Let t, h, f, b be as above. Then

$$D^k h = \sum_{q_0 i + j = k} (D^i f)^{q_0} D^j (b^q - b)$$

and

$$D^i t = -D^i h - (D^{i/q} h)^q.$$

Taking into account the supports of f and b and using (6) and (7) to make simplifications, we compute the derivatives of t which appear in equations (A1)–(A10):

$$\begin{aligned}
D^{kq_0+1} t &= f^{q_0} D^{kq_0+1} b + (D^1 f)^{q_0} D^{(k-1)q_0+1} b, \qquad k = 1,2,3 \\
D^q t &= f^{q_0} D^q b + (D^1 f)^{q_0} \ell^{q_0} D^q (b^q) + (D^{3q_0+1} f)^{q_0} \ell^{q_0+1} D^1 b \\
D^{q+1} t &= f^{q_0} D^{q+1} b - (D^{3q_0+1} f)^{q_0} \ell^{q_0} D^1 b \\
D^{q+q_0} t &= f^{q_0} D^{q+q_0} b - (D^1 f)^{q_0} D^q (b^q - b) \\
&\quad + (D^{3q_0+1} f)^{q_0} (\ell^{q_0+1} D^{q_0+1} b - \ell D^1 b) \\
D^{q+2q_0} t &= f^{q_0} D^{q+2q_0} b + (D^1 f)^{q_0} D^{q+q_0} b \\
&\quad + (D^{3q_0+1} f)^{q_0} (\ell^{q_0+1} D^{2q_0+1} b - \ell D^{q_0+1} b) \\
D^{q+3q_0} t &= f^{q_0} D^{q+3q_0} b + (D^1 f)^{q_0} D^{q+2q_0} b - (D^{3q_0+1} f)^{q_0} \ell D^{2q_0+1} b \\
D^{2q} t &= f^{q_0} D^{2q} b + (D^{3q_0+1} f)^{q_0} \ell^{q_0} D^q (b^q - b) \\
D^{2q+q_0} t &= f^{q_0} D^{2q+q_0} b + (D^1 f)^{q_0} D^{2q} b \\
&\quad - (D^{3q_0+1} f)^{q_0} (\ell^{q_0} D^{q+q_0} b + D^q (b^q - b)) \\
D^{3q} t &= f^{q_0} D^{3q} b - (D^{3q_0+1} f)^{q_0} \ell^{q_0} D^{2q} b \\
D^{qq_0} t &= f^{q_0} D^{qq_0} b + (D^1 f)^{q_0} \ell^{q_0} D^{qq_0} (b^q) \\
&\quad - (D^q f)^{q_0} \ell D^1 b - (D^q f^q)^{q_0} \ell^q D^q (b^q) \\
D^{qq_0+1} t &= f^{q_0} D^{qq_0+1} b + (D^q f)^{q_0} D^1 b \\
D^{qq_0+q_0} t &= f^{q_0} D^{qq_0+q_0} b - (D^1 f)^{q_0} D^{qq_0} (b^q - b) \\
&\quad - (D^q f)^{q_0} \ell D^{q_0+1} b - (D^{q+1} f)^{q_0} \ell D^1 b \\
D^{qq_0+2q_0} t &= f^{q_0} D^{qq_0+2q_0} b + (D^1 f)^{q_0} D^{qq_0+q_0} b \\
&\quad - (D^q f)^{q_0} \ell D^{2q_0+1} b - (D^{q+1} f)^{q_0} \ell D^{q_0+1} b. \\
D^{qq_0+3q_0} t &= f^{q_0} D^{qq_0+3q_0} b - (D^{q+1} f)^{q_0} \ell D^{2q_0+1} b \\
D^{qq_0+q} t &= f^{q_0} D^{qq_0+q} b + (D^1 f)^{q_0} \ell^{q_0} D^{qq_0+q} (b^q) \\
&\quad - (D^{3q_0+1} f)^{q_0} \ell^{q_0} (D^{qq_0} b - D^{qq_0} (b^q)) - (D^q f)^{q_0} D^q (b^q - b) \\
&\quad - (D^{q+3q_0} f)^{q_0} \ell D^1 b + (D^q f^q)^{q_0} D^q (b^q) \\
D^{qq_0+q+q_0} t &= f^{q_0} D^{qq_0+q+q_0} b - (D^1 f)^{q_0} D^{qq_0+q} (b^q - b) \\
&\quad - (D^{3q_0+1} f)^{q_0} (\ell^{q_0} D^{qq_0+q_0} b + D^{qq_0} (b^q - b)) + (D^q f)^{q_0} D^{q+q_0} b \\
&\quad - (D^{q+1} f)^{q_0} D^q (b^q - b) + (D^{q+3q_0} f)^{q_0} D^{q_0} b - (D^{q+3q_0+1} f)^{q_0} \ell D^1 b
\end{aligned}$$

$$\begin{aligned}D^{qq_0+2q}t = f^{q_0}D^{qq_0+2q}b + (D^{3q_0+1}f)^{q_0}\ell^{q_0}D^{qq_0+q}(b^q-b)\\ - (D^qf)^{q_0}D^{2q}(b^q-b) - (D^{q+3q_0}f)^{q_0}D^q(b^q-b).\end{aligned}$$

We will also use a few of the values D^ib, which we collect in the following table:

i	D^iy	D^iz	D^iw_4
1	x^{q_0}	x^{2q_0}	$-x^{2q_0+1}-x^{q_0}y-z$
q_0+1	1	$-x^{q_0}$	$x^{q_0+1}-y$
$2q_0+1$	0	1	$-x^q$
$q+1$	0	0	$-\ell^{2q_0}$
$q+q_0$	-1	x^{q_0}	$\ell^{q_0+1}-x^{q_0+1}+y$
$2q$	0	0	ℓ^{2q_0}
qq_0+1	0	0	$-\ell^{q+q_0}$
qq_0+q_0	0	0	$-\ell^{q+1}$

(14)

Now we verify each of the equations (A1)–(A10) in turn. Since each b_i associated with w appears before w in the list

$$x,\ y,\ z,\ w_4,\ w_5,\ w_7,\ w_9,\ w_{10},$$

and since each of (A1)–(A10) has already been verified for x, y, and z, we may assume by induction that these equations are satisfied by each b_i which appears in the proof.

Proof of (A1). We show that the desired equation holds for all t. Since $D^{3q_0+1}b=0$, we have

$$\begin{aligned}\sum_{k=1}^{3}\ell^{kq_0}D^{kq_0+1}t &= f^{q_0}\sum_{k=1}^{3}\ell^{kq_0}D^{kq_0+1}b + (D^1f)^{q_0}\ell^{q_0}(D^1b+\sum_{k=1}^{3}\ell^{kq_0}D^{kq_0+1}b)\\ &= f^{q_0}(\ell D^{q+1}b+D^qb) + (D^1f)^{q_0}\ell^{q_0}(D^1b+D^qb+\ell D^{q+1}b)\\ &= f^{q_0}(\ell D^{q+1}b+D^qb) + (D^1f)^{q_0}\ell^{q_0}D^q(b^q)\\ &= D^qt+\ell D^{q+1}t,\end{aligned}$$

where we used (8) in the third line. □

Proof of (A2). Let

$$\Delta_t := \ell^{q_0}(D^{q+2q_0}t + D^{2q_0+1}t) - (D^{q+q_0}t + D^{q_0+1}t).$$

Here it is not the case that $\Delta_t = 0$ for all t, so we need to show that $\Delta_{t_1}=\Delta_{t_2}$ for each $w=t_1-t_2$. The contribution to Δ_t of those terms involving f^{q_0} is zero by our assumption on b. Moreover, by (8) the contributed coefficient of $(D^1f)^{q_0}$ is

$$\begin{aligned}&\ell^{q_0}D^{q+q_0}b + \ell^{q_0}D^{q_0+1}b + D^q(b^q-b) - D^1b\\ &= \ell^{q_0}D^{q+q_0}b + \ell^{q_0}D^{q_0+1}b + \ell D^{q+1}b,\end{aligned}$$

which is zero by (14). Therefore, the only terms giving a contribution to Δ_t are those involving $(D^{3q_0+1}f)^{q_0}$. The coefficient of $(D^{3q_0+1}f)^{q_0}$ in Δ_t is

$$\sum_{k=0}^{2}\ell^{kq_0+1}D^{kq_0+1}b = \ell(D^1b+D^qb+D^{q+1}b) = \ell D^q(b^q),$$

where we have used the fact that $D^{3q_0+1}b = 0$, along with (8) and (A1). With the exception of $w = w_7$ we have $f_i^q - f_i = b_j^{3q_0}(x^q - x)$ for $i \neq j$, so that

$$(D^{3q_0+1} f_i)^{q_0} = (D^{3q_0}(b_j^{3q_0}))^{q_0} = D^q(b_j^q).$$

Therefore, for $w \neq w_7$ we have

$$\Delta_{t_i} = D^q(b_j^q) \cdot \ell D^q(b_i^q),$$

and so $\Delta_{t_1} = \Delta_{t_2}$ as desired. Finally, for $w = w_7$ we check that

$$D^q(y^q) \cdot \ell D^q(y^q) = x^{2qq_0}\ell = D^q(z^q) \cdot \ell D^q(x^q). \qquad \square$$

Proof of (A3). We show that the desired equation holds for all t. Since $D^{q+3q_0}b = 0$, we have

$$\sum_{k=0}^{3} \ell^{kq_0} D^{q+kq_0} t = f^{q_0} \sum_{k=0}^{3} \ell^{kq_0} D^{q+kq_0} b + (D^1 f)^{q_0} \ell^{q_0} \sum_{k=0}^{3} \ell^{kq_0} D^{q+kq_0} b + (D^{3q_0+1} f)^{q_0} \cdot 0 = 0. \qquad \square$$

Proof of (A4). We show that the desired equation holds for all t. The contribution of the terms involving f^{q_0} is zero by our assumption on b. By considering the contribution of terms involving $(D^1 f)^{q_0}$ and $(D^{3q_0+1} f)^{q_0}$, it will suffice to show that the equations

$$D^q(b^q - b) + D^1 b = \ell D^{2q} b$$
$$\ell^{q_0} D^{q_0} b + \ell D^1 b = \ell^{q_0+1} D^{q+q_0} b + \ell D^q(b^q - b)$$

hold for $b = x, y, z, w_4$. These may be rewritten using (8) as

$$D^{2q} b + D^{q+1} b = 0$$
$$\ell^{q_0} D^{q+q_0} b + \ell D^{q+1} b + \ell^{q_0} D^{q_0+1} b = 0,$$

and these follow from (14). $\square$

Proof of (A5). We show that the desired equation holds for all t. By our assumption on b, the contribution of all terms involving f^{q_0} is zero. By comparing the coefficients of $(D^{3q_0+1} f)^{q_0}$ and using (8), it suffices to show that

$$\ell D^{2q} b = D^1 b - D^q(b^q - b) = -\ell D^{q+1} b.$$

This follows from (14). $\square$

Proof of (A6). Let

$$\Delta_t = \ell D^{qq_0+1} t + D^{qq_0} t - \ell^q(\ell^{q_0} D^{2q_0+1} t - D^{q_0+1} t).$$

By our assumption on b, the terms in Δ_t involving f^{q_0} sum to zero. Moreover, the terms involving $(D^q f)^{q_0}$ also give no contribution to Δ_t. After some simplification, the sum of the remaining terms is ℓ^q times

$$(D^1 f)^{q_0} D^1 b - ((D^1 f)^{q_0} D^1 b)^q - (D^1 f)^{q_0} \ell^{q_0} (D^{q_0+1} b + (D^{q_0+1} b)^q).$$

Note that $(D^1 f)^{q_0} = c^q$, where $f^q - f = c^{3q_0}(x^q - x)$ and $c \in \{x, y, z, w_4\}$. Thus, Δ_t is ℓ^q times

$$c^q D^1 b - (c^q D^1 b)^q - c^q \ell^{q_0} (D^{q_0+1} b + (D^{q_0+1} b)^q).$$

That $\Delta_1 = \Delta_2$ may now be checked in each case by using (14). $\square$

Proof of (A7). Let

$$\Delta_t = \ell^{2q_0} D^{qq_0+2q_0} t - \ell^{q_0} D^{qq_0+q_0} t - \ell^q D^{q_0+1} t - \ell^q D^{q+q_0} t.$$

By our assumption on b, the contribution to Δ_t of those terms involving f^{q_0} is zero. The contribution of the terms involving $(D^1 f)^{q_0}$ is

$$\begin{aligned} &\ell^{2q_0} D^{qq_0+q_0} b + \ell^{q_0} D^{qq_0}(b^q - b) + \ell^q D^q(b^q - b) - \ell^q D^1 b \\ &= \ell^{2q_0} D^{qq_0+q_0} b + \ell^{q_0+1} D^{qq_0+1} b + \ell^{q+1} D^{q+1} b. \end{aligned}$$

This is also zero by (14). Then using (12), we may rewrite the sum of the remaining terms as

$$\begin{aligned} \Delta_t = (\ell^{3q_0} D^{3q_0+1} f)^{q_0} \ell^{q_0} D^{q_0} b \\ - (\ell^{3q_0} D^{3q_0+1} f)^{q_0} (\ell^{2q_0} D^{2q_0} b + \ell^{q_0} D^{q_0} b - \ell D^1 b) \\ + (\ell D^{q+1} f)^{q_0} (\ell^{2q_0} D^{2q_0} b + \ell^{q_0} D^{q_0} b - \ell D^1 b). \end{aligned}$$

As in the proof of (A2), we have

$$\ell^{2q_0} D^{2q_0} b + \ell^{q_0} D^{q_0} b - \ell D^1 b = \ell D^q(b^q).$$

Furthermore, if $c \in \{x, y, z, w_4\}$ with $f^q - f = c^{3q_0}(x^q - x)$, then $(D^{3q_0+1} f)^{q_0} = D^q(c^q)$ and $(D^{q+1} f)^{q_0} = (D^{q_0} c)^q$. Therefore,

$$\Delta_t = \ell^{q_0} D^q(c^q) D^{q_0} b - \ell D^q(c^q) D^q(b^q) + \ell(D^{q_0} c)^q D^q(b^q).$$

Now (14) may be used to verify in each case that $\Delta_{t_1} = \Delta_{t_2}$. □

Proof of (A8). By using (12), the terms on the left hand side of the desired equation may be written as

$$\begin{aligned} \ell^{q_0} D^{qq_0+q_0} t &= f^{q_0} \ell^{q_0} D^{qq_0+q_0} b - (D^1 f)^{q_0} \ell^{q_0} D^{qq_0}(b^q - b) \\ &\quad - (\ell^{3q_0} D^{3q_0+1} f)^{q_0} \ell D^1 b + (D^q f)^{q_0} (\ell^{q_0} D^{q_0} b + \ell D^1 b) \\ \ell^{2q_0} D^{qq_0+2q_0} t &= f^{q_0} \ell^{2q_0} D^{qq_0+2q_0} b + (D^1 f)^{q_0} \ell^{2q_0} D^{qq_0+q_0} b \\ &\quad + (\ell^{3q_0} D^{3q_0+1} f)^{q_0} \ell^{q_0} D^{q_0} b + (D^q f)^{q_0} (\ell^{2q_0} D^{2q_0} b - \ell^{q_0} D^{q_0} b) \\ \ell^{3q_0} D^{qq_0+3q_0} t &= f^{q_0} \ell^{3q_0} D^{qq_0+3q_0} b \\ &\quad + (\ell^{3q_0} D^{3q_0+1} f)^{q_0} \ell^{2q_0} D^{2q_0} b - (D^q f)^{q_0} \ell^{2q_0} D^{2q_0} b. \end{aligned}$$

By our assumption on b, the contribution of the terms involving f^{q_0} is zero. The terms involving $(D^q f)^{q_0}$ sum to zero. The sum of the terms involving $(D^{3q_0+1} f)^{q_0}$ is ℓ^q times the quantity which was dealt with (A2), so by the same argument these give no contribution to $\Delta_{t_1} - \Delta_{t_2}$. It remains only to show that the terms involving $(D^1 f)^{q_0}$ sum to zero, i.e., that

$$\ell^{2q_0} D^{qq_0+q_0} b = \ell^{q_0} D^{qq_0}(b^q - b) = \ell^{q_0+1} D^{qq_0+1} b.$$

But this follows from (14). □

Proof of (A9). We show that the desired equation holds for all t. Use (12) and (13) to write each side of the desired equation as

$$\begin{aligned} \ell^{q_0} D^{qq_0+q_0} t = f^{q_0} \ell^{q_0} D^{qq_0+q_0} b - (D^1 f)^{q_0} \ell^{q_0} D^{qq_0}(b^q - b) \\ - \ell^q (D^{3q_0+1} f)^{q_0} (b^q - b) + (D^q f)^{q_0} (\ell^{q_0} D^{q_0} b + (b^q - b)) \end{aligned}$$

and

$$\begin{aligned}\ell^{q+q_0} D^{qq_0+q+q_0} t = {} & f^{q_0} \ell^{q+q_0} D^{qq_0+q+q_0} b - (D^1 f)^{q_0} \ell^{q+q_0} D^{qq_0+q} (b^q - b) \\ & - \ell^q (D^{3q_0+1} f)^{q_0} (\ell^{2q_0} D^{qq_0+q_0} b \\ & \quad + \ell^{q_0} D^{qq_0} (b^q - b) + \ell^q D^q (b^q - b) - \ell D^1 b) \\ & + (D^q f)^{q_0} \left(\ell^{q+q_0} D^{q+q_0} b + \ell^q D^q (b^q - b) - \ell^{q_0} D^{q_0} b - \ell D^1 b \right).\end{aligned}$$

By our assumption on b, the contribution of the terms involving f^{q_0} is zero. After using (6) to rewrite the coefficients of $(D^1 f)^{q_0}$, $\ell^q (D^{3q_0+1} f)^{q_0}$, and $(D^q f)^{q_0}$ completely in terms of derivatives of b and doing some simplification, it will suffice to show that the equations

$$\begin{aligned}D^{qq_0+1} b + \ell^q D^{qq_0+q+1} b &= 0 \\ \ell^{q+1} D^{q+1} b + \ell^{2q_0} D^{qq_0+q_0} b + \ell^{q_0+1} D^{qq_0+1} b &= 0 \\ \ell^{q_0} D^{q_0+1} b + \ell D^{q+1} b + \ell^{q_0} D^{q+q_0} b &= 0\end{aligned}$$

hold for $b = x, y, z, w_4$. This is easily verified by consulting (14). □

Proof of (A10). We show that the desired equation holds for all t. By using (13) to replace occurrences of $D^{qq_0+3q_0} f$ with $D^q f$, we find that

$$\begin{aligned}\ell^q D^{qq_0+q} t = {} & f^{q_0} \ell^q D^{qq_0+q} b - (D^1 f)^{q_0} \ell^{q_0} D^{qq_0} (b^q) \\ & + (D^{3q_0+1} f)^{q_0} \ell^{q+q_0} D^{qq_0} (b^q - b) \\ & - (D^q f)^{q_0} (\ell^q D^q (b^q - b) - \ell D^1 b) + (D^q f^q)^{q_0} \ell^q D^q (b^q) \\ \ell^{2q} D^{qq_0+2q} t = {} & f^{q_0} \ell^{2q} D^{qq_0+2q} b + (D^{3q_0+1} f)^{q_0} \ell^{2q+q_0} D^{qq_0+q} (b^q - b) \\ & + (D^q f)^{q_0} (\ell^q D^q (b^q - b) - \ell^{2q} D^{2q} (b^q - b)).\end{aligned}$$

By our assumption on b, the contribution of the terms involving f^{q_0} is zero. The terms involving $(D^1 f)^{q_0}$ and $(D^q f^q)^{q_0}$ also give no contribution.

By comparing the coefficients of $(D^{3q_0+1} f)^{q_0}$ and $(D^q f)^{q_0}$ and doing some minor simplification, it will suffice to show that the equations

$$\begin{aligned}\ell^{q+1} D^{qq_0+q} (b^q - b) &= (\ell^q - \ell) D^{qq_0} (b^q - b) \\ \ell D^{2q} (b^q - b) + D^1 b &= D^q (b^q - b)\end{aligned}$$

hold for $b = x, y, z, w_4$. Each of these is easily verified using (6) and (14). □

This completes the proof of Lemma 8, and hence of Theorem 1.

6. Weierstrass points

As a consequence of Theorem 1, we determine the Weierstrass points of $\mathcal{D}$. Recall that the $\mathcal{D}$-Weierstrass points are those $P \in X$ satisfying $j_i(P) \neq \epsilon_i(P)$ for some i. These points make up the support of a divisor $R_{\mathcal{D}}$ with

$$\deg R_{\mathcal{D}} = (2g - 2) \sum \epsilon_i + (13 + 1) m.$$

Since the sequence $\nu_i(\mathcal{D})$ of Frobenius orders differs from $\epsilon_0, \ldots, \epsilon_{13}$, Corollary 2.10 of [15] implies that every rational point of X is a $\mathcal{D}$-Weierstrass point. We claim that in fact $\operatorname{Supp} R_{\mathcal{D}} = X(\mathbb{F}_q)$.

Corollary 9. *The set of Weierstrass points of $\mathcal{D}$ consists of the $\mathbb{F}_q$-rational points of X.*

Proof. By Theorem 1, we have

$$\deg R_{\mathcal{D}} = (2g-2)\sum \epsilon_i + (13+1)m = (3qq_0 + 9q + 23q_0 + 12)N,$$

and so it will suffice to show that $v_P(R) = 3qq_0 + 9q + 23q_0 + 12$ for $P \in X(\mathbb{F}_q)$. This will follow from the inequality

$$v_P(R_{\mathcal{D}}) \geq \sum_{i=0}^{r} (j_i(P) - \epsilon_i). \tag{15}$$

Since the automorphism group acts doubly transitively on the $\mathbb{F}_q$-rational points of X, it will suffice to show this for the point P_0 with $x = y = z = 0$. By expanding out the functions in $\mathcal{B}$ in terms of x, we find that they vanish at P_0 to the orders

$$\begin{aligned}
j_0 &= 0, & j_7 &= 1 + 3q_0 + 2q,\\
j_1 &= 1, & j_8 &= 1 + 2q_0 + q + qq_0,\\
j_2 &= 1 + q_0, & j_9 &= 1 + 3q_0 + q + qq_0,\\
j_3 &= 1 + 2q_0, & j_{10} &= 1 + 3q_0 + 2q + qq_0,\\
j_4 &= 1 + 3q_0, & j_{11} &= 1 + 3q_0 + 2q + 2qq_0,\\
j_5 &= 1 + 2q_0 + q, & j_{12} &= 1 + 3q_0 + 2q + 3qq_0,\\
j_6 &= 1 + 3q_0 + q, & j_{13} &= 1 + 3q_0 + 2q + 3qq_0 + q^2.
\end{aligned}$$

Inserting these values into (15) completes the proof. □

The same argument shows that the $\mathcal{E}$-Weierstrass points are exactly the $\mathbb{F}_q$-rational points as well. In this case, $v_P(R_{\mathcal{E}}) = 3qq_0 + 4q + 12q_0 + 5$ for $P \in X(\mathbb{F}_q)$.

7. Appendix

In the following tables, an asterisk in row i and column f indicates that i is in the set S_f described in section 3. In particular, $D^i f = 0$ wherever there is a blank entry in the table. The first table involves functions of type 1 and the second involves functions of type 2.

i	x	w_1	w_2	w_3	w_6	w_8
0	*	*	*	*	*	*
1	*	*	*	*	*	*
$3q_0$		*	*	*	*	*
$3q_0+1$		*	*	*	*	*
q		*	*	*	*	*
$q+1$			*	*	*	*
$q+3q_0$		*	*	*	*	*
$q+3q_0+1$			*	*	*	*
$2q$			*	*	*	*
$2q+1$				*	*	*
$2q+3q_0$			*	*	*	*
$2q+3q_0+1$				*	*	*
$3q$				*	*	*
$3q+3q_0$				*	*	*
$3qq_0$		*	*	*	*	*
$3qq_0+1$			*	*	*	*
$3qq_0+3q_0$					*	*
$3qq_0+3q_0+1$					*	*
$3qq_0+q$		*	*	*	*	*
$3qq_0+q+1$			*	*	*	*
$3qq_0+q+3q_0$					*	*
$3qq_0+q+3q_0+1$					*	*
$3qq_0+2q$			*	*	*	*
$3qq_0+2q+1$				*	*	*
$3qq_0+2q+3q_0$					*	*
$3qq_0+2q+3q_0+1$					*	*
$3qq_0+3q$				*	*	*
$3qq_0+3q+3q_0$					*	*
$6qq_0$					*	*
$6qq_0+1$					*	*
$6qq_0+q$					*	*
$6qq_0+q+1$					*	*
$6qq_0+2q$					*	*
$6qq_0+2q+1$					*	*
$6qq_0+3q$					*	*
q^2		*	*	*	*	*

i	y	z	w_4	w_7	w_5	w_9	w_{10}
0	*	*	*	*	*	*	*
1	*	*	*	*	*	*	*
q_0	*	*	*	*	*	*	*
q_0+1	*	*	*	*	*	*	*
$2q_0$		*	*	*	*	*	*
$2q_0+1$		*	*	*	*	*	*
$3q_0$					*	*	*
$3q_0+1$					*	*	*
q	*	*	*	*	*	*	*
$q+1$			*	*	*	*	*
$q+q_0$	*	*	*	*	*	*	*
$q+q_0+1$			*	*	*	*	*
$q+2q_0$		*	*	*	*	*	*
$q+2q_0+1$			*	*	*	*	*
$q+3q_0$					*	*	*
$q+3q_0+1$					*	*	*
$2q$			*	*	*	*	*
$2q+1$						*	*
$2q+q_0$			*	*	*	*	*
$2q+2q_0$			*	*	*	*	*
$2q+3q_0$					*	*	*
$2q+3q_0+1$						*	*
$3q$						*	*
$3q+3q_0$						*	*
qq_0	*	*	*	*	*	*	*
qq_0+1			*	*	*	*	*
qq_0+q_0			*	*	*	*	*
qq_0+q_0+1			*	*	*	*	*
qq_0+2q_0				*	*	*	*
qq_0+2q_0+1				*	*	*	*
qq_0+3q_0					*	*	*
qq_0+3q_0+1					*	*	*
qq_0+q	*	*	*	*	*	*	*
qq_0+q+1			*	*	*	*	*
qq_0+q+q_0			*	*	*	*	*
qq_0+q+q_0+1			*	*	*	*	*
qq_0+q+2q_0				*	*	*	*
qq_0+q+2q_0+1				*	*	*	*
qq_0+q+3q_0					*	*	*
qq_0+q+3q_0+1					*	*	*
qq_0+2q			*	*	*	*	*
qq_0+2q+1						*	*
qq_0+2q+q_0			*	*	*	*	*
$qq_0+2q+2q_0$				*	*	*	*
$qq_0+2q+3q_0$					*	*	*
$qq_0+2q+3q_0+1$						*	*
qq_0+3q						*	*
$qq_0+3q+3q_0$						*	*
$2qq_0$		*	*	*	*	*	*
$2qq_0+1$			*	*	*	*	*
$2qq_0+q_0$				*	*	*	*
$2qq_0+q_0+1$				*	*	*	*
$2qq_0+2q_0$					*	*	*
$2qq_0+2q_0+1$					*	*	*
$2qq_0+3q_0$							*
$2qq_0+3q_0+1$							*

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