ROBUST FEEDBACK CONTROL OF A SINGLE SERVER QUEUEING SYSTEM

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ABSTRACT. This paper extends previous work of Ball-Day-Kachroo [2] to control of a model of a simple queueing server. There are n queues of customers to be served by a server who can service only one queue at a time. Each queue is subject to a unknown arrival rate, called a "disturbance" in accord with standard usage from H_∞ theory. An H_∞-type performance criterion is formulated. The resulting control problem has several novel features distinguishing it from the standard smooth case already studied in the control literature: the presence of constraining dynamics on the boundary of the state space to insure the physical property that queue lengths remain nonnegative, and jump discontinuities in any nonconstant state-feedback law caused by the finiteness of the admissible control set (choice of queue to be served). We arrive at the solution to the appropriate Hamilton-Jacobi equation via an analogue of the stable invariant manifold for the associated Hamiltonian flow (as was done by A. J. van der Schaft for the smooth case) and relate this solution to the (lower) value of a restricted differential game, similar to that formulated by Soravia for problems without constraining dynamics. An additional example is included which shows that the projection dynamics used to maintain nonnegativity of the state variables must be handled carefully in the more general models involving interactions among the different queues. Primary motivation comes from the application to traffic signal control. Other application areas, such as manufacturing systems and computer networks, are mentioned.

1. INTRODUCTION

The H_∞ approach to control of linear systems has in recent years found a counterpart for nonlinear systems. This approach, called "robust" or "nonlinear H_∞" control, is expressible entirely in terms of state space constructs. The work of van der Schaft [12] gives a nice exposition of the connections with the theory of dissipative systems as formulated by Willems [13], the role of storage functions, and Hamilton-Jacobi equations as nonlinear analogues of the matrix Riccati equations of linear systems theory. In [2] these ideas were applied to a simple model for signal control of a traffic intersection. In that context, features not appearing in [12] arise due to various discontinuities in the dynamics. These were identified and explored. In the present paper we extend that analysis to higher dimensional versions of the same models. We obtain an explicit state-feedback control which implements the L_2-gain performance criterion formulated in the control problem, but only in an explicitly computable, bounded region Ω_γ of state space. A feature quite different from conventional H_∞-theory is that the control law is independent of the choice

of gain tolerance γ. The choice of γ only influences the extent of the region Ω_γ in which the γ-level L_2-gain performance is guaranteed. Since the control necessarily has some discontinuities as a function of the state, solutions of the closed-loop system must be interpreted as a solutions of a regularized differential inclusion in the sense of Filippov.

We explore other qualitative features and provide mathematical proofs for some of the assertions that were supported in [2] only by a computed example. In particular we obtain a solution of the fundamental Hamilton-Jacobi equation by connecting it with a stable invariant manifold for the associated Hamiltonian flow. This was done in [12] for the smooth case. Here we will have to accommodate discontinuities present in the Hamiltonian vector field. We will explain why the available storage function that we obtain by this construction is also the value of a restricted version of the differential game formulation of Soravia [10]. This provides a sense in which our feedback control is optimal. In the final section we consider an example with some new features that [2] did not include. This example illustrates that the verification theorem of [2] used hypotheses that were overly strong. An improved verification result, Theorem 2.1 below, is developed in its place.

We consider the control system given by (1.6) below. The rest of this section is devoted to explaining the meaning of that equation and its interpretation in the context of some sample specific applications. The control analysis is carried out in Section 3 for the n-dimensional generalization of the model from [2]. Section 4 describes a 2-dimensional example with some more complicated features.

1.1. **The Nominal Model.** We begin by considering an input-state system

$$\dot{x}(t) = q(t) - Gu(t). \tag{1.1}$$

Here $x(t) \in \mathbb{R}^n$ is the state and the term $q(t) \in \mathbb{R}^n$ is a disturbance or input, which we assume belongs to $L^2[0, T]$ for every $0 < T < \infty$. The control $u(t)$ belongs to the finite set of admissible control values

$$U_0 = \{e_i : \ i = 1, \ldots, n\},$$

where e_i, $i = 1, \ldots, n$ are the standard basis vectors in $\mathbb{R}^n$. G is a constant matrix. In Section 3 we consider the diagonal matrix

$$G = \begin{bmatrix} s_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & s_n \end{bmatrix} \tag{1.2}$$

where $s_1, \ldots, s_n$ are positive parameters. In Section 4 a specific nondiagonal G is considered; see (4.1).

The $x_i(t)$ in (1.1) are interpreted as the lengths of queues $i = 1, \ldots, n$. Each value of the control variable $u \in U_0$ corresponds to a choice of queue to be served. The parameter s_i indicates the (constant) rate of service performed by the server when servicing queue i. The disturbance $q_i(t)$ represents the rate of arrivals to (or departures from, if $q_i(t) < 0$) the ith queue.

We must modify the system equations (1.1) to prevent them from producing $x_i(t) < 0$. This is described in Section 1.2. We must also extend the set of admissible control values from the finite set U_0 to its convex hull:

$$u(t) \in \mathcal{U} = \{u = (u_1, \ldots, u_n) : \ u_i \geq 0, \ \sum u_i = 1\}. \tag{1.3}$$

This extension is necessary for the existence of an optimal feedback control $u(t) = u_*(x(t))$, as we will see.

1.2. **Projected Dynamics.** The nominal system equations $\dot{x} = q - Gu$ are not appropriate when one of the queues becomes empty, $x_i(t) = 0$. Either the disturbance or control term in (1.1) could lead to $x_i(t) < 0$, which is untenable physically. Thus we want to enhance the system equations to insure that $x(t)$ remains in the nonnegative orthant

$$K = \{x \in \mathbb{R}^n : \ x_i \geq 0 \text{ for all } i\}.$$

The approach taken in [2] and continued here is to use the notion of projected dynamical systems as developed by Dupuis and Ishii [5] as well as Dupuis and Nagurney [6]. We refer to [5] for the full technical description, and merely summarize the intuitive idea here. For each face of ∂K,

$$\partial_i K = \{x \in \partial K : \ x_i = 0\},$$

we specify a fixed unit vector d_i as the *direction of constraint* for that face. Each such face also has a natural *interior normal vector* $n_i = e_i$. For x belonging to only one face $\partial_i K$, and any velocity vector v, one can define a *projected velocity* $\pi_K(x, v)$ as follows:

Interior Velocities: If v is such that $x + hv \in K$ for all sufficiently small $h > 0$ (i.e. $n_i \cdot v \geq 0$) then $\pi_K(x, v) = v$.

Exterior Velocities: If $n_i \cdot v < 0$, then $\pi_K(x, v) = v + \beta d_i$ where $\beta \geq 0$ is smallest possible so that $n_i \cdot (v + \beta d_i) \geq 0$, namely $\beta = -\frac{n_i \cdot v}{n_i \cdot d_i}$.

The definition of $\pi_K(x, v)$ when x belongs to more than one face is more involved (see [5]). In general $\pi_K(x, v) - v$ will be a member of the convex cone generated by d_i for those faces $\partial_i K$ which contain x. Finally, for x interior to K, $\pi_K(x, v) = v$. With π_K properly defined, one can take a nominal dynamical system

$$(1.4) \qquad\qquad\qquad \dot{x} = b(x, t)$$

and form a *projected dynamical system*

$$(1.5) \qquad\qquad\qquad \dot{x} = \pi_K(x, b(x, t)),$$

which can be viewed as a version of (1.4) which has been modified on ∂K to have solutions $x(t)$ which never leave K.

The system (1.5) is really a cover for a more intricate mathematical mechanism (called a *Skorokhod Problem*) for adding "restorative" terms to (1.4) which keep the solution in K, come into play only when $x \in \partial K$, and act in directions parallel to the appropriate d_i. The analysis of (1.5) in [5] is based on the equivalent expression

$$\dot{y} = b(t, x); \quad x(\cdot) = \Gamma(y(\cdot))$$

where $x(\cdot) = \Gamma(y(\cdot))$ denotes the Skorokhod map on paths. Again we refer to [5] for full technical details. Here we can describe it heuristically as the solution map of $\dot{x} = \pi_K(x, \dot{y})$ with $x(0) = y(0) \in K$. In general there are technical conditions on the constraint directions d_i and their relations to the normals n_i that must be satisfied to insure that $\Gamma(\cdot)$ has adequate properties to develop an existence and uniqueness theory for initial value problems using (1.5). These conditions, which we will call the *Dupuis-Ishii conditions*, are always satisfied for the case of *normal constraint*, $d_i = n_i$, that we use in Section 3. In Section 4 we use an "oblique" d_i, and will need to verify these conditions directly.

The control systems that we consider are of the form

$$(1.6) \qquad \dot{x} = \pi_K(x, q - Gu), \quad u \in \mathcal{U}.$$

For Section 3 we are interested in the diagonal G of (1.2) with normal constraint, $d_i = n_i$. In this case the equations of (1.6) decompose as

$$\dot{x}_i = \delta(x_i, q_i - s_i u_i)$$

where

$$\delta(y, v) = \begin{cases} v & \text{if } y > 0, \text{ or } y = 0 \text{ and } v > 0 \\ 0 & \text{otherwise.} \end{cases}$$

In this way (1.6) is seen to be a reasonable model for controlled queue dynamics.

1.3. **The Control Problem.** Our goal for the system (1.6) is to design a state-feedback control $u = u(x)$ using the criteria of nonlinear H_∞ control theory. This control should make $x = 0$ asymptotically stable for (1.6) for the null disturbance $q(t) \equiv 0$, i.e. it should be capable of emptying all the queues if no new customers arrive. Secondly, for a prescribed tolerance level $\gamma > 0$, (1.6) should satisfy an L_2-gain condition

$$(1.7) \qquad) \int_0^T \|x(t)\|^2 \, dt \leq \gamma^2 \int_0^T \|q(t)\|^2 \, dt + \alpha(x_0)$$

for some nonnegative function $\alpha(x_0)$ with $\alpha(0) = 0$. Section 2 will explain this more carefully. We derive a set-valued state-feedback control $u_*(x)$ in terms of a certain solution $\alpha(x) = V(x)$ solving the relevant Hamilton-Jacobi equation (2.5). While much of the paper is devoted to the derivation and verification of all the requisite properties for $V(x)$, the resulting control is quite simple; see (3.18) for the case where G is given by (1.2), and (4.8) for the case of our specific example of nondiagonal G (4.1). The reader will note that in both cases the control law $x \to u_*(x)$ is in general set-valued; solutions of the closed-loop dynamics must be understood in the sense of differential inclusions, the topic of the next subsection. Also, as mentioned in the Introduction, these control laws are independent of the preassigned attenuation level γ; what does depend on γ is the region Ω_γ in state space over which the desired L_2-gain condition (1.7) is valid (see (3.14) and (4.7).

1.4. **Filippov-Sense Solutions.** The feedback control law that we will obtain is discontinuous. (This is due to using a polyhedral control set $\mathcal{U}$.) Thus the typical nominal system (1.4) that we want to consider will have a discontinuous right-hand side:

$$(1.8) \qquad b(x, t) = q(t) + Gu(x).$$

There are also discontinuities in the Hamiltonian system (2.16) that we will use in the course of constructing our solution of (2.5). We will follow the general approach of Filippov [7] to (ordinary) differential equations with discontinuous right-hand sides. In that approach one interprets an equation $\dot{y} = b(y)$ as an almost sure differential inclusion

$$(1.9) \qquad \dot{y} \in B(y)$$

where $B(y)$ is a set-valued extension of the vector field $b(x)$ with the following properties:

   a) $B(y) \subseteq \mathbb{R}^n$ is a nonempty, compact convex set for each y;

  b) $B(y)$ is (contained in) the closed convex hull of the set of limit points of $b(z)$ as $z \to y$;

  c) $B(\cdot)$ is upper semicontinuous with respect to inclusion (which means that the graph of B is a closed subset of $\mathbb{R}^n \times \mathbb{R}^n$).

Various constructions of $B(\cdot)$ from $b(\cdot)$ are described in the literature, some which are insensitive to changes in $b(\cdot)$ on sets of measure 0. The following definition is adequate for our purposes:

$$(1.10) \qquad\qquad B(y) = \cap_{\epsilon > 0} \overline{\mathrm{conv}} \{ b(z) : \ \|z - y\| \leq \epsilon \}.$$

Assuming $b(\cdot)$ is locally bounded, one may easily check that a), b) and c) are satisfied, and that $B(y) = \{b(y)\}$ at continuity points of $b(\cdot)$. In particular, this is the sense in which we will understand solutions of (2.16), with $y = (x, p)$ and $b(y)$ representing the combined right-hand sides of (2.16).

  We need to be more careful when we consider our control system (1.8) in conjunction with projection dynamics (1.5). If we were to simply apply (1.10) to (1.8), so that the "Filippov regularization" affects both the t and x variables, then a solution of $\dot{x} = b(x, t)$ might actually be a function with

$$\dot{x}(t) = \bar{q}(t) + G\bar{u}(t),$$

where $\bar{q}(t)$ is is obtained from limiting values of $q(s)$ as $s \to t$ but might actually differ from $q(t)$ on a set of positive measure. Thus we could end up actually changing the disturbance to which the control system is responding. In addition, when we apply the uniqueness condition (1.14) to (1.8) we will want the q(t) terms to cancel, so that only differences $u(x_a) - u(x_b)$ are involved. This will not work if we have used a Filippov regularization (1.10) which acts on the t variable. In general for time-dependent problems $b = b(x, t)$ there are existence results for Filipov-sense soltuions of $\dot{x} = b(x, t)$ that only require the Filippov regularization operate on the x variable, i.e. in c) above the semicontinuity need only be with respect to x in $y = (x, t)$, while t is held fixed. (See [7, Theorem 8 pg. 85].) However we will need existence results for (1.8) in conjunction with the projection dynamics (1.5). For this we must rely on the existence argumuent given in [6]. That result, when redeveloped for a time-dependent $b(x, t)$, requires Filippov regularization jointly in $y = (t, x)$, essentialy because it is based on a weak convergence argument in time-state-velocity space. However, for the particular structure of our (1.8) we can avoid the Filippov regularization in t by reformulating our control system before applying the Dupuis-Nagurney argument. We now describe that.

  Suppose we are given a feedback control: $u(x) \in \mathcal{U}$. We form its Filippov regularization $U(x) \subseteq \mathcal{U}$ following (1.10). (Our optimal controls $u_*(x)$ below are already set-valued and satisfy a) – c) above, so $u_*(x) = U(x)$ when we are considering them.) For a given integrable disturbance function $q(t)$ and an initial condition $x_0 \in K$ we are interested in absolutely continuous functions $x(t) \in K$ with $x(0) = x_0$ so that for almost all t we have

$$(1.11) \qquad\qquad \dot{x}(t) \in \pi_K(x, q(t) - GU(x)).$$

Such an $x(\cdot)$ is what we will mean by a solution of the initial value problem for the control system. This is (1.5) with (1.8) in which we have used the Filippov regularization only in x:

$$(1.12) \qquad\qquad B(x, t) = q(t) + GU(x),$$

By the argument given in [5, Theorem 5.1] and [6, Theorem2] we find that (1.11) is equivalent to the existence of an absolutely continuous $y(\cdot)$ so that (almost surely)

$$\dot{y}(t) \in q(t) - GU(x(t)),$$
$$x(\cdot) = \Gamma(y(\cdot)).$$

The existence argument in [6] refers to a problem in this type of format. However we first want to change variables so that the disturbance only occurs in an integrated form. Let $Q(t) = \int_0^t q(s)\,ds$ and define $z(t) = y(t) - Q(t)$. The above is clearly equivalent to

(1.13)
$$\dot{z}(t) \in -GU(x(t)) \text{ (a.s.)},$$
$$x(\cdot) = \Gamma(z(\cdot) + Q(\cdot)).$$

The argument of [6, Theorem 3] can now carried out in the context of (1.13), to extablish the existence of a solution for a given initial $x(0) = x_0 \in K$. This requies some minor adjustments to the argument as given in [6], due to the presence of the extra $Q(\cdot)$ term. The key feature in the weak convergence argument is that the set

$$\{(s, x, \beta) : \ \beta \in Q(s) - GU(x)\}$$

is closed, which follows from the upper-semicontinuity of $U(x)$ and the continuity of $Q(t)$. In this way we find that solutions of (1.11) exist for any (locally) integrable disturbance, and any initial $x_0 \in K$.

A sufficient condition for uniqueness of the initial value problem for (1.9) is described in [7], namely that there exist a constant L so that for all $v_a \in B(y_a)$ and $v_b \in B(y_b)$

(1.14)
$$(y_a - y_b) \cdot (v_a - v_a) \le L\|y_a - y_b\|^2.$$

When we apply this with to (1.12) the $q(t)$ terms cancel, so that it is sufficient to verify

$$(x_a - x_b) \cdot (u_a - u_b) \le L\|x_a - x_b\|^2.$$

for all $u_a \in U(x_a)$ and $u_b \in U(x_b)$. If $U(\cdot)$ is constructed from a single-vlued $u(x)$ by means of (1.10), then the above will follow if we can just check that

(1.15)
$$(x_a - x_b) \cdot (u(x_a) - u(x_b)) \le L\|x_a - x_b\|^2.$$

Uniqueness results are more difficult when projection dynamics are also included, as in (1.11). In the case of (1.5) with a single-valued $b(x, t)$ satisfying the usual Lipshitz condition in x, then solutions are unique by [5, Theorem 5.1]. For differential inclusions, in the special case of normal constraint directions, the Filippov condition (1.14) implies uniqueness (in the increasing time direction), by [6, Theorem 2]. Thus (1.15) is sufficient for uniqueness of (1.11) in Section 3. However in Section 4 we will be using an oblique constraint direction, so we can not assert uniqueness on the basis of these general results.

1.5. **Comments.** There are many obvious shortcomings to (1.6) as a model of traffic flow at a real signalized intersection. It does not incorporate a yellow-light transition period to switch from one signal setting to another, nor does it account for the maximum/minimum green times that are part of many signal light algorithms; see [8]. There is no speed-up or slow-down time for motion through the intersection as the signal settings change. However, we are able to complete a true optimal control analysis, as opposed to the comparative evaluation of some collection of preconceived (i.e. "open loop") control strategies that is common in the traffic

control literature; see [8]. This offers the prospect of being able to develop optimal H_∞ feedback controls in models that incorporate the more complicated patterns of interconnections in urban traffic networks.

The same mathematical model (1.6) is relevant for other applications. Consider for instance, flexible manufacturing systems, in which manufacturing cells and layouts can be configured in different ways to serve different manufacturing needs. An example would be a robotic cell which is expected to perform a variety of tasks for different products. After drilling a hole for one product, the robotic cell may be retooled and reprogrammed to weld a joint on a different product. The various types of objects or parts which need to be worked on form queues waiting for service by the robot. For example the parts to be drilled will form one queue, and the parts to be welded will form another queue. Real-time decision has to be made when to switch from serving one queue to another based on the queue lengths. A more complex manufacturing system will have a network of flexible cells which can serve various queues and the parts might travel from one cell to another for various tasks to be performed on them. This problem fits in the same model formulation we have used in this paper. The retooling time would constitute a switching cost, similar to that associated with changing traffic light setting, so that our model would have that a shortcoming analogous to the absence of a yellow light phase mentioned above.

Problems in communication networks provide additional applications for our model (1.6) in which the shortcomings mentioned above may be less relevant. General refrences are [4, 11]. One such problem is resource sharing among virtual circuts for a common link in an ATM computer network. Before two nodes in an ATM can communicate they are assigned a path of intermediate nodes through the network. This assigned path is called a *virtual circut.* Two or more such virtual circuts may share a common link in the network. That common link must decide which traffic stream to transmit at a given time, giving rise to the same problem as we have described for traffic signal control. Unlike the traffic application however, there is esentially no cost or lost time needed to switch from serving one queue to another; no need for a yellow light phase. There is nothing like the speed-up or slow-down time of traffic flow. Transmission through the link can be consided to be instantaneous so that there is essentially no delay between departure from one intersection and arrival at the next in a communication network. Thus the models we study below fit this aplication very well. In an ATM network different virtual circuts may may also have different priorities for service. This feature could be represented in our models by weighting the different queues; i.e. replacing $\|x(t)\|$ with a different norm in the control problem formulation. Such modifications are not developed here.

2. The H_∞ Control Approach

The general features of our approach to the the control problem for (1.6) are presented in this section. We explain the relation of our analysis to the work of van der Schaft [12] and Soravia [10].

2.1. The Hamilton-Jacobi Equation and Storage Function. Central to our approach is the Hamiltonian function $H^\pi(x, p)$ determined by the system dynamics:

$$(2.1) \qquad H^\pi(x, p) = \sup_q \inf_{u \in \mathcal{U}} K^\pi(x, p; u, q),$$

where

$$K^\pi(x, p; u, q) = p \cdot \pi_K(x, q - Gu) - \frac{1}{2}(\gamma^2 \|q\|^2 - \|x\|^2)$$

The inclusion of projection dynamics π_K in this definition complicates some aspects of the standard theory. Although most of our analysis will deal with the simpler interior Hamiltonian of (2.12) below, we do need the general version (2.1) for Section 4. One complication is the existence of a saddle point u^*, q^* in (2.1). In (2.12) the $\sup_q$ and $\inf_u$ separate, so it will be simple there. For the general case we make the following definitions:

$$(2.2) \qquad U_*(x, p) = \{u \in \mathcal{U} : \ \sup_q K^\pi(x, p; u, q) = H^\pi(x, p)\}$$

$$(2.3) \qquad Q^*(x, p) = \{q \in \mathbb{R}^n : \ \inf_{u \in \mathcal{U}} K^\pi(x, p; u, q) = H^\pi(x, p)\}.$$

Any pair $u^* \in U_*(x, p)$ and $q^* \in Q^*(x, p)$ is a saddle point, because (suppressing the x, p)

$$H^\pi = \inf_{u \in \mathcal{U}} K^\pi(u, q^*) \le K^\pi(u^*, q^*) \le \sup_q K^\pi(u^*, q) = H^\pi.$$

These imply that for all $q \in \mathbb{R}^n$ and $u \in \mathcal{U}$

$$(2.4) \qquad K^\pi(x, p; u^*, q) \le K^\pi(x, p; u^*, q^*) \le K^\pi(x, p; u, q^*)$$
$$\|$$
$$H^\pi(x, p) \qquad\qquad .$$

The basic task is to construct an appropriate solution of the Hamilton-Jacobi equation

$$(2.5) \qquad\qquad H^\pi(x, \nabla V(x)) = 0$$

and then use it to define a (set-valued) feedback control

$$(2.6) \qquad\qquad u_*(x) = U_*(x, \nabla V(x)).$$

Solutions of the resulting feedback control system (2.8) must be understood in the sense of Filippov, as described above. (In both Sections 3 and 4 we will obtain $u_*(\cdot)$ which is upper semi-continuous in x, so that for almost all t we will have $B(x, t) = q(t) + u_*(x)$ in the Filippov definition.) Stability and L_2-gain properties of this control follow from $H^\pi(x, \nabla V(x)) \le 0$ and the left inequality in (2.4). The following theorem summarizes these assertions in our context.

Theorem 2.1. *Let $H^\pi(x, p)$ be the Hamiltonian function defined in (2.1). Suppose Ω is a neighborhood of 0 relative to K on which is defined a C^1 real-valued function $V(x)$ satisfying*

$$(2.7) \qquad\qquad H^\pi(x, \nabla V(x)) \le 0.$$

Suppose the set-valued state-feedback control (2.6) is upper semicontinuous and define the dynamics of the closed-loop system as solutions of the differential inclusion

$$(2.8) \qquad\qquad \dot{x}(t) \in \pi_K(x, q - Gu_*(x)),$$

in the Filippov sense described in Section 1.4. Then the dissipation inequality

$$(2.9) \qquad V(x(T)) - V(x(0)) \le \int_0^T \frac{1}{2}(\gamma^2 \|q(t)\|^2 - \|x(t)\|^2) \ dt$$

is satisfied along all trajectories $(x(t), q(t))$ of the closed loop system (2.8) for which the state vector $x(t)$ remains in the region Ω over the time interval $[0, T]$. If $V(x) \geq 0$ in Ω, then the L_2-gain property

$$(2.10) \qquad \int_0^T \frac{1}{2} \|x(t)\|^2 \, dt \leq \gamma^2 \int_0^T \frac{1}{2} \|q(t)\|^2 \, dt + V(x(0))$$

holds for all such trajectories. If $V(0) = 0 < V(x)$ for all $x \neq 0$, then $x = 0$ is asymptotically stable for (2.8) with $q(t) \equiv 0$.

We emphasize that it is a hypothesis for this theorem that such a $V(x)$ exists. The actual existence is the subject of Section 3.

Proof. Suppose that $V(x)$ is a C^1 solution of the Hamilton-Jacobi inequality (2.7) in the region Ω, and suppose that $(x(t), q(t))$ is a solution of the differential inclusion (2.8). For any $u_* \in U_*(x, \nabla V(x))$ we have

$$0 \geq H^\pi(x, \nabla V(x))$$
$$(2.11) \qquad \geq K^\pi(x, \nabla V(x); u_*, q)$$
$$= \nabla V(x) \cdot \pi_K(q - Gu_*) - \frac{1}{2}(\gamma^2 \|q\|^2 - \|x\|^2),$$

for all q. In particular, if $(x(t), q(t))$ is a solution of the differential inclusion (2.8) and $u^*(t)$ is the element in the set $U_*(x(t), \nabla V(x(t)))$ associated with this solution, then (2.11) implies

$$0 \geq \nabla V(x) \cdot \dot{x}(t) - \frac{1}{2}(\gamma^2 \|q(t)\|^2 - \|x(t)\|^2)$$
$$= \frac{d}{dt} V(x(t)) - \frac{1}{2}(\gamma^2 \|q(t)\|^2 - \|x(t)\|^2),$$

as long as $x(t)$ remains in the region Ω. Integration from 0 to T yields

$$V(x(T)) - V(x(0)) \leq \frac{1}{2} \int_0^T (\gamma^2 \|q(t)\|^2 - \|x(t)\|^2) \, dt,$$

which is the dissipation inequality (2.9). If we rearrange, this becomes

$$V(x(T)) + \int_0^T \frac{1}{2} \|x(t)\|^2 \, dt \leq \gamma^2 \int_0^T \frac{1}{2} \|q(t)\|^2 \, dt + V(x(0)).$$

Under the assumption that $V(x(T)) \geq 0$ we arrive at the L_2-gain property (2.10). Asymptotic stability follows from the standard argument using $V(x)$ as a Lyapunov function. $\qquad \square$

2.2. **The Interior Hamiltonian.** We have mentioned that Theorem 2.1 differs from [2, Theorem 1]. In [2] the boundary projection π_K was not incorporated in the Hamiltonian; the discussion only used the *interior Hamiltonian*,

$$(2.12) \qquad H(x, p) = \sup_q \inf_{u \in \mathcal{U}} K(x, p; u, q)$$

where

$$K(x, p; u, q) = p \cdot (q - Gu) - \frac{1}{2}(\gamma^2 \|q\|^2 - \|x\|^2).$$

Note that the $\sup_q$ and $\inf_{u \in \mathcal{U}}$ in (2.12) can be performed in either order, and that the $\sup_q$ is achieved precisely at $q^* = \frac{1}{\gamma^2}p$, yielding

$$H(x,p) = \inf_{u \in \mathcal{U}} \{\frac{1}{2\gamma^2}\|p\|^2 - p \cdot Gu + \frac{1}{2}\|x\|^2\}.$$

For x in the interior of K, H and H^π agree. They differ for $x \in \partial K$. Instead of (2.7), the hypotheses of [2, Theorem 1] used the interior Hamiltonian and a boundary condition on ∂K:

$$(2.13) \qquad\qquad\qquad H(x, \nabla V(x)) \leq 0, \text{ all } x \in K;$$

$$(2.14) \qquad\qquad\qquad n_i \cdot \nabla V(x) \leq 0, \text{ if } x \in \partial_i K$$

The next lemma shows that these together are sufficient for hypothesis (2.7) above. The lemma accommodates the possibility of oblique as well as normal constraint directions assumed in (2.14). The example of Section 4 below shows, however, that (2.13) and (2.14) are not necessary; (2.14) with d_i in place of n_i can fail even though (2.7) holds.

Lemma 2.2. *Suppose $x \in \partial K$. If*

$$(2.15) \qquad\qquad\qquad p \cdot d \leq 0$$

holds for the constraint direction $d = d_i$ of each face $\partial_i K$ which contains x, then

$$H^\pi(x,p) \leq H(x,p).$$

If we strengthen (2.15) to assume $p \cdot d = 0$ for each such $d = d_i$, then it follows that $H^\pi(x,p) = H(x,p)$.

Proof. If x belongs to just one face $\partial_i K$, then given a velocity v we know that

$$\pi_K(x,v) = v + \beta d$$

for some scalar $\beta \geq 0$ and the constraint direction $d = d_i$ associated with the face. Therefore

$$p \cdot \pi_K(x,v) = p \cdot v + \beta\, p \cdot d \leq p \cdot v.$$

If x is contained in more than one face, then $\pi_K(x,v) = v + \beta d$ for some $\beta \geq 0$ and d a convex combination of the constraint directions d_i of those faces containing x. In that case $p \cdot d \leq 0$ follows and we conclude the same inequality. In either case it follows that

$$K^\pi(x,p;u,q) \leq K(x,p;u,q).$$

Taking $\sup_q \inf_u$ implies

$$H^\pi(x,p) \leq H(x,p) \leq 0.$$

The strengthening to $p \cdot d = 0$ is clear. $\square$

2.3. **Solution by Bicharacteristics.** A. J. van der Schaft [12] has developed a very nice approach to producing solutions of Hamilton-Jacobi equations

$$H(x, \nabla V(x)) = 0$$

arising in nonlinear H_∞ control problems. The basic idea is that, under appropriate assumptions, the Hamiltonian system

$$(2.16) \qquad\qquad \begin{aligned} \dot{x} &= H_p(x,p) \\ \dot{p} &= -H_x(x,p) \end{aligned}$$

will have a hyperbolic equilibrium at $x = 0$, $p = 0$, and its stable invariant manifold
will provide the graph of $p = \nabla V(x)$ for the desired solution, at least in some domain
Ω containing 0. Moreover, for the feedback control resulting as in (2.6), one can
show that $V(x)$ will be the "available storage" function for the closed-loop system
associated with the control $u_*(x)$:

$$(2.17) \qquad\qquad V(x) = \sup_{T, q(\cdot)} \frac{1}{2} \int_0^T \|x(t)\|^2 - \gamma^2 \|q(t)\|^2 \, dt,$$

where $q(\cdot)$ ranges over all disturbance functions, with the restriction that the re-
sponse $x(t)$ produced by the control $u_*(x)$ from initial condition $x(0) = x$ remains
inside Ω for all $t \geq 0$, and T ranges over all $0 \leq T < \infty$.

  Although the special features of our problem (projection dynamics on ∂K and
bounded, polyhedral control set $\mathcal{U}$) place it outside the scope of van der Schaft's
discussion, we will still be able to carry out this general construction process for our
class of problems in the next section. We will work initially with the interior Hamil-
tonian H, constructing a solution to $H(x, \nabla V(x)) = 0$ and then check compatibility
with the true Hamiltonian H^π on ∂K. We will have to accept Filippov solutions of
(2.16), since our H is only piecewise smooth in p. Nonetheless, we will be able to
identify a manifold of solutions to (2.16) (all with $x \in K$) described by a continuous
function $P(x)$, having all the needed properties: $H^\pi(x, P(x)) = H(x, P(x)) = 0$,
$P(x) = \nabla V(x)$ for a function with $V(0) = 0 < V(x)$ if $x \neq 0$, $u_*(x) = U_*(x, \nabla V(x))$
well-defined and satisfying (2.17).

2.4. **Optimality.** Given a tolerance level γ, a feedback control $u = u(x)$ meet-
ing the criteria of th H_∞-control problem described in Section 1.3 is certainly
not unique. We will construct a particular control $u_*(x)$ by the method just de-
scribed. A natural question is whether this particular control is optimal in some
sense, or uniquely determined by some structural control property. In the case of
a closed-loop system in which there is no control variable (or a feedback control
has already been prescribed and implemented) one can only analyze the closed-loop
disturbance-to-error system for stability and L_2-gain $\leq \gamma$. This is an optimization
problem with respect to the disturbance $q(t)$. In [12] van der Schaft showed that the
solution of the L_2-gain Hamilton-Jacobi equation arising from the stable invariant
manifold of the associated Hamiltonian flow is the available storage function (2.17),
which is minimal among all storage functions. This $V(x)$ can be interpreted as the
value function for the associated optimization problem with respect to the distur-
bance. In problems such as ours we are selecting a control strategy $u(\cdot)$ as well as
considering possible disturbances $q(\cdot)$. To provide a sense in which a control $u_*(\cdot)$
can be considered optimal requires a differential game formulation. Soravia [10] has
presented such a game formulation for nonlinear H_∞-control problems, in a general
context. He showed that the lower value of this game is a viscosity solution of the
relevant Hamilton-Jacobi equation, in fact that solution which is minimal among
all viscosity subsolutions. His Hamilton-Jacobi equation is the same as our (2.5).
Thus the obvious question is whether this lower value function can also be identified
with our solution, constructed from the stable invariant manifold of the Hamilton-
ian flow. For the smooth, nonlinear case (and even for the linear-quadratic case),
the literature does not address this question. Nevertheless, we will show that $V(x)$
constructed by our generalization of van der Schaft's ideas is in fact the value of a
restricted version of Soravia's game. The restriction involves limiting the class of

controls considered. To make this precise we offer the following definition. Again, (2.18) is understood in the sense of (1.11).

Definition 2.3. *Suppose $\Omega \subseteq \mathbb{R}^n$ is open relative to K and for each $x \in K$ we have a specified disturbance set $Q(x) \subseteq \mathbb{R}^n$. A feedback control $u : \Omega \to \mathcal{U}$ will be called Q-tolerant in Ω if every solution of the closed loop system*

$$(2.18) \qquad\qquad \dot{x}(t) \in \pi_K(x(t), q(t) - Gu(x(t))$$

with $x(0) \in \Omega$ and $q(t) \in Q(x(t))$ for all $t \in [0, \infty)$ is defined for all $0 \leq t$ and is contained in a compact subset of Ω.

This means that the control $u(\cdot)$ can respond to any disturbance whose values are limited to $Q(x)$ when $x(t) = x$, without driving the system out of Ω, or even letting $x(t)$ approach points of the boundary of Ω relative to K. (In the case of (3.14) below these relative boundary points are just those x for which $\sum x_i/s_i = \gamma$. The $x \in \partial K$ are *not* relative boundary points.) We will be able to prove the following for the cases studied in Sections 3 and 4 below.

Theorem 2.4. *Let G be given either by (1.2) or (4.1). Let $V(x)$ be the nonnegative solution of (2.5) constructed in Sections 3 or refS:4, let Ω_γ be the region given by (3.14) or (4.7), let $u_*(\cdot)$ be the state feedback law given either by (3.18) or (4.8), respectively, and take $Q(x) = \{\frac{1}{\gamma^2} \nabla V(x)\}$.*

a: *The control $u_*(\cdot)$ is Q-tolerant in Ω_γ and for all $x \in \Omega_\gamma$,*

$$V(x) = \sup_{T, q(\cdot)} \frac{1}{2} \int_0^T \|x(t)\|^2 - \gamma^2 \|q(t)\|^2 \, dt,$$

where the supremum is over all $0 \leq T < \infty$, all disturbances $q(\cdot)$ whcih are locally L^2, and all solutions $x(t)$ of (2.8) with $x(0) = x$ such that $x(t) \in \Omega_\gamma$ on $[0, T]$. Moreover $V(x) \geq 0$ for all $x \in \Omega_\gamma$ and solutions of (2.8) satisfy the L_2-gain condition

$$\int_0^T \|x(t)\|^2 \, dt \leq \gamma^2 \int_0^T \|q(t)\|^2 \, dt + V(x(0))$$

as long as $x(t)$ remains in Ω_γ.

b: *For any feedback control $u(\cdot)$ which is Q-tolerant in Ω_γ,*

$$V(x) \leq \sup_{T, q(\cdot)} \frac{1}{2} \int_0^T \|x(t)\|^2 - \gamma^2 \|q(t)\|^2 \, dt,$$

where the supremum is as in part (a) except that now $x(t)$ ranges over solutions of (2.18) with $x(0) = x$.

Theorem 2.4 says that, for all $x \in \Omega_\gamma$,

$$V(x) = \inf_{u(\cdot)} \sup_{T, q(\cdot)} \frac{1}{2} \int_0^T \|x(t)\|^2 - \gamma^2 \|q(t)\|^2 \, dt,$$

where $u(\cdot)$ ranges over all feedback controls which are Q-tolerant in Ω_γ. This should be compared to the differential game (2.6) and (2.7) in Soravia [10]. In his formulation the disturbances considered in the supremum are limited to those to which the control is able to respond without leaving Ω_γ. To prove our theorem above we need to know that the control can respond to a particular disturbance, $q(t) \in Q(x(t))$ specifically, so we have formulated the differential game differently

by considering only Q-tolerant controls. Moreover it is important that the state response to this particular disturbance does not approach the relative boundary of Ω_γ. Hence our game formulation is somewhat different than that of [10].

Much of the proof of Theorem 2.4 depends on the specifics of the constructions in Sections 3 and 4. However we will present the key part of the proof of part (b) here, since it may be of some independent interest. Suppose $u(\cdot)$ is Q-tolerant in Ω with $Q(x) = \{q^*(x)\}$ where $q^*(x)$ is the continuous function $\frac{1}{\gamma^2}\nabla V(x)$. We will see in the following sections that $q^*(x) \in Q^*(x, \nabla V(x))$. The key is to generate a disturbance $q(t)$ and associated solution pair $(q(t), x(t))$ for which $q(t) = q^*(x(t))$. Since $q^*(x)$ is continuous, we can do this by applying the existence result of [6] to the differential inclusion

$$(2.19) \qquad \dot{x}(t) \in \pi_K(x(t), q^*(x(t)) - GU(x(t))),$$

Let $\bar{u}(t) \in U(x(t))$ be the associated control process and define $q(t) = q^*(x(t))$, so that

$$\dot{x}(t) = \pi_K(x(t), q(t) - G\bar{u}(t)) \text{ (a.s.)}.$$

Since $q(t) \in Q^*(x, \nabla V(x))$, we can use the right inequality of (2.4):

$$\begin{aligned}
0 &= H^\pi(x, \nabla V(x)) \\
&\leq K^\pi(x(t), \nabla V(x(t)); \bar{u}(t), q(t)) \\
&= \nabla V(x) \cdot \pi_K(x(t), q(t) - G\bar{u}(t)) - \frac{1}{2}(\gamma^2 \|q(t)\|^2 - \|x(t)\|^2)
\end{aligned}$$

In other words,

$$-\nabla V(x(t)) \cdot \dot{x}(t) \leq \frac{1}{2}(\|x(t)\|^2 - \gamma^2\|q(t)\|^2).$$

Integrating both sides yields, for all $0 < T < \infty$,

$$(2.20) \qquad V(x(0)) - V(x(T)) \leq \frac{1}{2}\int_0^T \|x(t)\|^2 - \gamma^2\|q(t)\|^2 \, dt.$$

Now if 0 is a limit point of $x(t)$ then using a sequence of T_n with $x(T_n) \to 0$ it follows from (2.20) and $V(0) = 0$ that

$$V(x(0)) \leq \sup_T \frac{1}{2}\int_0^T \|x(t)\|^2 - \gamma^2\|q(t)\|^2 \, dt,$$

which implies part (b) of the theorem. Suppose that 0 is not a limit point of $x(t)$. By hypothesis $x(t)$ has no limit points on the relative boundary of Ω either. Therefore there is a compact subset $M \subseteq \Omega \setminus \{0\}$ which contains $x(t)$ for all $t > 0$. We will show in our construction of $V(x)$ below that

$$\|x\|^2 - \gamma^{-2}\|\nabla V(x)\|^2 > 0$$

holds for every $x \in \Omega$ except $x = 0$; see (3.23). In particular there is a positive lower bound on M:

$$\|x\|^2 - \gamma^{-2}\|\nabla V(x)\|^2 > \delta > 0, \quad x \in M.$$

Therefore

$$\begin{aligned}
\frac{1}{2}\int_0^T \|x(t)\|^2 - \gamma^2\|q(t)\|^2 \, dt &= \frac{1}{2}\int_0^T \|x(t)\|^2 - \gamma^{-2}\|\nabla V(x(t))\|^2 \, dt \\
&\geq \delta T/2.
\end{aligned}$$

It follows that

$$\infty = \sup_T \frac{1}{2} \int_0^T \|x(t)\|^2 - \gamma^2 \|q(t)\|^2 \, dt,$$

which again implies part b of the Theorem.

3. The Construction

Assume now that G is the diagonal matrix (1.2) and that the projection dynamics are based on normal constraint directions: $d_i = n_i$. We will carry out a construction generalizing that of [2] to produce a solution $V(\cdot)$ to (2.5) satisfying all the hypotheses of Theorem 2.1. We will verify all the assertions of Theorem 2.4, showing that the feedback control in (3.18) below is optimal.

There are two types of difficulties that we must deal with in solving (2.5). One is the complications due to the projection dynamics. We will be able to deal with this after the fact, using Lemma 2.2. Thus we will work with the interior Hamiltonian (2.12) for most of the construction.

3.1. Interfaces and Bicharacteristics.
The second difficulty, which we must deal with head on, is the fact that $H(x, p)$ is only piecewise smooth in p. This is because the minimizing u in (2.12) will jump between extreme points of $\mathcal{U}$ as p varies. The set of p for which $u = e_i$ is a minimizing control value defines a convex cone

$$P_i = \{p : \ p \cdot Ge_i \geq p \cdot Gu \text{ all } u \in \mathcal{U}\}$$
$$= \{p = (p_1, \ldots, p_n) : \ s_i p_i = \max_j s_j p_j\}.$$

Let $N = \{1, 2, \ldots, n\}$ be the set of coordinate indices. For any subset $I \subseteq N$ of indices define

$$P_I = \cap_{i \in I} P_i.$$

When $p \in P_I$ we can write

$$H(x, p) = \frac{1}{2\gamma^2} \|p\|^2 - p \cdot Gu + \frac{1}{2} \|x\|^2$$

using any $u \in \mathcal{U}$ which can be written as a convex combination of e_i, $i \in I$. Define a specific such u by

$$u_I = c_I \sum_{i \in I} \frac{1}{s_i^2} e_i, \quad \text{where } c_I \sum_{i \in I} \frac{1}{s_i^2} = 1.$$

For each of these control values u_I define the associated vector

$$\eta_I = Gu_I = c_I \sum_{i \in I} \frac{1}{s_i} e_i.$$

Thus for $p \in P_I$ we have

$$H(x, p) = H_I(x, p)$$

where H_I is defined by

(3.1) $$H_I(x, p) \doteq \frac{1}{2\gamma^2} \|p\|^2 - p \cdot \eta_I + \frac{1}{2} \|x\|^2.$$

Notice that H_I is smooth everywhere, although it only agrees with H for $p \in P_I$.

Notice also that $\eta_I \cdot \eta_J = 0$ if I and J are disjoint, while

(3.2) $$\eta_I \cdot \eta_J = c_J, \quad \text{for } I \subseteq J.$$

In particular $c_I = \|\eta_I\|^2$. When $I = \{i\}$ is a singleton then $u_I = e_i$. For any $I \subseteq N$ observe that $\eta_J \in P_I$ for all $I \subseteq J$. To see this note that

$$\eta_J \cdot Ge_i = \eta_J \cdot \eta_{\{i\}} = \begin{cases} c_J & \text{if } i \in J \\ 0 & \text{if } i \notin J. \end{cases}$$

Define the duplicate cones in state space:

$$X_i = \{x = (x_1, \ldots, x_n) : \ s_i x_i = \max_j s_j x_j\}; \quad X_I = \cap_{i \in I} X_i.$$

The fact that $\eta_I = Gu_I$ belongs to P_I is significant because it means that the *interface* $X_I \times P_I$ is a (locally) invariant set for the H_I Hamiltonian system:

(3.3)
$$\dot{x} = \frac{1}{\gamma^2} p - \eta_I$$
$$\dot{p} = -x.$$

This feature plays a critical role in our construction below. We point out that this is a Filippov-sense solution of (2.16), since $p \in P_I$ is a limit point of p in the interiors of the P_i for $i \in I$, and $\eta_I = Gu_I$ where u_I is a convex combination of the corresponding u_i.

3.1.1. *Staged Bicharacteristics.* The heart of our construction is to identify a particular family of bicharacteristics (i.e. solutions $x(t), p(t)$ of (2.16) in the sense of Filippov) that will play the role of the stable invariant manifold in van der Schaft's development. For us this will be the family of staged bicharacteristics described below. For a smooth system (2.16) the solutions on the stable manifold converge to $(0, 0)$ as $t \to \infty$. In contrast, our staged bicharacteristics reach $(0, 0)$ at a finite time. We will find a continuous function $P(x)$, defined in the domain (3.14), which gives the state-costate relationship for our staged bicharacteristics: $p(t) = P(x(t))$. Thus the graph of $P(x)$ will be our analogue of the stable manifold. We will see that $P(x) = \nabla V(x)$ for the desired solution of (2.5).

Here then is the description of a *staged bicharacteristic* $x(t), p(t)$ with an initial state $x = x(0)$. Order the coordinates x_i of x according to

(3.4)
$$s_{i_1} x_{i_1} \geq s_{i_2} x_{i_2} \geq \cdots \geq s_{i_n} x_{i_n} \geq 0.$$

and define

$$I_1 = \{i_1\}, \quad I_2 = \{i_1, i_2\}, \ldots, \quad I_n = \{i_1, \ldots, i_n\} = N.$$

In particular the initial point $x(0)$ belongs to X_{I_1}. We require a sequence of times $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n = T \leq \frac{\pi}{2}\gamma$ so that $x(t)$ maintains (3.4) on $[0, T]$ as well as the analogous inequalities among the $p_i(t)$:

(3.5)
$$s_{i_1} p_{i_1} \geq s_{i_2} p_{i_2} \geq \cdots \geq s_{i_n} p_{i_n} \geq 0.$$

The evolution of $x(t), p(t)$ is in a sequence of n stages between the t_i is as follows:

1: on $[0, t_1]$ $(x(t), p(t)) \in X_{I_1} \times P_{I_1}$ and is a bicharacteristic of the H_{I_1} system, reaching the interface $X_{I_2} \times P_{I_2}$ at $t = t_1$;

2: on $[t_1, t_2]$ $(x(t), p(t)) \in X_{I_2} \times P_{I_2}$ and is a bicharacteristic of the H_{I_2} system, reaching the interface $X_{I_3} \times P_{I_3}$ at $t = t_2$;

$\vdots$

k: on $[t_{k-1}, t_k]$ $(x(t), p(t)) \in X_{I_k} \times P_{I_k}$ and is a bicharacteristic of the H_{I_k} system, reaching the interface $X_{I_{k+1}} \times P_{I_{k+1}}$ at $t = t_k$;

$$\vdots$$

n: on $[t_{n-1}, t_n]$ $(x(t), p(t)) \in X_{I_n} \times P_{I_n}$ and is a bicharacteristic of the H_{I_n} system, reaching $x(T) = p(T) = 0$ at $t = t_n$.

As pointed out after (3.3) such a $x(t), p(t)$ pair is indeed a Filippov solution to (2.16). Our goal is to find the function $P(x)$ that uniquely identifies the initial costate $p(0) = P(x)$ of the staged bicharacteristic for a given $x = x(0)$.

3.1.2. *Coefficient Expressions.* There are $n!$ possible orderings in (3.4), one for each permutation σ of $N = \{1, 2, \ldots, n\}$. To each corresponds a subregion of K in which (3.4) holds with the ordering determined by σ: $i_j = \sigma(j)$. We will describe the construction of $P(x)$ for the simplest ordering:

$$(3.6) \qquad s_1 x_1 \geq s_2 x_2 \geq \cdots \geq s_n x_n \geq 0.$$

In this case the interfaces are just $I_k = \{1, \ldots, k\}$. In the general case of (3.4) I_k would be replaced by $\{i_1, \ldots, i_k\}$ in what follows. The structure of the staged bicharacteristics for (3.6) works out nicely in terms of the basis vectors

$$\eta_{I_1} = c_{I_1} (\frac{1}{s_1}, 0, \ldots, 0)^T$$

$$\eta_{I_2} = c_{I_2} (\frac{1}{s_1}, \frac{1}{s_2}, 0, \ldots, 0)^T$$

$$\vdots$$

$$\eta_{I_n} = c_{I_n} (\frac{1}{s_1}, \ldots, \frac{1}{s_n})^T$$

Specifically we will write

$$(3.7) \qquad x(t) = \gamma \sum_1^n a_i(t) \eta_{I_i}, \qquad p(t) = \gamma^2 \sum_1^n \alpha_i(t) \eta_{I_i}.$$

In these coordinates (3.6) is equivalent to

$$a_i \geq 0 \text{ all } i,$$

and $x \in X_{I_k}$ if and only if

$$a_1 = a_2 = \ldots a_k = 0.$$

Analogous statements hold for the α_i, p_i and P_I. The H_{I_k} system equations become

$$(3.8) \qquad \begin{cases} \gamma \dfrac{d}{dt} a_k = \alpha_k - 1 \\ \gamma \dfrac{d}{dt} \alpha_k = -a_k \end{cases}, \quad \text{and for } i \neq k: \quad \begin{cases} \gamma \dfrac{d}{dt} a_i = \alpha_i \\ \gamma \dfrac{d}{dt} \alpha_i = -a_i \end{cases}.$$

We can describe the coefficient equations (3.8) conveniently in terms of the rotation matrix

$$R(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$

The solution of the system

$$(3.9) \qquad \begin{aligned} \gamma \frac{d}{dt} a &= \alpha \\ \gamma \frac{d}{dt} \alpha &= -a \end{aligned}$$

is given in general by clockwise rotation about the origin:

$$\begin{bmatrix} a(t) \\ \alpha(t) \end{bmatrix} = R(t/\gamma) \begin{bmatrix} a(0) \\ \alpha(0) \end{bmatrix}.$$

The system

$$
\begin{aligned}
\gamma \frac{d}{dt} a &= \alpha - 1 \\
\gamma \frac{d}{dt} \alpha &= -a
\end{aligned}
$$

(3.10)

constitutes a clockwise rotation about the point $(0,1)^T$; its general solution is given by the affine operator

$$\begin{bmatrix} a(t) \\ \alpha(t) \end{bmatrix} = T(t/\gamma) \begin{bmatrix} a(0) \\ \alpha(0) \end{bmatrix} \doteq \begin{bmatrix} 0 \\ 1 \end{bmatrix} + R(t/\gamma) \left(\begin{bmatrix} a(0) \\ \alpha(0) \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right).$$

We will summarize (3.8) by saying that

$$(a_k(t), \alpha_k(t))^T \text{ follows } \frac{d}{dt} T(t/\gamma),$$

and that for $i \neq k$

$$(a_i(t), \alpha_i(t))^T \text{ follows } \frac{d}{dt} R(t/\gamma).$$

In these coordinates we can describe the desired staged bicharacteristics as follows: all $a_i(t) \geq 0$, $\alpha_i(t) \geq 0$ on $[0, T]$, $T \leq \frac{\pi}{2}\gamma$, and

1: $(a_1(t), \alpha_1(t))^T$ follows $\frac{d}{dt} T(t/\gamma)$ on $[0, t_1]$, reaching $(0,0)^T$ for the first time at $t = t_1$; $a_1(t) \equiv \alpha_1(t) \equiv 0$ for $t_1 \leq t \leq T$;

2: $(a_2(t), \alpha_2(t))^T$ follows $\frac{d}{dt} R(t/\gamma)$ on $[0, t_1]$ and then $\frac{d}{dt} T(t/\gamma)$ on $[t_1, t_2]$ reaching $(0,0)^T$ for the first time at $t = t_2$; $a_2(t) \equiv \alpha_2(t) \equiv 0$ for $t_2 \leq t \leq T$;

$\vdots$

k: $(a_k(t), \alpha_k(t))^T$ follows $\frac{d}{dt} R(t/\gamma)$ on $[0, t_{k-1}]$ and then $\frac{d}{dt} T(t/\gamma)$ on $[t_{k-1}, t_k]$ reaching $(0,0)^T$ for the first time at $t = t_k$; $a_k(t) \equiv \alpha_k(t) \equiv 0$ for $t_k \leq t \leq T$;

$\vdots$

n: $(a_n(t), \alpha_n(t))^T$ follows $\frac{d}{dt} R(t/\gamma)$ on $[0, t_{n-1}]$ and then $\frac{d}{dt} T(t/\gamma)$ on $[t_{n-1}, t_n]$ reaching $(0,0)^T$ for the first time at $t = t_n = T$.

3.1.3. *Solution Formulas.* Our task in constructing $P(x)$ is to determine how, for a given set of $a_i(0) \geq 0$, values of $\alpha_i(0) \geq 0$ are uniquely determined by these conditions. The key is the following basic lemma. Note for this purpose that $R(\theta)$ and $T(\theta)$ constitute matrix groups on $\mathbb{R}^2$:

$$R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2); \quad R(0) = \text{ identity}.$$
$$T(\theta_1)T(\theta_2) = T(\theta_1 + \theta_2); \quad T(0) = \text{ identity}.$$

Lemma 3.1. *Suppose $\theta \in [0, \pi/2]$ and $a \geq 0$ are given. The equation*

(3.11)
$$T(\Delta\theta)R(\theta) \begin{bmatrix} a \\ \alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

can be solved for $\Delta\theta$, α if and only if $\sin(\theta) + a \leq 1$ and has a unique solution with $\theta + \Delta\theta \in [0, \pi/2]$, described by

(3.12) $\quad \sin(\theta + \Delta\theta) = \sin(\theta) + a, \quad 1 - \cos(\theta + \Delta\theta) = 1 - \cos(\theta) + \alpha.$

If $\sin(\theta) + a < 1$ *there is a second solution, but with* $\theta + \Delta\theta > \pi/2$.

Proof. First note that the set of $(b, \beta)^T$ such that there exists λ with

$$T(\lambda) \begin{bmatrix} b \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is the circle

$$\left\| \begin{bmatrix} b \\ \beta \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\|^2 = 1.$$

Since $R(\theta)(a, \alpha)^T$ as α varies is a line, there can be at most two values of α for which a $\Delta\theta$ exists solving (3.11). In fact as an equation for α,

$$\left\| R(\theta) \begin{bmatrix} a \\ \alpha \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\|^2 = 0$$

is quadratic with discriminant $1 - (a + \sin(\theta))^2$. Thus we see there is no solution if $a + \sin(\theta) > 1$ and two distinct solutions if $a + \sin(\theta) < 1$. We will exhibit both solutions.

The first solution is the one described by (3.12) and the constraint $0 \le \theta + \Delta\theta \le \pi/2$. Notice that $a \ge 0$ implies $\Delta\theta \ge 0$. To verify that (3.12) is a solution, define

$$M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and note the identity

$$MR(-\theta) = R(\theta)M.$$

Now notice that (3.12) is equivalent to

$$\begin{bmatrix} a \\ \alpha \end{bmatrix} = M(R(\theta + \Delta\theta) - R(\theta)) \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore, using the M-identity above and the group property,

$$R(\theta) \begin{bmatrix} a \\ \alpha \end{bmatrix} = MR(-\theta)(R(\theta + \Delta\theta) - R(\theta)) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= MR(\Delta\theta) \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and so

$$T(\Delta\theta)R(\theta) \begin{bmatrix} a \\ \alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + R(\Delta\theta)MR(\Delta\theta) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + M \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

By hypothesis $0 \le \sin(\theta) + a$. If $\sin(\theta) + a < 1$ we can identify the second solution by finding $\theta + \bar{\Delta}\theta \in [\frac{\pi}{2}, \pi]$ so that

$$\sin(\theta + \bar{\Delta}\theta) = \sin(\theta) + a$$

and then $\bar{\alpha}$ from

$$1 - \cos(\theta + \bar{\Delta}\theta) = 1 - \cos(\theta) + \bar{\alpha}.$$

Since (3.12) is again satisfied, with $\bar{\Delta}\theta, \bar{\alpha}$ in place of $\Delta\theta, \alpha$, it follows by the same argument as above that $\bar{\Delta}\theta, \bar{\alpha}$ is the other solution of (3.11), distinct from that of (3.12) because

$$\cos(\theta) - \alpha = \cos(\theta + \Delta\theta) > \cos(\theta + \bar{\Delta}\theta) = \cos(\theta) - \bar{\alpha}$$

□

We now apply the lemma iteratively to find the unique initial costate coordinates $\alpha_i(0)$ for a staged bicharacteristic. For the first stage, $[0, t_1]$, apply the lemma with $a = a_1(0)$ and $\theta = 0$ to determine $t_1/\gamma = 0 + \Delta\theta$ and $\alpha_1(0)$ from

$$\sin(t_1/\gamma) = a_1(0), \quad 1 - \cos(t_1/\gamma) = \alpha_1(0).$$

For the second stage, $[t_1, t_2]$, apply the lemma with $\theta = t_1/\gamma$ and $a = a_2(0)$ to determine $t_2/\gamma = t_1/\gamma + \Delta\theta$ and $\alpha_2(0)$ from

$$\sin(t_2/\gamma) = a_1(0) + a_2(0), \quad 1 - \cos(t_2/\gamma) = \alpha_1(0) + \alpha_2(0).$$

Continue this through the successive stages. For stage k, $[t_{k-1}, t_k]$, apply the lemma with $\theta = t_{k-1}/\gamma$ and $a = a_k(0)$ to determine $t_k/\gamma = t_{k-1}/\gamma + \Delta\theta$ and $\alpha_k(0)$ from

(3.13) $$\sin(t_k/\gamma) = a_1(0) + \cdots + a_k(0), \quad 1 - \cos(t_k/\gamma) = \alpha_1(0) + \cdots + \alpha_k(0).$$

For the final stage, $[t_{n-1}, t_n]$ we find

$$\sin(t_n/\gamma) = a_1(0) + \cdots + a_n(0), \quad 1 - \cos(t_n/\gamma) = \alpha_1(0) + \cdots + \alpha_n(0).$$

The requirement that $0 \le t_i/\gamma \le T/\gamma \le \frac{\pi}{2}$ gives the uniqueness at each stage. According to the lemma this is possible at all stages if and only if $\sum_1^n a_i(0) \le 1$. Notice however that, using (3.2),

$$\frac{1}{\gamma}\sum_1^n x_i(0)/s_i = \frac{1}{\gamma c_N}x(0) \cdot \eta_N = \frac{1}{\gamma}\sum_1^n a_i(0)\frac{\gamma}{c_N}\eta_{I_i} \cdot \eta_N = \sum_1^n a_i(0).$$

Hence the solvability condition $\sum_1^n a_i(0) \le 1$ is equivalent to $\sum_1^n x_i(0)/s_i \le \gamma$, or $x(0) \cdot \eta_N \le \gamma c_N$. We define Ω_γ to be the interior of this set relative to K:

(3.14) $$\Omega_\gamma = \{x \in K : \sum_{i=1}^n x_i/s_i < \gamma\}$$

Thus starting with $x = x(0) \in \Omega_\gamma$ satisfying (3.6) the $a_i(0)$ are uniquely determined by the first formula of (3.7). Then (3.13), $0 \le t_1 \le \cdots \le t_n \le \frac{\pi}{2}\gamma$ and the second formula of (3.7) determine the unique $p(0)$ for a staged bicharacteristic associated with the ordering (3.6). This is $P(x)$ if x satisfies (3.6). One may check using (3.8) that the evolution of this staged bicharacteristic is described by

(3.15)
$$\sin(\gamma^{-1}\max(0, t_k - t)) = a_1(t) + \cdots + a_k(t)$$
$$1 - \cos(\gamma^{-1}\max(0, t_k - t)) = \alpha_1(t) + \cdots + \alpha_k(t),$$

holding for all $k = 1, \ldots n$. This makes it clear that $p(t) = P(x(t))$ for all $0 \le t \le T$. In particular we see that all $a_i(t), \alpha_i(t) \ge 0$, confirming that (3.6) and the analogous ordering for the components of p are maintained for all $0 \le t \le T$. Observe that $x_n = 0$ iff $a_n = 0$ iff $t_n = t_{n-1}$ iff $\alpha_n = 0$ iff $p_n = 0$. When generalized to all possible orderings (3.4) this says

(3.16) $$x_i = 0 \text{ iff } P_i(x) = 0,$$

which will be significant when we apply Lemma 2.2 below. In a similar fashion notice the following chain of equivalences:

(3.17) $$s_k x_k = s_{k+1}x_{k+1} \text{ iff } a_k = 0 \text{ iff } \alpha_k = 0 \text{ iff } s_k p_k = s_{k+1}p_{k+1}.$$

If this holds at $t = 0$ then it is maintained for all $0 \le t \le T$. This observation explains why $P(x)$ is continuous, as we now discuss.

It is clear that (3.13) determines a continuous function $P(x)$ for those $x \in \Omega_\gamma$ satisfying (3.4) for a specific ordering σ. But suppose x satisfies (3.4) for two different orderings, σ and τ. It is conceivable that the value $P(x)$ might depend on whether σ or τ is used. Note however that the rearrangement $\sigma \circ \tau^{-1}$ of (3.4) from the τ-ordering to the σ-ordering can only rearrange indices within sequences of equalities: $s_{\tau(i)} x_{\tau(i)} = s_{\tau(i+1)} x_{\tau(i+1)} \cdots = s_{\tau(i+m)} x_{\tau(i+m)}$. We have just explained that these equalities are maintained for all $0 \le t \le T$, for both the τ-bicharacteristic and the σ-bicharacteristic. This means that the rearrangement $\sigma \circ \tau^{-1}$ can be applied to the τ-bicharacteristic for all t, showing that the σ-ordering is also satisfied for all t. At those $t \ge t_k$ for which

$$s_{\tau(1)} x_{\tau(1)}(t) = \cdots = s_{\tau(k)} x_{\tau(k)}(t) \text{ (and likewise for } p(t)\text{)},$$

the same will be true in the $\sigma \circ \tau^{-1}$ rearrangement. This all says that the τ-staged bicharacteristic is also a σ-staged bicharacteristic, and therefore agrees with the staged bicharacteristic constructed assuming σ. By uniqueness for a given ordering, we conclude that $P(x)$ must be the same regardless of which of the possible orderings in (3.4) we use in the construction (3.13). As a consequence, $P(x)$ is continuous in all of Ω_γ.

3.2. The Hamilton-Jacobi Equation. The above provides a continuous $P(x)$, $x \in \Omega_\gamma$ describing the staged bicharacteristics, which we view as the analogue of the stable manifold for (2.16). Now we will explain why

$$H(x, P(x)) = H^\pi(x, P(x)) = 0$$

and why $P(x)$ is the gradient $P(x) = \nabla V(x)$ of a solution to the Hamilton-Jacobi equation (2.5).

Consider any $x \in \Omega_\gamma$ and the unique staged bicharacteristic $x(t), p(t) = P(x(t))$ with $x = x(0)$. Since on each $[t_{k-1}, t_k]$ we know $p(t) \in P_{I_k}$, it follows that $H(x(t), p(t)) = H_{I_k}(x(t), p(t))$ there. Since the bicharacteristic follows the H_{I_k} system (3.3) in this time interval, $H_{I_k}(x(t), p(t))$ is constant over $[t_{k-1}, t_k]$. Therefore, since $x(T) = p(T) = 0$, we conclude that

$$H(x, P(x)) = H(x(t), p(t)), \text{ for } 0 \le t \le T$$
$$= H(0, 0) = 0.$$

This shows that $H(x, P(x)) = 0$ for all $x \in \Omega_\gamma$.

Next consider x in a face $\partial_i K$. We observed in (3.16) that $P_i(x) = 0$. Because we are using the normal constraint, $d_i = n_i$, we find that $d_i \cdot P(x) = 0$ for all $x \in \partial_i K$. Lemma 2.2 now tells us that

$$H^\pi(x, P(x)) = H(x, P(x)) = 0 \quad \text{for all } x \in \Omega_\gamma.$$

In fact from the argument of the lemma we see that for all $q \in \mathbb{R}^n$ and $u \in \mathcal{U}$ we have

$$K^\pi(x, P(x); u, q) = K(x, P(x); u, q).$$

From this it follows that

$$Q^*(x, P(x)) = \{q^*(x)\}, \text{ where } q^*(x) = \gamma^{-2} P(x).$$

We also see that

$$U_*(x, P(x)) = \text{conv}\{e_i : \ i \in I(x)\},$$

where

$$I(x) = \{i : s_i p_i(x) = \max_j s_j p_j(x)\},$$

i.e., $I(x)$ is the maximal $I \subseteq N$ with $P(x) \in P_I$. Since (3.17) tells us that $P(x) \in P_I$ if and only if $x \in X_I$, we see that $I(x)$ is also the maximal $I \subseteq N$ with $x \in X_I$. Thus the feedback control associated with $P(x)$ is

$$(3.18) \qquad u_*(x) = U_*(x, P(x)) = \text{conv}\{e_i : s_i x_i = \max_j s_j x_j\}.$$

That this is compact, convex and upper semicontinuous in x is clear.

3.2.1. *Exactness.* We define $V(x)$ for $x \in \Omega_\gamma$ by integration of $p(t) \cdot \dot{x}(t)$ along the staged bicharacteristic with $x(0) = x$, using $V(0) = 0$:

$$(3.19) \qquad 0 - V(x(0)) = \int_0^T p(t) \cdot \dot{x}(t) \, dt.$$

This defines a continuous function $V(x)$, $x \in \Omega_\gamma$. We need to establish that $\nabla V(x) = P(x)$ throughout Ω_γ.

This issue is fundamental for constructing solutions to a first order equation $H(x, \nabla V(x)) = 0$ by the method of characteristics. The classical treatment of this (see for instance Hartman [9], (9.7) pg. 138) goes as follows. We have a family of bicharacteristics ("characteristic strips") with initial values on a manifold Γ_0, parameterized smoothly by $\gamma \in \mathbb{R}^d$ ($d < n$): $x(0; \gamma) \in \Gamma_0$, with costate values $p(0; \gamma)$ and initial data $u(0; \gamma) = w(x(0; \gamma))$, satisfying the compatibility condition

$$(3.20) \qquad \frac{\partial}{\partial \gamma_i} u(0; \gamma) = p(0; \gamma) \cdot \frac{\partial}{\partial \gamma_i} x(0; \gamma).$$

In other words, $dw = p \cdot dx$ on Γ_0. These initial values determine the bicharacteristic family

$$\frac{\partial}{\partial t} x(t; \gamma) = H_p(x(t; \gamma), p(t; \gamma))$$

$$\frac{\partial}{\partial t} p(t; \gamma) = -H_x(x(t; \gamma), p(t; \gamma))$$

$$(3.21) \qquad \frac{\partial}{\partial t} u(t; \gamma) = p \cdot H_p(x(t; \gamma), p(t; \gamma))$$

Now a calculation shows that

$$\lambda_i(t) = \frac{\partial}{\partial \gamma_i} u(t; \gamma) - p(t; \gamma) \frac{\partial}{\partial \gamma_i} x(t; \gamma)$$

satisfies $\frac{d}{dt} \lambda_i(t) = 0$, so from (3.20) we conclude that $\lambda_i(t) \equiv 0$. This and (3.21) mean that $u(t; \gamma) = w(x(t; \gamma))$ and $dw = p \cdot dx$ for a function $w(x)$ defined on the larger submanifold covered by $x(t; \gamma), p(t; \gamma)$. In other words if Γ is a manifold simply covered by $x(t; \gamma)$ (t, γ ranging over some domain with $x(t; \gamma)$ having nonvanishing Jacobian) then $dw = p \cdot dx$ in Γ. (In fact that $x(t; \gamma)$ simply covers some manifold is superfluous; see Arnold [1, §44] for a nice exposition of this basic property of Hamiltonian systems in terms of differential forms.)

We need to apply this property successively to our interface Hamiltonians H_{I_k}. Consider any of the possible orderings $\sigma(j) = i_j$ and the interior of the subregion associated with this ordering:

$$\Gamma_1 = \{x \in \Omega_\gamma : s_{i_1} x_{i_1} > s_{i_2} x_{i_2} > \cdots > s_{i_n} x_{i_n} > 0\} \subseteq X_{I_1}.$$

This region is simply covered by the H_{i_1} bicharacteristics with initial (actually terminal) conditions in

$$\Gamma_2 = \{x \in \Omega_\gamma : \ s_{i_1} x_{i_1} = s_{i_2} x_{i_2} > \cdots > s_{i_n} x_{i_n} > 0\} \subseteq X_{I_2}.$$

So $dV = p \cdot dx$ will hold in Γ_1 if it does in the $n - 1$ dimensional manifold Γ_2. Proceeding through the successive stages, the $n - k + 1$ dimensional submanifold Γ_{k-1} is covered by $H_{I_{k-1}}$ bicharacteristics with initial conditions in

$$\Gamma_k = \{x \in \Omega_\gamma : \ s_{i_1} x_{i_1} = \cdots = s_{i_k} x_{i_k} > s_{i_{k+1}} x_{i_{k+1}} > \cdots > s_{i_n} x_{i_n} > 0\} \subseteq X_{I_k}.$$

So $dV = p \cdot dx$ holds in Γ_{k-1} if it does in Γ_k. Continuing, we are eventually reduced to considering the validity of $dV = p \cdot dx$ in Γ_{I_n}, which is true since this is 1-dimensional, covered by a single bicharacteristic of the H_{I_n} system. (In Γ_{I_n}, $dV = p \cdot dx$ is equivalent to $\frac{d}{dt}V(x(t)) = p \cdot \dot{x}$.) In this way we eventually conclude that $dV = p \cdot dx$ in each subregion of Ω_γ associated with a specific ordering σ; i.e., $\nabla V(x) = P(x)$ in each. Since p and V are both continuous throughout Ω_γ, we conclude that $\nabla V(x) = P(x)$ throughout.

We have $V(0) = 0$ by definition (3.19). We want to show $V(x) > 0$ if $x \neq 0$. This follows simply by observing as in (3.16) that the coordinates $P_i(x) > 0$ are positive whenever $x_i > 0$, so that $P(x) \cdot x > 0$ for any nonzero $x \in \Omega_\gamma$. Integrating along a straight line from the origin to x implies that $V(x) > 0$ for $x \neq 0$.

Consider once again a staged bicharacteristic $x(t), p(t)$. Notice that in each stage $[t_{k-1}, t_k]$ we have $x(t) \in X_{I_k}$, $\eta_{I_k} = Gu_{I_k}$ and $u_{I_k} \in u_*(x(t))$. This means we can write (3.3) as

$$\dot{x}(t) = q^*(x(t)) - Gu(t)$$

with $u(t) \in u_*(x(t))$ for each t, i.e., our staged bicharacteristic solves the differential inclusion (2.8) on $[0, T]$ for the disturbance $q(t) = q^*(x(t))$. Moreover since $q(t) \in Q^*(x, P(x))$ and $p(t) = P(x(t))$ we know

$$0 = H^\pi(x, P(x)) = K^\pi(x, P(x); q(t), u(t))$$

$$= p(t) \cdot (q(t) - Gu(t)) - \frac{1}{2}(\gamma^2 \|q(t)\|^2 - \|x\|^2),$$

or $-P(x(t)) \cdot \dot{x}(t) = \frac{1}{2}(\|x(t)\|^2 - \gamma^2 \|q(t)\|^2)$. Integrating over $[0, T]$ gives

(3.22)
$$V(x(0)) = \int_0^T -p(t) \cdot \dot{x}(t) \, dt$$
$$= \frac{1}{2} \int_0^T \|x(t)\|^2 - \gamma^2 \|q(t)\|^2 \, dt.$$

Moreover this, and the uniqueness of solutions to (3.16) discussed below, shows that u_* is Q-tolerant. Together with (2.10), this implies the first assertion of part (a) of Theorem 2.4. Since $V(x)$ is a nonnegative solution of the Hamilton-Jacobi equation (2.5), the second assertion in part (a) is an immediate consequence of Theorem T:Verification.

3.3. **Optimality.** The only aspect of Theorem 2.4 which has not been addressed is the inequality

(3.23) $$\|x\| > \gamma^{-1} \|P(x)\| \text{ for all nonzero } x \in \Omega_\gamma$$

that was used in the proof for part b given earlier. (Note that by (3.22) this also provides an alternate proof that $V(x) > 0$ for $x \neq 0$, which will be a more convenient

approach in Section 4.) Using our formula $\sin(t_k/\gamma) = \sum_1^k a_i$ and summing by parts we obtain

$$\gamma^{-1}x = \sum_1^n a_k \eta_{I_k}$$

$$= \eta_{I_n} \sin(t_n/\gamma) + \sum_1^{n-1}(\eta_{I_k} - \eta_{I_{k+1}})\sin(t_k/\gamma)$$

Now from (3.2) we know that $\eta_{I_i} \cdot \eta_{I_j} = \|\eta_{I_{\max(i,j)}}\|^2$, which implies that the vectors

$$\mu_i = \eta_{I_i} - \eta_{I_{i+1}}, \ i = 1, \ldots, n-1; \ \mu_n = \eta_{I_n}$$

are orthogonal, and also non-zero as one may check. Therefore

$$\|x\|^2 = \gamma^2 \sum_1^n \|\mu_k\|^2 \sin^2(t_k/\gamma).$$

The analogous calculation for $P(x)$ yields

$$\|P(x)\|^2 = \gamma^4 \sum_1^n \|\mu_k\|^2 (1 - \cos(t_k/\gamma))^2.$$

Noticing that $1 - \cos(\theta) < \sin(\theta)$ for all $0 < \theta < \pi/2$, we see that

$$\|x\|^2 \geq \gamma^{-2}\|P(x)\|^2,$$

with equality holding only if all t_k/γ are either 0 or $\pi/2$, which implies that either $x = 0$ or $x \cdot \eta_N = \gamma c_N$. However we have excluded x with $x \cdot \eta_N = \gamma c_N$ from Ω_γ, by definition. Theorem 2.4 now follows as in Section 2.

We close this section with some observations about the closed-loop system

$$(3.24) \qquad \dot{x} = \pi_K(x, q(t) - Gu_*(x))$$

resulting from our optimal control (3.18). Notice first that the domain Ω_γ of (3.14) depends on the gain rate γ, increasing to all of K as $\gamma \to +\infty$. Likewise $V(x)$ depends on γ (although there is a simple scaling relationship that can be deduced from (3.7): $V_\gamma(x) = \gamma^3 V_1(x/\gamma)$). However the optimal control (3.18) does not depend on the gain tolerance γ, giving an additional sense in which (3.18) is robust. Thus we can consider the system (3.24) for any initial $x(0) \in K$. The size of $x(0)$ implies a lower bound on the γ values for which we can discuss L_2-gain properties, but otherwise the closed-loop system is independent of γ.

Since we are using normal constraint directions for the projection dynamics, the uniqueness result [6, Theorem 2] will apply to (3.24) provided we can verify the Filippov condition (1.15). For this we only need check that

$$(3.25) \qquad (-s_{i_0}e_{i_0} + s_{j_0}e_{j_0}) \cdot (x - y) \leq L\|x - y\|^2$$

assuming

$$(3.26) \qquad s_{i_0}x_{i_0} \geq s_j x_j \text{ and } s_{j_0}y_{j_0} \geq s_j y_j \text{ for all } j.$$

Simplifying and using (3.26) in (3.25) gives

$$(-s_{i_0}e_{i_0} + s_{j_0}e_{j_0}) \cdot (x - y) = -s_{i_0}x_{i_0} + s_{j_0}x_{j_0} + s_{i_0}y_{i_0} - s_{j_0}y_{j_0}$$

$$\leq -s_{j_0}x_{j_0} + s_{j_0}x_{j_0} + s_{j_0}y_{j_0} - s_{j_0}y_{j_0} = 0$$

and hence (3.25) holds with $L = 0$. Thus given an integrable function $q : [0, T] \to \mathbb{R}^n$ and $x(0) \in K$, there is a unique solution to (3.24) defined for $0 \leq t \leq T$.

4. A NONDIAGONAL EXAMPLE

We present in this final section a simple example (with G given by (4.1) which illustrates that in general the boundary conditions in both the state equations as well as the verification theorem can be more complicated than what we encountered in Section 3. We will be able to deal with these issues in the context of this particular example, but we will leave their general resolution for the future. Much of the analysis is the same as in Section 3. We focus mostly on the parts where something new is involved. The discussion will lead to the proof of Theorem 2.4 for this case.
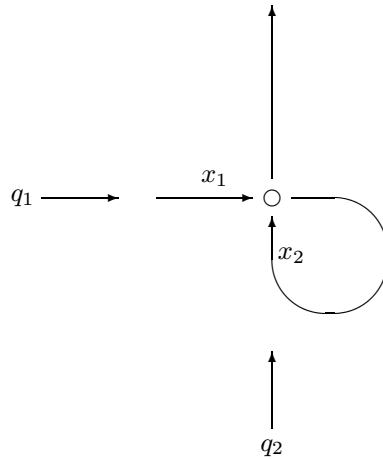

FIGURE 1. Loop Intersection Example

4.1. **The Loop Model.** The example is suggested by the traffic intersection illustrated in Figure 1. There are two queues, x_1 and x_2, each with their own disturbance q_i. The unique feature is that as customers from x_1 are served and pass through the intersection they feed into queue x_2, where they need service through the intersection a second time. Thus there is an interconnection between the two queues, a feature that will be encountered as one begins to consider models describing networks of several interconnected intersections. The controlled dynamics in the interior of K are $\dot{x} = q - Gu$ where

$$(4.1) \qquad\qquad G = \begin{bmatrix} s_1 & 0 \\ -s_1 & s_2 \end{bmatrix},$$

and $u \in \mathcal{U}$ as in (1.3) for $n = 2$. Of course $s_i > 0$ are parameters specifying the maximum flow rates for the two signal phases. Giving x_1 the "green light" corresponds to $u = e_1$. The explanation for the off-diagonal term $-s_1$ in G is the simultaneous effect of the traffic departing the queue x_1 at rate s_1 and joining the queue x_2 at the same rate.

We again want the queue length vector $x(t) = (x_1(t), x_2(t))^T$ to remain in the first quadrant

$$K = \{x : \ x_i \geq 0\}.$$

Care is needed to pick the constraint directions for the faces $\partial_i K$ to get an appropriate projected dynamical system

$$\dot{x} = \pi_K(x, q - Gu).$$

Along the horizontal face $\partial_2 K = \{x : \ x_2 = 0, \ x_1 \geq 0\}$, the normal constraint $d_2 = e_2$ remains the natural choice. Consider however the vertical face, $\partial_1 K = \{x : x_1 = 0, \ x_2 \geq 0\}$. Suppose the operative control is $u = e_1$, so that the equations (without projection) would be

$$(4.2) \qquad\qquad\qquad \dot{x}_1 = q_1 - s_1$$

$$(4.3) \qquad\qquad\qquad \dot{x}_2 = q_2 + s_1.$$

Consider in particular what would happen if

$$(4.4) \qquad\qquad\qquad 0 \leq q_1 < s_1.$$

The queue $x_1 = 0$ is empty but has a green light. Traffic is arriving in this queue at a nonnegative rate but which is less than the intersection's capacity for this phase, and so immediately passes through the intersection and joins queue x_2. Equation (4.2) would not only have x_1 becoming negative, but (4.3) has more traffic entering x_2 from x_1 than is actually passing through the intersection: $s_1 > q_1$. Instead of (4.2) and (4.3) the appropriate equations under the circumstances (4.4) are

$$\dot{x}_1 = 0 \ (= q_1 - q_1)$$
$$\dot{x}_2 = q_2 + q_1.$$

This corresponds to using the oblique constraint direction $d_1 = \frac{1}{\sqrt{2}}(1, -1)^T$ for the vertical face. (The normal constraint direction would give $\dot{x}_2 = q_2 + s_1$ which is again incorrect.)

This oblique constraint direction is correct for (4.4) with $u = e_1$, but is unrealistic in other circumstances. For instance if $u = e_2$, $x_1 = 0$ but $q_1 < 0$, then $\dot{x} = \pi_K(x, q - Gu)$ gives $\dot{x}_1 = 0$ and $x_2 = q_2 - q_1$ which would indicate traffic arriving in x_2 "through the loop" at a rate $-q_1$ even though x_1 has a red light, so that there should be no traffic in the loop. However, for the following reasons we choose to stay with the oblique constraint direction described above.

- To accommodate all mathematical possibilities in a realistic way would require a more intricate projection mechanism than is possible with π_K in general. For instance we would want to use constraint directions that depend on the operative control, or that change with the magnitude of the required constraining term βd. To develop such a mechanism seems unwarranted for this one simple example.
- All the unrealistic consequences of the oblique constraint direction occur for $x_1 = 0$ and $q_1 < 0$, which is an unphysical possibility in the first place. The situation (4.4) is quite realistic physically, and so is the most important to be accurately modeled.
- For the control and storage function $V(x)$ resulting from our analysis of the model with oblique constraint direction, it will turn out that the the "improvements" to the model that we might contemplate will not alter the fact that $H^\pi(x, \nabla V(x)) \leq 0$, i.e., the contemplated modifications really don't

matter — the storage function and feedback control would continue to satisfy Theorem 2.1 regardless.

4.2. **Dupuis-Ishii Conditions.** We proceed then to consider our analysis of the model

$$(4.5) \qquad\qquad \dot{x} = \pi_K(x, q - Gu)$$

with the projection $\pi_K(x, v)$ as determined using the normal constraint direction $d_2 = e_2$ for the horizontal face $\partial_2 K$, and the oblique direction $d_1 = \frac{1}{\sqrt{2}}(1, -1)^T$ for the vertical face $\partial_1 K$.

The first thing we need to do is insure that these constraint directions satisfy the sufficient conditions of [5] for π_K and the projected dynamical system (4.5) to be well-defined. There are two conditions that need to be checked. The first is [5, Assumption 2.1], related to Lipschitz continuity of the Skorokhod Problem. A sufficient condition for this is given in [5, Theorem 2.1 part 2], namely that there exist $a_i > 0$ so that

$$a_1 n_1 \cdot d_1 > a_2 |n_1 \cdot d_2|$$
$$a_2 n_2 \cdot d_2 > a_1 |n_2 \cdot d_1|.$$

Since $n_1 \cdot d_1 = \frac{1}{\sqrt{2}}$, $n_2 \cdot d_2 = 1$, $n_2 \cdot d_1 = -\frac{1}{\sqrt{2}}$ and $n_1 \cdot d_2 = 0$, this reduces to

$$a_2 > \frac{1}{\sqrt{2}} a_1 > 0,$$

so that positive such a_i certainly do exist. Secondly, there is [5, Assumption 3.1] related to the existence of solutions, for which [5, Theorem 3.1 part 2] provides a sufficient condition: that $d_i = Dn_i$ where $n_i \cdot Dn_i > 0$ for both n_i. For us $D = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 1 \end{bmatrix}$. The $n_i \cdot Dn_i > 0$ are just the diagonal entries, which are indeed positive.

4.3. **Interior Analysis.** We can now carry out the construction of a solution $V(x)$ of (2.5) analogous to the construction in Section 3 for the case of diagonal G. The set of p for which $u = e_i$ acheives the $\inf_{u \in \mathcal{U}}$ in (2.12) is

$$P_i = \{p : \ p \cdot Ge_i \geq p \cdot Gu \text{ all } u \in \mathcal{U}\}.$$

Let

$$\eta_{\{1\}} = \begin{bmatrix} s_1 \\ -s_1 \end{bmatrix} \quad \text{and} \quad \eta_{\{2\}} = \begin{bmatrix} 0 \\ s_2 \end{bmatrix}$$

be the two basic control vectors $\eta_{\{i\}} = Ge_i$. Define the averaged control

$$u_{\{1,2\}} = \frac{1}{s_1^2 + (s_1 + s_2)^2} \begin{bmatrix} s_2(s_1 + s_2) \\ s_1^2 + s_1(s_1 + s_2) \end{bmatrix},$$

and the "interface direction"

$$\eta_{\{1,2\}} = Gu_{\{1,2\}} = \frac{s_1 + s_2}{s_1^2 + (s_1 + s_2)^2} \begin{bmatrix} s_1 + s_2 \\ s_1 \end{bmatrix}.$$

As in Section 3, $u_{\{1,2\}}$ is determined by the requirement that $\eta_{\{1,2\}} = Gu_{\{1,2\}}$ be parallel to the interface $P_{\{1,2\}} = P_1 \cap P_2$.

For $p \in P_i$ the (interior) Hamiltonian and associated Hamiltonian system are

$$H(x, p) = H_{\{i\}}(x, p) = \frac{1}{2\gamma^2} \|p\|^2 - p \cdot \eta_{\{i\}} + \frac{1}{2} \|x\|^2,$$

and

$$(4.6) \qquad \begin{aligned} \dot{x} &= \frac{1}{\gamma^2} p - \eta_{\{i\}} \\ \dot{p} &= -x. \end{aligned}$$

We define the cones X_i in state space which duplicate the P_i:

$$X_i = \{x : \ x \cdot Ge_i \geq x \cdot Gu \text{ all } u \in \mathcal{U}\},$$

and $X_{\{1,2\}} = X_1 \cap X_2$. For $x \in X_i$ ($i = 1$ or 2) we want $p = P(x) \in P_i$ so that $x = x(0)$, $p = p(0)$ is the initial point of a staged bicharacteristic: $x(t), p(t)$ solves (4.6) for $0 \leq t \leq t_1$ at which time it contacts $X_{\{1,2\}} \times P_{\{1,2\}}$. Then on a final time segment $t_1 \leq t \leq t_2 = T$ we use the averaged system

$$\begin{aligned} \dot{x} &= \frac{1}{\gamma^2} p - \eta_{\{1,2\}} \\ \dot{p} &= -x, \end{aligned}$$

which will stay in the interface $X_{\{1,2\}} \times P_{\{1,2\}}$ until reaching $x(T) = p(T) = 0$ at the final time. As before, using the representation (3.7) we can determine $P(x)$ from $0 \leq t_1 \leq t_2 \leq \frac{\pi}{2}\gamma$ and the equations

$$\begin{aligned} a_1(0) &= \sin(t_1/\gamma) & \alpha_1(0) &= 1 - \cos(t_1/\gamma) \\ a_1(0) + a_2(0) &= \sin(t_2/\gamma) & \alpha_1(0) + \alpha_2(0) &= 1 - \cos(t_2/\gamma). \end{aligned}$$

The solvability condition $a_1(0) + a_2(0) \leq 1$ corresponds to $x \cdot \eta_{\{1,2\}} \leq \gamma \|\eta_{\{1,2\}}\|^2$. We define the solvability region Ω_γ to consist of those $x \in K$ for which this inequality is strict, i.e.,

$$(4.7) \qquad \Omega_\gamma = \{x : \ x \cdot \eta_{\{1,2\}} < \gamma \|\eta_{\{1,2\}}\|^2.$$

The a_i, α_i equations above determine $P(x)$ which corresponds to $P(x) = \nabla V(x)$ solving the equation $H(x, \nabla V(x)) = 0$ for $x \in \Omega_\gamma$ with $0 = V(0) < V(x)$. The corresponding feedback signal control is

$$u_*(x) = \text{conv}\{e_i : \ x \in X_i\},$$

which works out to be the compact, convex upper semicontinuous set-valued function

$$(4.8) \qquad u_*(x) = \begin{cases} \{e_1\} \ (\text{i.e. green for } x_1) & \text{if } x_1 s_1 > x_2(s_1 + s_2) \\ \text{all } (u_1, u_2) \in \mathcal{U} & \text{if } x_1 s_1 = x_2(s_1 + s_2) \\ \{e_2\} \ (\text{i.e. green for } x_2) & \text{if } x_1 s_1 < x_2(s_1 + s_2). \end{cases}$$

4.4. **Confirmation on the Boundary.** We now need to check that $P(x) = \nabla V(x)$ for $x \in \partial K$ is compatible with the projection dynamics, so that we can conclude $H^\pi(x, \nabla V(x)) = 0$ as desired. At $x = 0$, $p(0) = 0$, $H^\pi(0,0) = 0$ is easily confirmed. We will look at the vertical and horizontal faces separately.

4.4.1. *The Vertical Face.* The region X_2 above the interface $X_{\{1,2\}}$, including the vertical boundary where $x_1 = 0$, is similar to what we found in Section 3. In $X_2 \times P_2$ we use the representation

$$x = \gamma(a_1 \eta_{\{2\}} + a_2 \eta_{\{1,2\}}), \quad p = \gamma^2(\alpha_1 \eta_{\{2\}} + \alpha_2 \eta_{\{1,2\}}).$$

Since $\eta_{\{2\}}$ is collinear with the vertical boundary, $x_1 = 0$ implies $a_2 = \alpha_2 = 0$ so that $P(x)$ is also collinear with the boundary: $P(x) = \gamma^2 \alpha_1(0, s_2)^T$ with $\alpha_1 > 0$ for

$x_2 > 0$. This allows us to invoke Lemma 2.2 to conclude that $H^\pi(x, \nabla V(x)) \leq 0$, since

$$P(x) \cdot d_1 = -\frac{1}{\sqrt{2}}\gamma^2 \alpha_1 s_2 \leq 0.$$

Observe that we would have come to the same conclusion had we used the normal constraint direction $d_1 = n_1$ on this boundary face, since then $P(x) \cdot n_1 = 0$. This was the boundary where we observed above that the oblique constraint direction led to unnatural projected dynamics under some mathematically possible but unphysical sets of circumstances. We abandoned the possibility of developing a more elaborate velocity projection mechanism to accommodate these, but commented above that even if we had it would not affect our final conclusion. Now we see why: if $\pi_K(x, v)$ is replaced by any more elaborate mechanism that produces a value

$$\pi_K(x, v) = v + \beta_d d_1 + \beta_n n_1$$

with $\beta_d, \beta_n \geq 0$, depending in some elaborate way on x and v, we will still have

$$P(x) \cdot \pi_K(x, v) \leq P(x) \cdot v,$$

for all v, so that $H^\pi(x, \nabla V(x)) \leq 0$ will again follow as in Lemma 2.2.

Theorem 2.4 on the optimality of $u_*(\cdot)$ in (4.8) requires more than the inequality (2.7). We need to show that $H^\pi(x, P(x)) = 0$, $u_*(x) = U_*(x, P(x))$ and $q^*(x) = \frac{1}{\gamma^2}P(x) \in Q^*(x, P(x))$. In the interior of K (and at $x = 0$) these follow from our construction using the interior Hamiltonian $H(\cdot, \cdot)$. We have to consider the boundary points more carefully. Consider $x \in \partial_1 K \setminus \{0\}$ and let $q^* = \frac{1}{\gamma^2}P(x)$. We know from the comments above that $q^* = (0, q_2^*)$. Consider any $u = (u_1, u_2) \in \mathcal{U}$. Then

$$(4.9) \qquad q^* - Gu = \begin{bmatrix} 0 \\ q_2^* - u_2 s_2 \end{bmatrix} + u_1 \begin{bmatrix} -s_1 \\ s_1 \end{bmatrix}.$$

Since the projection dynamics use the constraint direction $d_1 = \frac{1}{\sqrt{2}}(1, -1)^T$ on this face, we will have $\pi_K(x, q^* - Gu) = q^* - Gu + \beta d$ where βd exactly cancels the u_1 term in (4.9). Thus

$$\pi_K(x, q^* - Gu) = q^* - u_2 G e_2,$$

and so for any $u \in \mathcal{U}$ we find

$$(4.10) \qquad K^\pi(x, P(x); u, q^*) = \frac{1}{2\gamma^2}\|P(x)\|^2 - u_2 s_2 P(x) \cdot e_2 + \frac{1}{2}\|x\|^2.$$

Since $P(x) \cdot e_2 > 0$ we see that the infimum over $u \in \mathcal{U}$ is achieved uniquely for $u_2 = 1$, i.e. at $u = u_*(x) = e_2$. Therefore

$$\inf_{u \in \mathcal{U}} K^\pi(x, P(x); u, q^*) = \frac{1}{2\gamma^2}\|P(x)\|^2 - s_2 P(x) \cdot (1, 0)^T + \frac{1}{2}\|x\|^2$$

$$= \frac{1}{2\gamma^2}\|P(x)\|^2 - P(x) \cdot \eta_{\{2\}} + \frac{1}{2}\|x\|^2$$

$$= H_{\{2\}}(x, P(x)) = 0.$$

It follows that

$$0 \geq H^\pi(x, P(x)) = \sup_q \inf_{u \in \mathcal{U}} K^\pi(x, P(x); u, q) \geq \inf_{u \in \mathcal{U}} K^\pi(x, P(x); u, q^*) = 0$$

so that

$$0 = H^\pi(x, P(x)) = \inf_{u \in \mathcal{U}} K^\pi(x, P(x); u, q^*) = 0.$$

This implies $q^* \in Q^*(x, P(x))$. Also,

$$0 = \sup_q \inf_{u \in \mathcal{U}} K^\pi(x, P(x); u, q)$$

$$\leq \sup_q K^\pi(x, P(x); e_2, q)$$

$$\leq \sup_q K(x, P(x); e_2, q)$$

$$= H_{\{2\}}(x, P(x)) = 0,$$

which tells us that $\{e_2\} = u_*(x) \subseteq U_*(x, P(x))$. Finally, (2.4) says that any other $\bar{u} \in U_*(x, P(x))$ must achieve $\inf_u K^\pi(x, P(x); u, q^*)$. We know from (4.10) that $\bar{u} = e_2$ is the only possibility. Thus, $u^*(x) = \{e_2\} = U_*(x, P(x))$.

4.4.2. *The Horizontal Face.* The horizontal face, where $x_2 = 0$, is part of the region X_1 below the interface $X_{\{1,2\}}$. The confirmation that $H^\pi(x, \nabla V(x)) = 0$ is fundamentally different here, because it turns out that $P(x) \cdot d_2 = P(x) \cdot e_2 > 0$, so that Lemma 2.2 can not be invoked. This explains why, in contrast to [2], using (2.12) and a boundary condition like (2.14): $d_i \cdot \nabla V(x) \leq 0$ on $\partial_i K$ is too strong a hypothesis for our Theorem 2.1 in general. The representation we use in X_1 is

$$x = \gamma(a_1 \eta_{\{1\}} + a_2 \eta_{\{1,2\}}), \quad p = \gamma^2(\alpha_1 \eta_{\{1\}} + \alpha_2 \eta_{\{1,2\}}).$$

Since $\eta_{\{1\}}$ is *not* collinear with the horizontal boundary, $x_2 = 0$ does not correspond simply to taking one of the $a_i = 0$. The resulting expression for $P(x)$ along the horizontal boundary is rather complicated. It does turn out that $P(x) \cdot e_2 > 0$ for $x_2 = 0 < x_1$. This can be verified algebraically, or by numerical exploration of test cases. We omit this however and proceed to investigating $H^\pi(x, \nabla V(x)) = 0$ in light of it.

The key is to consider $\sup_q K^\pi(x, P(x); e_1, q)$ carefully . On the face $\partial_2 K$ we use normal constraint $d_2 = e_2$. So for $x \in \partial_2 K \setminus \{0\}$ and any $v = (v_1, v_2)^T$ we have

$$\pi_K(x, v) = \begin{bmatrix} v_1 \\ \max(0, v_2) \end{bmatrix}.$$

Thus for

$$q - Ge_1 = \begin{bmatrix} q_1 - s_1 \\ q_2 + s_1 \end{bmatrix}$$

we find, writing $P(x) = (p_1, p_2)^T$ and $(q_1^*, q_2^*) = \frac{1}{\gamma^2} P(x)$, that

$$K^\pi(x, P(x); e_1, q) = p_1(q_1 - s_1) - \frac{1}{2}\gamma^2 q_1^2 + \frac{1}{2}\|x\|^2$$
$$+ p_2 \max(0, q_2 + s_1) - \frac{1}{2}\gamma^2 q_2^2.$$

Using the fact that $-s_1 < 0 \leq q_2^*$, we find that the maximum with respect to $q = (q_1, q_2)$ occurs (uniquely) at $q = q^* = \frac{1}{\gamma^2} P(x)$, yielding

(4.11) $$\sup_q K^\pi(x, P(x); e_1, q) = \sup_q K(x, P(x); e_1, q) = H_{\{1\}}(x, P(x)) = 0.$$

Therefore

$$H^\pi(x, P(x)) = \sup_q \inf_{u \in \mathcal{U}} K^\pi(x, P(x); u, q) \leq \sup_q K^\pi(x, P(x); e_1, q) = 0.$$

On the other hand, $P(x) \cdot d_2 \geq 0$ implies (just as in Lemma 2.2) that

$$K^\pi(x, P(x); u, q) \geq K(x, P(x); u, q),$$

so that in general we can say

$$H^\pi(x, P(x)) \geq H(x, P(x)) = 0.$$

Putting these inequalities together, we see that

$$H^\pi(x, P(x)) = 0,$$

and therefore (4.11) implies $e_1 = u_*(x) \in U_*(x, P(x))$. Also, since

$$K^\pi(x, P(x); e_1, q^*) = K(x, P(x); e_1, q^*) = 0,$$

we have

(4.12) $$0 \geq \inf_{u \in \mathcal{U}} K^\pi(x, P(x); u, q^*) \geq \inf_{u \in \mathcal{U}} K(x, P(x); u, q^*) = 0.$$

Therefore, $q^* \in Q^*(x, P(x))$ as well. Finally, by (2.4), any $\bar{u} \in U_*(x, P(x))$ must satisfy

$$0 = K^\pi(x, P(x); \bar{u}, q^*) \geq K(x, P(x); \bar{u}, q^*) \geq K(x, P(x); e_1, q^*) = 0$$

This implies $\bar{u} = e_1$, so that $U_*(x, P(x)) = \{e_1\}$.

4.5. **The Optimal Signal Control.** The closed-loop system resulting from the feedback control (4.8)

(4.13) $$\dot{x} = \pi_K(x, q - Gu_*(x))$$

has features similar to those of (3.24). In particular the optimal control (4.8) is independent of γ, so that (4.13) can be considered for any $x(0) \in K$, without regard to a particular choice of γ. We can again verify the Filippov condition (1.14) by checking that

(4.14) $$(-\eta_{\{1\}} + \eta_{\{2\}}) \cdot (x - y) \leq L\|x - y\|^2$$

for $x \in X_1$ and $y \in X_2$. Any $x \in X_1$ can be written as

$$x = \gamma(a_1 \eta_{\{1\}} + a_2 \eta_{\{1,2\}}),$$

for some $a_i \geq 0$. Since

$$(\eta_{\{1\}} - \eta_{\{2\}}) \cdot \eta_{\{1,2\}} = 0 = \eta_{\{1\}} \cdot \eta_{\{2\}},$$

we find that

$$(-\eta_{\{1\}} + \eta_{\{2\}}) \cdot x = -\gamma a_1 \|\eta_{\{1\}}\|^2 \leq 0.$$

A similar argument shows that

$$(-\eta_{\{1\}} + \eta_{\{2\}}) \cdot (-y) \leq 0.$$

Thus (4.14) again holds with $L = 0$. However, as we observed in Section 1.4, the literature does not contain a general uniqueness result that we can apply to (4.13) because we are using oblique constraint directions. Since the only boundary point of K at which the discontinuities of $u_*(\cdot)$ occur is the origin itself, one can argue, based on the two separate uniqueness results mentioned in Section 1.4, that for $x(0) \in K$, the solution of (4.13) will be unique at least up to the first time $x(t) = 0$.

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