ON NEUMANN TYPE BOUNDARY CONDITIONS FOR HAMILTON-JACOBI EQUATIONS IN SMOOTH DOMAINS

MARTIN V. DAY

Abstract. Neumann or oblique derivative boundary conditions for viscosity solutions of Hamilton-Jacobi equations are considered. As developed by P. L. Lions, such boundary conditions are naturally associated with optimal control problems for which the state equations employ "Skorokhod" or reflection dynamics to insure that the state remains in a prescribed set, assumed here to have smooth boundary. We develop connections between the standard formulation of viscosity boundary conditions and an alternative formulation using a naturally occurring discontinuous Hamiltonian which incorporates the reflection dynamics directly. At points of differentiability, equivalent conditions for the boundary conditions are given in terms of the Hamiltonian and the geometry of the state trajectories using optimal controls.

1. Introduction

P. L. Lions developed the notion of Neumann type boundary conditions for viscosity solutions of Hamilton-Jacobi equations in [9]. The now standard expression of these is given in (16) below. He showed that this provided the appropriate notion of solution for the Hamilton-Jacobi equations describing the value functions of control problems or differential games involving reflected deterministic processes, often called Skorokhod problems; see [6].

To discuss this we will work in the following setting. Assume that $\Omega \subseteq \mathbb{R}^n$ is open with $\partial\Omega$ smooth and unit outward normal $n(x)$. Also given on $\partial\Omega$ is a smooth vector field $\gamma : \partial\Omega \to \mathbb{R}^n$ on $\partial\Omega$, normalized by

$$\gamma(x) \cdot n(x) = 1. \tag{1}$$

Consider the *velocity projection map* defined by

$$\pi(x, v) = \begin{cases} v & \text{if } x \in \Omega \\ v - c\gamma(x) \text{ with } c = (n(x) \cdot v)^+ & \text{if } x \in \partial\Omega. \end{cases}$$

Notice that c is the smallest nonnegative value for which $n(x) \cdot (v - c\gamma(x)) \le 0$. Thus the effect of $\pi(x, v)$ is to correct v by adding the minimal nonnegative multiple of $-\gamma(x)$ such that $v - c\gamma(x)$ does not point outside $\bar\Omega$. Suppose $U \subset \mathbb{R}^m$ is a closed set of control values and

$$f : \bar\Omega \times U \to \mathbb{R}^n$$

is a controlled vector field. We are motivated by control problems whose state dynamics consist of f coupled with the velocity projection map:

$$\dot x(t) = \pi(x(t), f(x(t), u(t))); \quad x(0) = x_0. \tag{2}$$

This is what we refer to as *Skorokhod dynamics.* The result is a control system whose natural state space is $\bar\Omega$. See Dupuis and Ishii [6] for discussion of the general Skorokhod problem, its relation to $\pi(x, v)$, and the equivalence of (2) with the formulation of (47) in [9].

For appropriate running cost functions $L : \bar\Omega \times U \to \mathbb{R}$ a variety of control problems can be formulated in which the goal is to minimize an integral of $L(x(t), u(t))$. The resulting value function $V(x)$ will typically be a viscosity solution of a Hamilton-Jacobi equation involving the Hamiltonian

$$H(x, p) = \sup_u \{-p \cdot f(x, u) - L(x, u)\}. \tag{3}$$

For instance Lions uses the example of an infinite horizon discounted problem

$$V(x_0) = \inf_{u(\cdot)} \int_0^\infty e^{-\lambda t} L(x(t), u(t))\, dt, \tag{4}$$

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with positive discount rate λ. He showed that V will be the unique continuous viscosity solution of

$$\lambda V(x) + H(x, DV(x)) = 0 \text{ in } \Omega, \quad \gamma(x) \cdot DV(x) = 0 \text{ in } \partial\Omega, \tag{5}$$

the boundary conditions being understood in the viscosity sense of (16) below. However, the usual dynamic programming heuristics first suggest

$$\lambda V(x) + H^\pi(x, DV(x)) = 0, \quad x \in \bar{\Omega} \tag{6}$$

as the appropriate Hamilton-Jacobi equation, where H^π is the *reflected Hamiltonian*

$$H^\pi(x,p) = \sup_{u \in U} \{-p \cdot \pi(x, f(x,u)) - L(x,u)\}. \tag{7}$$

Note that $H^\pi(x,p) = H(x,p)$ if $x \in \Omega$, but they can be quite different for $x \in \partial\Omega$. The formulation (6) appears as [9, (53)] in Lions.

The assertions and arguments of Lions' Theorem 12 imply that (5) and (6) are equivalent (in his context). Indeed the value function representation (4) establishes the existence of solutions to (6). The last part of the proof of Theorem 12 argues that any solution of (6) is also a solution of (5). Solutions of (5) are unique, by his Theorem 11. Thus any solution of (5) must agree with the solution of (6) provided by (4). We hasten to note however that this argument for equivalence depends on both the existence and uniqueness results.

Our goal in this paper is to explore the connections between the boundary condition formulation and the reflected Hamiltonian formulation more directly, without recourse to uniqueness results for solutions or a restriction to solutions which can be interpreted as value functions for some control problem. We carry out our discussion for the simpler equation

$$H(x, DV(x)) = 0. \tag{8}$$

It is well known that there is generally no uniqueness theorem for equations of this form, which lack a λV or V_t term. (An instance of nonuniqueness is provided in Example 1 below.)

The specific problem formulations we will compare are the boundary condition formulation (14) and the reflected Hamiltonian formulation (17) below. Our discussion of the relation between the two formulations will not appeal to a control theoretic interpretation of the solutions. However the implications for control problems is a primary motivation for our study. Specifically, we hope to understand what the boundary conditions require in terms of the structure of the extremals associated with a solution V. Suppose that V is a smooth solution. By by an *extremal* we mean a trajectory $\dot{x}(t) = \pi(x(t), f(x(t), u^*(t))$ where at each t, $u^*(t)$ achieves the maximum over $u \in U$ of the expression

$$-DV(x(t)) \cdot \pi(x(t), f(x(t), u)) - L(x(t), u), \tag{9}$$

which defines $H^\pi(x, DV(x))$. The values u^* that maximize this expression at a boundary point x are the possible otimal controls when the system is at state x. We would like to be able to identify these u^* from properties of $\gamma \cdot DV(x)$, $f(x, u^*)$ and other quantities, avoiding if possible the more complicated nonlinear optimization problem (9). Hamilton-Jacobi equations with boundary conditions have arisen in recent examples from queueing theory, where the Skorokhod problem mechanism arises naturally; see [2] and [5] for instance. Work such as [5] in particular uses appropriate families of extremal trajectories to construct the value function V. This motivates our interest in the connection between the boundary conditions and the properties of extremals.

At interior points $x \in \Omega$ we have $\pi(x, f(x,u)) = f(x,u)$ and so the optimal controls would be those u^* which achieve the maximum of

$$-DV(x) \cdot f(x,u) - L(x,u),$$

which defines $H(x, DV(x))$. If H is also smooth, we see that these maximizing u^* satisfy

$$-\frac{\partial}{\partial p} H(x, DV(x)) = f(x, u^*).$$

Thus our extremal is a solution of

$$\dot{x} = -\frac{\partial}{\partial p} H(x, DV(x)).$$

This is the state component of the usual characteristic equations

$$\dot{x} = -\frac{\partial}{\partial p} H(x,p), \quad \dot{p} = \frac{\partial}{\partial x} H(x,p).$$

The classical method of characteristics associates each smooth solution V with the family of solutions to the characteristic equations for which $p(t) = DV(x(t))$. Thus by considering extremals we will be characterizing the boundary conditions (for smooth solutions) in terms the classical characteristics.

Hamilton-Jacobi equations with boundary conditions have arisen in recent examples from queueing theory, where the Skorokhod problem mechanism arises naturally; see [2] and [5] for instance. Work such as [5] in particular uses appropriate families of extremal trajectories to construct the value function V. This motivates interest in the connection between the boundary conditions and the properties of extremals.

In Section 2 give a complete specification of our hypotheses, state more carefully the two notions of solution, and collect some elementary inequalities. In Section 3 the equivalence of the two notions is considered. It is quite simple to show that a solution in the reflected Hamiltonian sense must be a solution in the boundary condition sense; this is Theorem 1. Theorem 2 will show that the two notions of subsolutions are indeed equivalent. To prove the analogous result for supersolutions, Theorem 3, we need to make stronger affine-convex assumptions (AC) on the dynamics and running cost. These are satisfied in the recent applications in queueing theory and our examples below. However, in the queuing theory applications Ω is generally a convex polygon, in which case $\partial\Omega$ is not smooth but has numerous edges and corners. Moreover those applications often lead to differential games rather than simple control problems. We have not attempted to deal with those features here. (Some of our simpler results are easy to establish in that setting however, as we will comment in Section 5.)

In Section 4 we assume that V is C^1 solution of $H(x, DV(x)) = 0$. We will see that the two notions of solution are equivalent under this hypothesis, without reference to (AC). We will be able to describe the boundary conditions in terms of simple geometric properties of extremals and identify optimal control values on $\partial\Omega$. Section 5 will close with some comments on yet another formulation that occurs in the work of Dupuis, Ishii and Soner [7].

2. Hypotheses and Preliminaries

We assume $\Omega \subset \mathbb{R}^n$ is open with $\partial\Omega$ of class C^2. (I.e. it is locally the level set of a C^2 function with nonvanishing gradient.) It follows that the unit outward normal $n(x)$ is C^1 on $\partial\Omega$. We assume $U \subseteq \mathbb{R}^m$ is closed (nonempty) and that $f : \bar{\Omega} \times U \to \mathbb{R}^n$ and $L : \bar{\Omega} \times U \to \mathbb{R}$ are continuous. In terms of these we define the Hamiltonian $H(x, p)$ as in (3) above. If U is unbounded we impose the growth condition

$$\frac{L(x,u)}{1 + |f(x,u)|} \to +\infty \text{ as } |u| \to \infty, \text{ uniformly over } |x| \le R, \text{ each } R. \tag{10}$$

Our continuity hypotheses are weaker than those of Lions [9, (50)] in that no Lipschitz or modulus of continuity assumptions are made on $f(\cdot, u)$, $L(\cdot, u)$, nor are they assumed bounded. On the other hand he assumes nothing like our (10). Observe that the growth assumption implies that

$$\begin{aligned} -p \cdot f(x,u) - L(x,u) &\le |p|\,|f(x,u)| - L(x,u) \\ &= -\left(\frac{L(x,u)}{1+|f(x,u)|} - |p|\right)|f(x,u)| - \frac{L(x,u)}{1+|f(x,u)|} \\ &\to -\infty \text{ as } |u| \to \infty. \end{aligned} \tag{11}$$

It follows that for $|x| \le R$ and $|p| \le R$ the $\sup_u$ in (3) can be restricted to a compact subset of U. As a consequence, $H : \bar{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ is continuous, convex in p for each $x \in \bar{\Omega}$, and the set

$$\{(x, p, u) : \; H(x,p) = -p \cdot f(x,u) - L(x,u)\} \tag{12}$$

is nonempty and closed. When (x, p, u) is in this set, we say that u *achieves the supremum in* $H(x, p)$.

Most of our results hold with no further hypotheses. However, for the supersolution equivalence result of Theorem 3 below, we assume the following affine-convex structure:

$$\begin{aligned} &U \text{ is convex,} \\ &L(x,u) \text{ is convex in } u \text{ for each } x \in \bar{\Omega}, \\ &f(x,u) = b(x) + G(x)u \text{ for some continuous functions } b, G \end{aligned} \tag{AC}$$

We will *only* assume (AC) where explicitly stated.

The vector field $\gamma : \partial\Omega \to \mathbb{R}^n$ is assumed C^1 and, as we said above, normalized by $\gamma(x) \cdot n(x) = 1$. The velocity reflection map π described above can be expressed as follows: if $x \in \Omega$ then $\pi(x, v) = v$, and if $x \in \partial\Omega$ then

$$\begin{aligned} \pi(x, v) &= v - c\gamma(x), \quad \text{where } c = \max(0, n(x) \cdot v) \\ &= \begin{cases} v & \text{if } n(x) \cdot v \le 0 \\ R(x)v & \text{if } n(x) \cdot v > 0 \end{cases}, \end{aligned} \tag{13}$$

where $R(x)$ is the *reflection matrix*

$$R(x) = I - \gamma(x) n^T(x).$$

It is easy to see that $\pi(x, v)$ is continuous with respect to v. The normalization $\gamma(x) \cdot n(x) = 1$ implies that $n(x)^T R(x) = 0$, from which we see that $n(x) \cdot \pi(x, v) \le 0$ for all v. The reflected Hamiltonian H^π is now defined as in (7) above.

The viscosity solution property is based on the semidifferential sets of a function V, assumed to be continuous on $\bar{\Omega}$. The semidifferential sets are defined in terms of smooth test functions, as follows. For $x \in \bar{\Omega}$ we say $\xi \in D^+V(x)$ $[D^-V(x)]$ if $\xi = D\phi(x)$ for some $\phi \in C^1(\bar{\Omega})$ such that x is a local maximum [minimum] of $V - \phi$ relative to $\bar{\Omega}$. It is well known that both $D^\pm V(x)$ are closed. (See [3] for typical arguments; e.g. Lemma 1.8 page 28.)

We now formulate the two versions of the Hamilton-Jacobi equation that we are interested in. First is the *boundary condition formulation*:

$$H(x, DV(x)) = 0 \text{ in } \Omega \text{ with } \gamma(x) \cdot DV(x) = 0 \text{ on } \partial\Omega. \tag{14}$$

For interior points $x \in \Omega$, we say V is a viscosity subsolution [supersolution] of (14) at x when

$$\begin{gathered} H(x, \xi) \le 0 \text{ for all } \xi \in D^+V(x), \\ [\ H(x, \xi) \ge 0 \text{ for all } \xi \in D^-V(x)\ ]. \end{gathered} \tag{15}$$

For boundary points $x \in \partial\Omega$, we say V is a viscosity subsolution [supersolution] of (14) at x when

$$\begin{gathered} \min(H(x, \xi), \gamma(x) \cdot \xi) \le 0 \text{ for all } \xi \in D^+V(x), \\ [\ \max(H(x, \xi), \gamma(x) \cdot \xi) \ge 0 \text{ for all } \xi \in D^-V(x)\ ]. \end{gathered} \tag{16}$$

This provides the sense in which $\gamma(x) \cdot DV(x) = 0$ is understood for viscosity solutions.

Example 1. As an example of the above consider

$$f(x, u) = u; \quad L(x, u) = \frac{1}{2}|x|^2 + \frac{1}{2}|u|^2,$$

with $x = (x_1, x_2) \in \mathbb{R}^2$ and $U = \mathbb{R}^2$. Take $\Omega = \{x : \ x_1 < 1\}$, so that $n(x) = (1, 0)$ on $\partial\Omega$. We take $\gamma(x) = (1, 0)$. The resulting Hamiltonian is

$$H(x, p) = \frac{1}{2}|p|^2 - \frac{1}{2}|x|^2.$$

We can easily write down several (smooth) solutions of $H(x, DV(x)) = 0$ in Ω:

$$\begin{aligned} V_A(x) &= \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \\ V_B(x) &= \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 \\ V_C(x) &= -\frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 \end{aligned}$$

We can examine (16) directly for each of these. For V_A we find that for an arbitrary $x = (1, x_2) \in \partial\Omega$,

$$\begin{aligned} D^+V_A(x) &= \{\xi = (\zeta, x_2) : \ \zeta \le 1\} \\ D^-V_A(x) &= \{\xi = (\zeta, x_2) : \ \zeta \ge 1\}. \end{aligned}$$

For $\xi \in D^- V_A(x)$ we have $\gamma(x) \cdot \xi = \zeta \geq 1$ so that the supersolution property (16) holds. For $\xi \in D^+ V_A(x)$ with $\zeta \leq 0$ the subsolution property holds because $\gamma(x) \cdot \xi = \zeta \leq 0$. For $0 < \zeta \leq 1$ the subsolution property requires $H(x, \xi) \leq 0$. This we easily check, since

$$H(x, \xi) = \frac{1}{2}(\zeta^2 - 1) \leq 0.$$

Thus V_A is indeed a viscosity solution of (14).

A nearly identical calculation (x_2 replaced by $-x_2$ throughout) shows that V_B is also a solution of (14). This provides an example of the nonuniqueness of solutions to (14). We note that both V_A and V_B are smooth, and in the context of smooth functions both the boundary condition formulation (14) and the reflected Hamiltonian formulation (17) are equivalent. See Section 4.

For V_C however the situation is different. Consider the supersolution property. We have

$$D^- V_C(x) = \{\xi = (\zeta, -x_2) : \ \zeta \geq -1\}.$$

If $-1 < \zeta < 0$ we have $\gamma(x) \cdot \xi = \zeta < 0$ so the supersolution property requires $H(x, \xi) \geq 0$. But we find

$$H(x, \xi) = \frac{1}{2}(\zeta^2 - 1) \leq 0.$$

Thus V_C is *not* a supersolution. (The subsolution property does hold however.) We will return to this example in Section 4 to see how these solution properties can be checked geometrically in terms of associated extremals.

Next we state the *reflected Hamiltonian formulation*:

$$H^\pi(x, DV(x)) = 0 \text{ for all } x \in \bar{\Omega}. \tag{17}$$

V is defined to be a viscosity subsolution [supersolution] of (17) at $x \in \bar{\Omega}$ when

$$\begin{gathered} H^\pi(x, \xi) \leq 0 \text{ for all } \xi \in D^+ V(x), \\ [\ H^\pi(x, \xi) \geq 0 \text{ for all } \xi \in D^- V(x)\ ]. \end{gathered} \tag{18}$$

For either formulation, we say V is a viscosity solution in $\bar{\Omega}$ when it is both a subsolution and supersolution at every $x \in \bar{\Omega}$. Note that for (18) and (15) are the same $x \in \Omega$, because $H(x, \cdot) = H^\pi(x, \cdot)$. Thus to say V is a solution (sub- or super-) of (14) at all interior points is the same as in reference to (17); it reduces to (15) for all $x \in \Omega$ in either case. We will usually refer to (14) when when talking exclusively about interior points, understanding that there is no distinction. Our interest is in the connection between (18) and (16) for boundary points $x \in \partial\Omega$.

The structure of $D^\pm V(x)$ for $x \in \partial\Omega$ is important to our arguments. The hypothesis that $\partial\Omega$ is C^2 implies that there exists a C^2 function $d(x)$ with the properties that $d(x) > 0$ for $x \in \Omega$, and $d(x) = 0$ and $Dd(x) = -n(x)$ for $x \in \partial\Omega$. ($d(x) = \text{dist}(x, \partial\Omega)$ near $\partial\Omega$, smoothed in the interior. See [8, Appendix] for details.) If $\phi \in C^1(\bar{\Omega})$ is such that $V - \phi$ has a local maximum at $x \in \partial\Omega$, then $V - (\phi + \lambda d)$ also has a local maximum at x for any $\lambda \geq 0$. Thus if $\xi \in D^+ V(x)$ then $\xi - \lambda n(x) \in D^+ V(x)$ for all $\lambda \geq 0$. Likewise, if $\xi \in D^- V(x)$ then $\xi - \lambda n(x) \in D^- V(x)$ for all $\lambda \leq 0$.

Given ξ in either of $D^\pm V(x)$ and $\lambda \in \mathbb{R}$, we will adopt the notation

$$\xi_\lambda = \xi - \lambda n(x).$$

Thus $\xi = \xi_0$. For $\xi_0 \in D^+ V(x)$ then our observations above say that $\xi_\lambda \in D^+ V(x)$ for all $\lambda \geq 0$. Define

$$\lambda^+ = \inf\{\lambda : \ \xi_\lambda \in D^+ V(x)\}.$$

Then $-\infty \leq \lambda^+ \leq 0$. Since $D^+ V(x)$ is closed, it follows that

$$\xi_\lambda \in D^+ V(x) \text{ for all finite } \lambda \geq \lambda^+.$$

Analogously, For $\xi_0 \in D^- V(x)$ define

$$\lambda^- = \sup\{\lambda : \ \xi_\lambda \in D^- V(x)\}.$$

Then $0 \leq \lambda^- \leq \infty$ and

$$\xi_\lambda \in D^- V(x) \text{ for all finite } \lambda \leq \lambda^-.$$

Consider any $x \in \partial\Omega$ and ξ_0 in either of $D^{\pm}V(x)$. We define

$$\lambda^* = \gamma(x) \cdot \xi_0.$$

The significance of λ^* is that

$$\begin{aligned} \gamma(x) \cdot \xi_\lambda > 0 \quad &\text{iff} \quad \lambda < \lambda^*, \\ \gamma(x) \cdot \xi_\lambda < 0 \quad &\text{iff} \quad \lambda^* < \lambda. \end{aligned} \tag{19}$$

Notice also that as a consequence of $n(x)^T R(x) = 0$ we have

$$\xi_\lambda^T R(x) = \xi_{\lambda^*}^T \quad \text{for all } \lambda. \tag{20}$$

Suppose $x \in \partial\Omega$ and $\gamma(x) \cdot \xi_\lambda \geq 0$. For any v we know $\pi(x, v) = v - c\gamma(x)$ for some $c \geq 0$, and therefore

$$\xi_\lambda \cdot \pi(x, v) \leq \xi_\lambda \cdot v.$$

It follows that $H(x, \xi_\lambda) \leq H^\pi(x, \xi_\lambda)$. In this way we see that

$$\begin{aligned} H(x, \xi_\lambda) \leq H^\pi(x, \xi_\lambda) \quad &\text{whenever } \lambda \leq \lambda^*, \\ H^\pi(x, \xi_\lambda) \leq H(x, \xi_\lambda) \quad &\text{whenever } \lambda^* \leq \lambda. \end{aligned} \tag{21}$$

Next consider any $\lambda_1 \leq \lambda_2$. For any v we have

$$(\xi_{\lambda_2} - \xi_{\lambda_1}) \cdot \pi(x, v) = (\lambda_1 - \lambda_2) n(x) \cdot \pi(x, v) \geq 0,$$

because $n(x) \cdot \pi(x, v) \leq 0$. Therefore $\xi_{\lambda_2} \cdot \pi(x, v) \geq \xi_{\lambda_1} \cdot \pi(x, v)$ for all v, which implies that $H^\pi(x, \xi_{\lambda_1}) \geq H^\pi(x, \xi_{\lambda_2})$. In other words,

$$\lambda \mapsto H^\pi(x, \xi_\lambda) \quad \text{is nonincreasing.} \tag{22}$$

The following lemma is proven as Lemma 3 in Lions [9]. (In the subsolution case, our λ^+ is the negative of Lions' λ_0.)

Lemma 1. *Assume $V \in C(\bar{\Omega})$ and $x \in \partial\Omega$. If V is a viscosity subsolution of* (14) *in the interior then for any $\xi_0 \in D^+V(x)$ with $-\infty < \lambda^+$ we have*

$$H(x, \xi_{\lambda^+}) \leq 0.$$

Likewise, if V is a viscosity supersolution of (14) *in the interior then for any $\xi_0 \in D^-V(x)$ with $\lambda^- < \infty$ we have*

$$H(x, \xi_{\lambda^-}) \geq 0.$$

Again we remind the reader that the interior solution properties are defined by (15) for $x \in \Omega$. The lemma says that certain Hamiltonian inequalities on $\partial\Omega$ follow from solution properties on the interior, without any express assumptions about the equation or conditions on the boundary.

The careful reader may also observe that for $x \in \partial\Omega$ our definition (16) of subsolution requires that

$$H(x, \xi) \leq 0 \quad \text{for all} \quad \xi \in D^+V(x) \text{ with } \gamma(x) \cdot \xi > 0,$$

whereas the formulation in Lions [9, (39)] is slightly different:

$$H(x, \xi) \leq 0 \quad \text{for all} \quad \xi \in D^+V(x) \text{ with } \gamma(x) \cdot \xi \geq 0.$$

An alternate formulation used by Lions is [9, (39')], which in our notation may be expressed as

$$\text{for each } \xi \in D^+V(x) \text{ there is } \min(0, \lambda^*) \leq \lambda \leq 0 \text{ such that } H(x, \xi_\lambda) \leq 0.$$

We will use yet another formulation in Section 3 below:

$$H(x, \xi_\lambda) \leq 0 \quad \text{for each } \xi \in D^+V(x) \text{ and all } \lambda^+ \leq \lambda \leq \max(\lambda^+, \lambda^*). \tag{23}$$

Obviously, [9, (39)] implies our (16). Both [9, (39')] and (23) imply [9, (39)] because if $\gamma(x) \cdot \xi \geq 0$ then $\lambda^* \geq 0$, so that for $\lambda = 0$ is the *only* possibility in [9, (39')] and *a* possibility in (23). (Recall that $\lambda^+ \leq 0$ in general.) So in either case we deduce $H(x, \xi) = H(x, \xi_0) \leq 0$.

Under the assumption that V is a subsolution in the interior, all of the above are equivalent. Lemma 1 is the key. The argument that [9, (39)] implies [9, (39')] is given by Lions. The following lemma records the equivalence of (16) and (23) for our use in Section 3.

Lemma 2. *Suppose V is a continuous subsolution of* (14) *in the interior and $x \in \partial\Omega$. Then V is a subsolution of* (14) *at x iff*

$$H(x, \xi_\lambda) \leq 0 \quad \text{for each } \xi \in D^+V(x) \text{ and all } \lambda^+ \leq \lambda \leq \max(\lambda^+, \lambda^*).$$

Likewise, if V is a continuous supersolution of (15) *in the interior and $x \in \partial\Omega$. Then V is a supersolution of* (14) *at x iff*

$$H(x, \xi_\lambda) \geq 0 \quad \text{for each } \xi \in D^+V(x) \text{ and all } \min(\lambda^-, \lambda^*) \leq \lambda \leq \lambda^-.$$

Proof. We consider only the subsolution case, the supersolution case being analogous. Suppose V is a continuous subsolution of (14) in the interior and $x \in \partial\Omega$. We know that $\lambda^+ \leq 0$ and $\xi_\lambda \in D^+V(x)$ for all $\lambda^+ \leq \lambda$. Therefore (16) implies

$$\min(H(x, \xi_\lambda), \gamma(x) \cdot \xi_\lambda) \leq 0.$$

If $\lambda^+ \leq \lambda < \lambda^*$, we know from (19) that $\gamma(x) \cdot \xi_\lambda > 0$, so that $H(x, \xi_\lambda) \leq 0$. If $\lambda^+ < \lambda^*$, this extends by continuity to $\lambda = \lambda^*$. Lemma 1 says that this holds for $\lambda = \lambda^+$, even if $\lambda^* \leq \lambda^+$. These facts collectively imply (23). The converse, that (23) implies (16), follows as indicated above. ☐

3. Equivalence Results

Our first result is that viscosity solutions of the reflected Hamiltonian formulation problem are necessarily solutions of the boundary condition formulation. This is contained in the proof of Lions' Theorem 12. We copy the elementary proof from [5].

Theorem 1. *Suppose that $V \in C(\bar{\Omega})$ is a viscosity subsolution [supersolution] of* (17) *at $x \in \partial\Omega$. Then V is a viscosity subsolution [supersolution] solution of* (14) *at x.*

Proof. Consider $\xi_0 \in D^+V(x)$. Then $H^\pi(x, \xi_0) \leq 0$. Suppose $\gamma(x) \cdot \xi_0 > 0$. From (19) and (21) it follows that $H(x, \xi_0) \leq H^\pi(x, \xi_0) \leq 0$. Thus

$$\min(H(x, \xi_0), \gamma(x) \cdot \xi_0) \leq 0.$$

Since this obviously holds if $\gamma(x) \cdot \xi_0 \leq 0$, V is a subsolution of (14). The supersolution argument is analogous. ☐

We are interested in a converse to Theorem 1. We will present the subsolution and supersolution parts separately, since the the extra hypotheses (AC) are only needed for the supersolution part. Note that both Theorems 2 and 3 assume that the sub- or supersolution properties hold in the interior as well as at the individual boundary point.

3.1. **Subsolutions.**

Theorem 2. *Suppose $V \in C(\bar{\Omega})$ is a viscosity subsolution of* (14) *in the interior and at $x \in \partial\Omega$. Then V is a viscosity subsolution of* (17) *at x.*

Proof. Consider $\xi_0 \in D^+V(x)$. Our goal is to show that

$$H^\pi(x, \xi_0) \leq 0. \tag{24}$$

By the hypotheses and Lemma 2 we know that

$$H(x, \xi_\lambda) \leq 0 \quad \text{for all } \lambda^+ \leq \lambda \leq \max(\lambda^+, \lambda^*). \tag{25}$$

We now deduce (24) from (25) in three cases, depending on the position of λ^* relative to $\lambda^+ \leq 0$.

First, suppose $\lambda^* \leq \lambda^+ \leq 0$. Then

$$\begin{aligned} H^\pi(x, \xi_0) &\leq H^\pi(x, \xi_{\lambda^+}), \quad \text{by (22)} \\ &\leq H(x, \xi_{\lambda^+}), \quad \text{by (21)} \\ &\leq 0, \quad \text{by (25)}. \end{aligned}$$

The case of $\lambda^+ < \lambda^* \leq 0$ is similar. We have

$$\begin{aligned} H^\pi(x, \xi_0) &\leq H^\pi(x, \xi_{\lambda^*}), \quad \text{by (22)} \\ &= H(x, \xi_{\lambda^*}), \quad \text{by (21)} \\ &\leq 0, \quad \text{by (25)}. \end{aligned}$$

In the remaining case of $\lambda^+ \le 0 < \lambda^*$, we claim that

$$H^\pi(x,\xi_0) \le \sup_{0\le\lambda\le\lambda^*} H(x,\xi_\lambda). \tag{26}$$

Since (25) implies $\sup_{0\le\lambda\le\lambda^*} H(x,\xi_\lambda) \le 0$, we see that (24) follows from (26).

To finish we need to establish (26), assuming $0 < \lambda^*$. Observe that we can use (13) to break the supremum defining $H^\pi(x,\xi)$ up into two halves, depending on the sign of $n(x)\cdot f(x,u)$.

$$H^\pi(x,\xi_0) = \max(Q_1,Q_2),$$

where Q_1 is the supremum of

$$-\xi_0\cdot f(x,u) - L(x,u)$$

over u for which $n(x)\cdot f(x,u) \le 0$, and Q_2 is the supremum of

$$-\xi_0\cdot R(x)f(x,u) - L(x,u)$$

over u for which $n(x)\cdot f(x,u) > 0$. Obviously, $Q_1 \le H(x,\xi_0)$ and from (20), $Q_2 \le H(x,R(x)^T\xi_0) = H(x,\xi_{\lambda^*})$. Thus

$$\max(Q_1,Q_2) \le \max(H(x,\xi_0),H(x,\xi_{\lambda^*})) \le \sup_{0\le\lambda\le\lambda^*} H(x,\xi_\lambda),$$

proving (26). $\square$

3.2. **Supersolutions.**

Theorem 3. *Suppose $V\in C(\bar\Omega)$ is a viscosity supersolution of* (14) *in the interior and at $x\in\partial\Omega$. If the hypotheses* (AC) *hold, then V is a viscosity supersolution of* (17) *at x.*

The proof follows the same outline. However the analogue of (26) is harder to prove, requiring the additional hypothesis (AC). We present that part of the argument separately as Lemma 3 following the main body of the proof.

Proof. Consider $\xi_0\in D^-V(x)$. Our goal is to show that

$$H^\pi(x,\xi_0)\ge 0.$$

The hypotheses and Lemma 2 imply

$$H(x,\xi_\lambda)\ge 0 \quad \text{for all } \min(\lambda^*,\lambda^-)\le\lambda\le\lambda^-. \tag{27}$$

Since $0\le\lambda^-$ there are three cases. First, if $0\le\lambda^-\le\lambda^*$, then

$$\begin{aligned} H^\pi(x,\xi_0) &\ge H^\pi(x,\xi_{\lambda^-}), \quad \text{by (22)}\\ &\ge H(x,\xi_{\lambda^-}), \quad \text{by (21)}\\ &\ge 0, \quad \text{by (27)}. \end{aligned}$$

Next, if $0\le\lambda^*<\lambda^-$ then

$$\begin{aligned} H^\pi(x,\xi_0) &\ge H^\pi(x,\xi_{\lambda^*}), \quad \text{by (22)}\\ &= H(x,\xi_{\lambda^*}), \quad \text{by (21)}\\ &\ge 0, \quad \text{by (27)}. \end{aligned}$$

In the remaining case $\lambda^*<0\le\lambda^-$ we appeal to Lemma 3 to complete the proof:

$$H^\pi(x,\xi_0)\ge \inf_{\lambda^*\le\lambda\le 0} H(x,\xi_\lambda)\ge 0,$$

by virtue of (27). $\square$

Lemma 3. *If $\lambda^*<0$ and the hypotheses* (AC) *hold, then $H^\pi(x,\xi_0)\ge\inf_{\lambda^*\le\lambda\le 0}H(x,\xi_\lambda)$.*

The proof is quite different than for (26). Indeed $\lambda\mapsto H(x,\xi_\lambda)$ is convex, so that the supremum in (26) only involved the endpoints. Here we need to consider interior minimizers as well.

Proof. By continuity there exists $\bar{\lambda}$ which minimizes $H(x, \xi_\lambda)$ over $\lambda^* \le \lambda \le 0$.

Suppose that $\lambda^* < \bar{\lambda} \le 0$. Then we can take a sequence $\lambda^* < \lambda_n < \bar{\lambda}$ with $\lambda_n \to \bar{\lambda}$. Since (12) is nonempty there exist u_n with

$$H(x, \xi_{\lambda_n}) = -\xi_{\lambda_n} \cdot f(x, u_n) - L(x, u_n).$$

Since $H(x, \xi_{\bar{\lambda}})$ is minimizing,

$$-\xi_{\lambda_n} \cdot f(x, u_n) - L(x, u_n) = H(x, \xi_{\lambda_n}) \ge H(x, \xi_{\bar{\lambda}}) \ge -\xi_{\bar{\lambda}} \cdot f(x, u_n) - L(x, u_n).$$

But this implies

$$-\xi_{\lambda_n} \cdot f(x, u_n) \ge -\xi_{\bar{\lambda}} \cdot f(x, u_n),$$

and therefore

$$n(x) \cdot f(x, u_n) \le 0.$$

The growth property (11) and the closure of (12) allow us to take a convergent subsequence $u_{n'} \to u_-$. It follows that

$$H(x, \xi_{\bar{\lambda}}) = -\xi_{\bar{\lambda}} \cdot f(x, u_-) - L(x, u_-), \quad \text{and } n(x) \cdot f(x, u_-) \le 0.$$

In particular, if $\bar{\lambda} = 0$ then $\pi(x, f(x, u_-)) = f(x, u_-)$ and we can say

$$\begin{aligned} H^\pi(x, \xi_0) &= H^\pi(x, \xi_{\bar{\lambda}}) \\ &\ge -\xi_{\bar{\lambda}} \cdot \pi(x, f(x, u_-)) - L(x, u_-) \\ &= -\xi_{\bar{\lambda}} \cdot f(x, u_-) - L(x, u_-) \\ &= H(x, \xi_{\bar{\lambda}}), \end{aligned}$$

completing the proof in the case of $\bar{\lambda} = 0$.

Next suppose $\lambda^* \le \bar{\lambda} < 0$. We can argue similarly from a sequence $\bar{\lambda} < \lambda_n < 0$ with $\lambda_n \to \bar{\lambda}$ that there exists u_+ with

$$H(x, \xi_{\bar{\lambda}}) = -\xi_{\bar{\lambda}} \cdot f(x, u_+) - L(x, u_+), \quad \text{and } n(x) \cdot f(x, u_+) \ge 0.$$

The latter inequality implies $\pi(x, f(x, u_+)) = R(x) f(x, u_+)$. Recall also from (20) that $\xi_0^T R(x) = \xi_{\lambda^*}^T$. Thus if $\bar{\lambda} = \lambda^*$ we have

$$\begin{aligned} H^\pi(x, \xi_0) &\ge -\xi_0 \cdot \pi(x, f(x, u_+)) - L(x, u_+) \\ &= -\xi_0 \cdot R(x) f(x, u_+) - L(x, u_+) \\ &= -\xi_{\lambda^*} \cdot f(x, u_+) - L(x, u_+) \\ &= H(x, \xi_{\lambda^*}), \end{aligned}$$

completing the proof in the case of $\bar{\lambda} = \lambda^*$.

There remains the case of $\lambda^* < \bar{\lambda} < 0$. Now both u_- and u_+ as described above exist. In this case we invoke the hypotheses (AC), which imply that the set of u satisfying

$$H(x, \xi_{\bar{\lambda}}) = -\xi_{\bar{\lambda}} \cdot f(x, u) - L(x, u)$$

is convex. Using the linearity of f in u we can take a convex combination $\bar{u}$ of $u_\pm$ for which

$$n(x) \cdot f(x, \bar{u}) = 0.$$

That means that $\xi_0 \cdot \pi(x, f(x, \bar{u})) = \xi_0 \cdot f(x, \bar{u}) = \xi_{\bar{\lambda}} \cdot f(x, \bar{u})$, and so

$$\begin{aligned} H^\pi(x, \xi_0) &\ge -\xi_0 \cdot \pi(x, f(x, \bar{u})) - L(x, \bar{u}) \\ &= -\xi_{\bar{\lambda}} \cdot f(x, \bar{u}) - L(x, \bar{u}) \\ &= H(x, \xi_{\bar{\lambda}}), \end{aligned}$$

completing the proof. □

We note for future reference that only the case of $\lambda^* < \bar{\lambda} < 0$ required the extra hypotheses (AC).

4. Smooth Solutions

We now restrict our attention to smooth functions: $V \in C^1(\bar{\Omega})$. For either notion of solution it is necessary that $H(x, DV(x)) = 0$ pointwise in Ω, and by continuity on $\partial\Omega$ as well. We are interested in understanding the additional properties of V on $\partial\Omega$ that make it a viscosity solution of (17). For $x \in \partial\Omega$ and $\xi_0 = DV(x)$, the differentiability of V at x implies that $\lambda^{\pm} = 0$ and $D^{\pm}V(x)$ can be described completely:

$$\begin{aligned} D^+V(x) &= \{\xi_\lambda : \ 0 \le \lambda\} \\ D^-V(x) &= \{\xi_\lambda : \ \lambda \le 0\}. \end{aligned} \tag{28}$$

An elementary consequence is that for smooth functions V, the viscosity interpretation of the reflected Hamiltonian equation (17) is equivalent to the pointwise or classical interpretation. This is *not* the case for (14); in Example 1 neither V_A nor V_B satisfy the boundary condition $\gamma(x) \cdot DV(x) = 0$ in a classical sense.

Theorem 4. *Suppose $V \in C^1(\bar{\Omega})$ and $x \in \partial\Omega$. Then V is a viscosity subsolution [supersolution] of* (17) *at x iff $H^\pi(x, DV(x)) \le 0$ [≥ 0].*

Proof. Let $\xi_0 = DV(x)$. Assume that $H^\pi(x, \xi_0) \le 0$. For $0 \le \lambda$, (22) implies

$$H^\pi(x, \xi_\lambda) \le H^\pi(x, \xi_0) \le 0.$$

By virtue of our identification of $D^+V(x)$ in (28), this means that V is a subsolution of (17) at x. Conversely, if V is a subsolution of (17) at x, then since $\xi_0 \in D^+V(x)$ it follows that $H^\pi(x, \xi_0) \le 0$. The supersolution version is analogous. □

Our smoothness assumption also eliminates the need for (AC) in Theorem 3. However we need to strengthen the hypothesis that V is a supersolution in the interior to the assumption that it is a solution in the interior.

Theorem 5. *Suppose $V \in C^1(\bar{\Omega})$ and is a solution of* (14) *in the interior and a viscosity supersolution of* (14) *at $x \in \partial\Omega$. Then V is a viscosity supersolution of* (17) *at x.*

Proof. Let $\xi_0 = DV(x)$. Inspecting the proof of Theorem 3 we see that (AC) was only needed for the case of $\lambda^* < \bar{\lambda} < 0$. Our hypotheses imply $H(x, \xi_0) = 0$. Lemma 2 and the supersolution property imply $H(x, \xi_\lambda) \ge 0$ for all $\lambda^* \le \lambda \le 0$. Thus $\lambda = 0$ minimizes $H(x, \xi_\lambda)$ over $\lambda^* \le \lambda \le 0$. I.e. $\bar{\lambda} = 0$. Therefore we can conclude $H^\pi(x, \xi_0) \ge 0$, without needing to resort to the extra hypotheses (AC). □

For $V \in C^1(\bar{\Omega})$ which solves (14) in the interior, Theorems 1, 2, 4 and 5 show that V is a viscosity solution of (14) al $x \in \partial\Omega$ iff $H^\pi(x, DV(x)) = 0$. As explained in the introduction, we want to explore conditions which are necessary and/or sufficient for this and which can be expressed in terms of $f(x, u^*)$ for those u^* which achieve the maximum in the expression

$$-DV(x) \cdot f(x, u) - L(x, u)$$

defining $H(x, DV(x))$. Proceeding, we fix $x \in \partial\Omega$ and $\xi_0 = DV(x)$. We know from continuity and our hypothesis that V is a solution in the interior that $H(x, \xi_0) = 0$. Since $\lambda^{\pm} = 0$, Lemma 2 tells us that the subsolution property (25) is equivalent to

$$H(x, \xi_\lambda) \le 0 \quad \text{for all } 0 \le \lambda < \lambda^*, \tag{29}$$

holding vacuously if $\lambda^* \le 0$. The supersolution property (27) is equivalent to

$$H(x, \xi_\lambda) \ge 0 \quad \text{for all } \lambda^* < \lambda \le 0, \tag{30}$$

holding vacuously if $\lambda^* \ge 0$. There are three cases to consider, depending on the sign of λ^*.

4.1. Case 1: $\gamma(x) \cdot DV(x) < 0$.

This is the case of $\lambda^* < 0$, so that the subsolution inequality (29) holds vacuously. Theorem 4 implies $H^\pi(x, \xi_0) \le 0$.

Suppose the supersolution inequality (30) holds. As in the proof of Lemma 3 above there exists a $u^* \in U$ achieving $H(x, \xi_0) = 0$ with

$$n(x) \cdot f(x, u^*) \le 0. \tag{31}$$

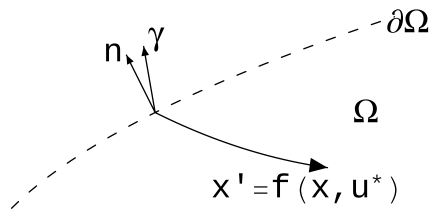

Figure 1. Extremal for Case 1

Conversely, if such a u^* exists then $\lambda \mapsto \xi_\lambda \cdot f(x,u^*)$ is nondecreasing, and so for $\lambda \le 0$ we have

$$\begin{aligned} H(x,\xi_\lambda) &\ge -\xi_\lambda \cdot f(x,u^*) - L(x,u^*) \\ &\ge -\xi_0 \cdot f(x,u^*) - L(x,u^*) \\ &= H(x,\xi_0) \\ &= 0, \end{aligned}$$

which implies (30). Thus the supersolution property is equivalent to the existence of $u^* \in U$ achieving $H(x,\xi_0) = 0$ and satisfying (31).

Given that the supersolution property holds, we have $H^\pi(x,\xi_0) = 0$. In a control problem we also care about the control values achieving the $\sup_u$ defining $H^\pi(x,\xi_0)$, since these give the optimal control values for (2) on $\partial\Omega$. In this regard consider any u^* as described above. Observe that (31) implies $\pi(x,f(x,u^*)) = f(x,u^*)$, and so

$$\begin{aligned} 0 &= -\xi_0 \cdot f(x,u^*) - L(x,u^*) \\ &= -\xi_0 \cdot \pi(x,f(x,u^*)) - L(x,u^*) \\ &\le H^\pi(x,\xi_0) \\ &\le H(x,\xi_0), \quad \text{by (21)} \\ &= 0. \end{aligned}$$

Thus the $\sup_u$ in $H^\pi(x,\xi_0) = 0$ is achieved by u^*. Conversely, for any $u^* \in U$ which achieves $H^\pi(x,\xi_0) = 0$ we find that (31) is necessary. Indeed otherwise $n(x) \cdot f(x,u^*) > 0$ would imply that $\pi(x,f(x,u^*)) = f(x,u^*) - c\gamma(x)$ for some $c > 0$. Since $\gamma(x) \cdot \xi_0 < 0$ this leads to

$$\xi_0 \cdot \pi(x,f(x,u^*)) = \xi_0 \cdot (f(x,u^*) - c\gamma(x)) > \xi_0 \cdot f(x,u^*),$$

which produces a contradiction:

$$0 = H^\pi(x,\xi_0) = -\xi_0 \cdot \pi(x,f(x,u^*)) - L(x,u^*) < -\xi_0 \cdot f(x,u^*) - L(x,u^*) \le H(x,\xi_0) = 0.$$

Thus (31) holds, which means that u^* achieves the supremum in $H(x,DV(x))$ as well. At such boundary points any extremal has $\pi(x,f(x,u^*)) = f(x,u^*)$; there are no active Skorokhod dynamics. Figure 1 illustrates the geometry of such an extremal.

Here are our conclusions in this case.

Lemma 4. *Suppose that $V \in C^1(\bar{\Omega})$ is a solution of (29) in the interior and that $x \in \partial\Omega$ is a boundary point at which $\gamma(x) \cdot DV(x) < 0$. Then V is a subsolution of (14) at x. The supersolution property holds iff there exists u^* achieving the supremum in $H(x,DV(x))$ and for which*

$$n(x) \cdot f(x,u^*) \le 0.$$

These u^* are precisely those which achieve the supremum in $H^\pi(x,DV(x)) = 0$.

Example 2. We reexamine V_C from Example 1 above. For $x = (1, x_2) \in \partial\Omega$ we have $\xi_0 = DV_C(x) = -x$. So $\gamma(x) \cdot \xi_0 = -1 < 0$, placing this example in Case 1. Lemma 4 tells us that V_C is a necessarily a supersolution, but to be a subsolution we need to check that $n(x) \cdot f(x, u^*) \leq 0$ for some u^* achieving $H(x, \xi_0) = 0$. We easily see that the unique u^* achieving the $\sup_u$ in $H(x, p)$ is $u^* = -p$. Thus for $p = \xi_0 = -x$ we have $u^* = x$ and so $f(x, u^*) = x$. Since $n(x) \cdot x = 1 \not\leq 0$, the subsolution property fails. We already knew this from our calculations in Example 1. The point is that Lemma 4 gives the same conclusions without reference to the semidifferential sets, using $f(x, u^*)$ instead.

4.2. **Case 2: $\gamma \cdot DV(x) > 0$.** This is the case of $\lambda^* > 0$, so the supersolution inequality (30) holds vacuously, and Theorem 4 implies $H^\pi(x, \xi_0) \geq 0$. Since $H(x, \xi_0) = 0$, the convexity of $\lambda \mapsto H(x, \xi_\lambda)$ makes (29) equivalent to

$$H(x, \xi_{\lambda^*}) \leq 0.$$

Suppose (29) does hold and u^* achieves the supremum in $H(x, \xi_0)$. Then for any $0 < \lambda < \lambda^*$ we have that

$$-\xi_0 \cdot f(x, u^*) - L(x, u^*) = H(x, \xi_0) = 0 \geq H(x, \xi_\lambda) \geq -\xi_\lambda \cdot f(x, u^*) - L(x, u^*).$$

This implies that $(\xi_0 - \xi_\lambda) \cdot f(x, u^*) \leq 0$ and therefore (31) again holds. As a consequence, $\pi(x, f(x, u^*)) = f(x, u^*)$. Therefore

$$0 = -\xi_0 \cdot f(x, u^*) - L(x, u^*) = -\xi_0 \cdot \pi(x, f(x, u^*)) - L(x, u^*).$$

Since (29) implies $H^\pi(x, \xi_0) = 0$, this means that u^* also achieves the supremum in $H^\pi(x, \xi_0) = 0$.

We need to consider if there might be other $\bar{u}$ achieving the supremum in $H^\pi(x, \xi_0) = 0$ for which $n(x) \cdot f(x, \bar{u}) > 0$. If so, $\pi(x, f(x, \bar{u})) = R(x) f(x, \bar{u})$ so that

$$\begin{aligned} 0 &= -\xi_0 \cdot R(x) f(x, \bar{u}) - L(x, \bar{u}) \\ &= -\xi_{\lambda^*} \cdot f(x, \bar{u}) - L(x, \bar{u}) \\ &\leq H(x, \xi_{\lambda^*}) \\ &\leq 0. \end{aligned}$$

Thus such a $\bar{u}$ can only exist if $H(x, \xi_{\lambda^*}) = 0$ and must achieve this supremum. Conversely, if $H(x, \xi_{\lambda^*}) = 0$ and $\bar{u}$ achieves this supremum, then we argue as above that

$$-\xi_{\lambda^*} \cdot f(x, \bar{u}) - L(x, \bar{u}) = H(x, \xi_{\lambda^*}) = H(x, \xi_0) \geq -\xi_0 \cdot f(x, \bar{u}) - L(x, \bar{u}).$$

Therefore $(\xi_0 - \xi_{\lambda^*}) \cdot f(x, \bar{u}) \geq 0$, which implies that $n(x) \cdot f(x, \bar{u}) \geq 0$. This in turn means that $\pi(x, f(x, \bar{u})) = R(x) f(x, \bar{u})$, so that

$$-\xi_{\lambda^*} \cdot \pi(x, f(x, \bar{u})) - L(x, \bar{u}) = -\xi_0 \cdot R(x) f(x, \bar{u}) - L(x, \bar{u}) = H(x, \xi_{\lambda^*}) = 0.$$

Thus $\bar{u}$ also achieves the supremum for $H^\pi(x, \xi_0) = 0$. The following collects these conclusions.

Lemma 5. *Suppose that $V \in C^1(\bar{\Omega})$ a solution of (14) in the interior and that $x \in \partial\Omega$ is a boundary point at which $\gamma(x) \cdot DV(x) > 0$. Then V is a supersolution of (14) at x. The subsolution property holds iff $H(x, \xi_{\lambda^*}) \leq 0$. Any u^* which achieves the supremum in $H(x, \xi_0)$ has $n(x) \cdot f(x_0, u^*) \leq 0$ and achieves the supremum defining $H^\pi(x, \xi_0)$. If $H(x, \xi_{\lambda^*}) < 0$ there are no other u achieving the supremum in $H^\pi(x, \xi_0)$. If $H(x, \xi_{\lambda^*}) = 0$ then any $\bar{u}$ which achieves the supremum in $H(x, \xi_{\lambda^*})$ has $n(x) \cdot f(x, \bar{u}) \geq 0$ and achieves the supremum defining $H^\pi(x, \xi_0)$. There are no u achieving the supremum in $H^\pi(x, \xi_0)$ other than the u^* and $\bar{u}$ just described.*

If $H(x_0, \xi_{\lambda^*}) < 0$ the geometry of the extremals are the same as in Figure 1. However if $H(x_0, \xi_{\lambda^*}) = 0$ there can be an additional *boundary extremal* through x with $n(x) \cdot f(x_0, \bar{u}) > 0$ so that $\dot{x} = \pi(x, f(x, \bar{u}))$ moves along $\partial\Omega$ with active Skorokhod dynamics, as illustrated in Figure 2.

Example 3. We reexamine V_A from Example 1 above. For $\xi_0 = DV_A(x) = (1, x_2)$ we have

$$\lambda^* = \gamma(x) \cdot \xi_0 = (1, 0) \cdot (1, x_2) = 1 > 0,$$

confirming that this falls in Case 2. Lemma 5 tells us that V_A is a supersolution, and that to be a subsolution we need $H(x, \xi_{\lambda^*}) \leq 0$. We calculate

$$\xi_{\lambda^*} = \xi - \lambda^* n(x) = (1, x_2) - (1, 0) = (0, x_2).$$

and $H(x, \xi_{\lambda^*}) = -\frac{1}{2} \leq 0$, confirming the subsolution property. The calculation for V_B works out similarly.

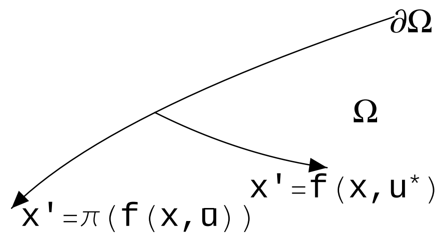

FIGURE 2. Extremal and boundary extremal for Case 2

4.3. **Case 3:** $\gamma(x) \cdot DV(x) = 0$. This is the case in which the boundary condition holds in the classical sense. Let $\xi_0 = DV(x)$. Since $\lambda^* = 0$, both of (29) and (30) now hold, and $H^\pi(x, \xi_0) = 0$. Moreover observe that $\gamma(x) \cdot \xi_0 = 0$ is equivalent to $\xi_0^T R(x) = \xi_0^T$, so that $\xi_0 \cdot \pi(x, v) = \xi_0 \cdot v$ for all v, which means that the u^* which achieve the minimum in $H^\pi(x, \xi_0)$ are the same as those achieve the minimum in $H(x, \xi_0)$. For consistency with the other cases, we state this as a lemma.

Lemma 6. *Suppose that $V \in C^1(\bar{\Omega})$ a solution of* (14) *in the interior and that $x \in \partial\Omega$ is a boundary point at which $\gamma(x) \cdot DV(x) = 0$. Then V is a viscosity solution of* (14) *at x and u^* achieves the supremum in $H^\pi(x, DV(x)) = 0$ iff it does in $H(x, DV(x)) = 0$.*

Notice that in the previous cases, $\lambda \mapsto H(x, \xi_\lambda)$ is nonincreasing on one side of $\lambda = 0$ or the other. That may be the situation in the present case as well, in which event $H^\pi(x, \xi_0)$ is again achieved by a u^* with $n(x) \cdot f(x, u^*) \leq 0$, as illustrated in Figure 1. However in the present case it is also possible that $\lambda \mapsto H(x, \xi_\lambda)$ be increasing at $\lambda = 0$. In that event, any u^* achieving the maximum in $H(x, \xi_0)$ must satisfy $n(x) \cdot f(x, u^*) > 0$: for $\lambda < 0$ we must have

$$-\xi_\lambda \cdot f(x, u^*) - L(x, u^*) \leq H(x, \xi_\lambda) < H(x, \xi_0) = -\xi_0 \cdot f(x, u^*) - L(x, u^*).$$

This implies $(\xi_0 - \xi_\lambda) \cdot f(x, u^*) < 0$, which implies $n(x) \cdot f(x, u^*) > 0$. Thus only in this case is it possible for extremals $x(t)$ to approach $x \in \partial\Omega$ from the interior Ω for increasing t, and then to follow $\dot{x} = \pi(x, f(x, u^*))$ on $\partial\Omega$ with nontrivial Skorokhod dynamics. Our final example provides an instance of this.

Example 4. We use the same f, L, H and Ω as in the previous examples, but now take $\gamma(x) = (1, 1)$. We wish to exhibit a solution for which extremals approach $\partial\Omega$ from the interior and then run along $\partial\Omega$ with active Skorokhod dynamics. This is this is possible only in Case 3. Thus along $\partial\Omega$ we want $\gamma \cdot DV(1, x_2) = 0$. This implies that

$$DV(1, x_2) = \rho(x_2)(-1, 1)$$

for some function $\rho(x_2)$. From $H(x, DV(x)) = 0$ we obtain $2\rho(x_2)^2 = 1 + x_2^2$, which determines $\rho(x_2)$ up to a $\pm$ sign. The optimal control is $u^* = -DV(x) = -\rho(x_2)(-1, 1)$ on $\partial\Omega$ so that $n(x) \cdot f(x, u^*) = \rho(x_2)$. Thus for an example with $n(x) \cdot f(x, u^*) \geq 0$ we want $\rho \geq 0$. We find therefore that

$$\rho(x_2) = 2^{-1/2}\sqrt{1 + x_2^2}.$$

The extremal $x(t) = (1, x_2(t))$ running along the boundary is

$$\begin{aligned}
\dot{x} &= \pi(x, u^*) \\
&= \pi(x, (\rho(x_2), -\rho(x_2))) \\
&= (\rho(x_2), -\rho(x_2)) - c\gamma, \quad \text{where } c = \rho(x_2) \\
&= (0, -2\rho(x_2)),
\end{aligned}$$

The solution (unique up to time translation) is

$$x(t) = (1, -\sinh(\sqrt{2}t)). \tag{32}$$

Integrating $\frac{\partial}{\partial x_2}V(1,x_2)=\rho(x_2)$ gives V on $\partial\Omega$:

$$V(1,x_2)=\frac{1}{2\sqrt{2}}\left(x_2\sqrt{1+x_2^2}+\sinh^{-1}(x_2)\right)+C$$

We take $C=0$. We extend V into the interior of Ω using the characteristic equations:

$$\begin{aligned}\dot{x}&=-p; \quad x(0)=(1,x_2)\\ \dot{p}&=-x; \quad p(0)=(-\rho(x_2),\rho(x_2)).\end{aligned}$$

Since $p(t)=DV(x(t))$ we propagate V along the characteristics using

$$\frac{d}{dt}V(x(t))=p(t)\cdot\dot{x}(t)=-|p(t)|^2; \quad V(x(0))=\frac{1}{2\sqrt{2}}\left(x_2\sqrt{1+x_2^2}+\sinh^{-1}(x_2)\right).$$

A collection of the resulting characteristics are plotted in Figure 3. A typical extremal $\dot{x}=f(x,u^*)$ follows a characteristic curve as it runs through the interior, and upon meeting the boundary follows the boundary extremal (32) downward.

Although the characteristic equations are not difficult to integrate, it does not appear possible to present a closed form expression for $V(x)$, since that would require solving for the parameters x_2, t in $x=x(t)$ for a given $x\in\Omega$. Secondly, the characteristics only provide a smooth V in a neighborhood of $\partial\Omega$. A complicated singularity arises as we follow them further into the interior.

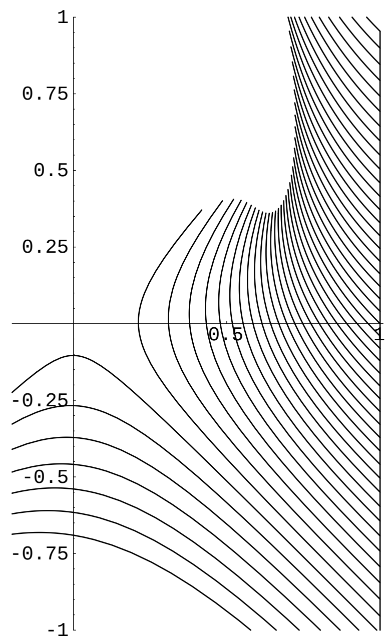

Figure 3. Characteristics for Example 4

5. Closing Remarks

We have assumed that $\partial\Omega$ is smooth and, since we only contemplated control problems (not games), that $H(x,p)$ is convex in p. Some of what we have said generalizes to the case in which Ω is a convex polygon and the more general H of a differential game. Theorem 1 part a) of [5] generalizes Theorem 4, and part b) generalizes Theorem 1.

We close with some comments on the work of Dupuis, Ishii and Soner [7]. They consider a problem of exponential asymptotics for exit probabilities of a queueing system, arriving at a Hamilton-Jacobi equation with boundary conditions and associated control problem to characterize the key limiting function. However the processes they consider are not associated with a pure Skorokhod problem. Even at smooth boundary points multiple constraint directions $\gamma(x)$ are involved, corresponding to independent components of the control. Rather than a single boundary condition like our $\gamma(x) \cdot DV(x) = 0$, each constraint direction is associated with a modified version of the Hamiltonian. These different Hamiltonians are combined at boundary points using a minimum or maximum similar to our (16); see (33) below. However we can compare their formulation to ours if we consider just boundary points in a smooth face of Ω (say their Γ_2, where $x_2 = 0$) and set one of their jump rate parameters to 0 (say $\gamma = 0$) so that only one jump direction is capable of of exiting across the boundary face, and consequently only a single constraint direction is relevant ($\gamma(x) = (1, -1)$, associated with their β).

In our notation the control can be considered in terms of two independent components:

$$u = (u_0, u_1).$$

The state velocity and running cost both depend separately on u_0 and u_1:

$$\begin{aligned} f(x, u) &= f_0(u_0) + f_1(u_1), \\ L(x, u) &= L_0(u_0) + L_1(u_1). \end{aligned}$$

To make this more specific, using our notation on the left and theirs on the right (see [7] (3.3a)),

$$\begin{aligned} u_0 &= t_2 \\ u_1 &= (t_1, t_4, t_5) \\ f_0(u_0) &= \beta t_2 v_\beta \\ f_1(u_1) &= \lambda t_1 v_\lambda + \alpha t_4 v_\alpha + \mu t_5 v_\mu \\ L_0(u_0) &= \beta h(t_2) \\ L_1(u_1) &= \lambda h(t_1) + \alpha h(t_4) + \mu h(t_5), \end{aligned}$$

where $h(t)$ is the entropy-cost

$$h(t) = \begin{cases} t \ln(t) - t + 1 & \text{for } t \geq 0 \\ +\infty & \text{for } t < 0, \end{cases}$$

and the individual jump vectors are (see [7] Figure 2)

$$\begin{aligned} v_\beta &= (1, -1) \\ v_\lambda &= (1, 0) \\ v_\alpha &= (-1, 0) \\ v_\mu &= (-1, 1). \end{aligned}$$

To make L finite valued, take $U = [0, \infty)^4$. The (interior) Hamiltonian works out to be (see [7] (2.8a))

$$H(x, p) = \beta g(-p \cdot v_\beta) + \lambda g(-p \cdot v_\lambda) + \alpha g(-p \cdot v_\alpha) + \mu g(-p \cdot v_\mu),$$

where our p replaces their (p, q) and

$$g(s) = e^s - 1.$$

For the face Γ_2 the outward normal is $n(x) = (0, -1)$ and our constraint vector would be

$$\gamma(x) = v_\beta.$$

Not all these specifics are important for what we have to say. The key features are the separation of f and L with respect to u_0 and u_1, that $n(x) \cdot f_1(u_1) \leq 0$ and that $f_0(u_0)$ is a nonnegative scalar multiple of $\gamma(x)$.

The modified Hamiltonian for Γ_2 is

$$H_{\partial,2}(x, p) = \sup_{u_1} \left\{ -p \cdot f_1(u_1) - L_1(u_1) \right\}.$$

In contrast to our (16), their formulation of the viscosity-sense boundary conditions is (see (2.19b))

$$\begin{aligned} &\min(H(x, \xi), H_{\partial,2}(x, \xi)) \leq 0 \text{ for all } \xi \in D^+V(x), \\ &[\ \max(H(x, \xi), H_{\partial,2}(x, \xi)) \geq 0 \text{ for all } \xi \in D^-V(x)\ ]. \end{aligned} \tag{33}$$

The Appendix in [7] explains why (33) implies our (16). Our observation in the next paragraph is that (33) in turn follows from (18) above. Thus their formulation falls between the two we have considered here. Moreover the definitions of $f(x, u)$ and $L(x, u)$ satisfy the affine-convex hypotheses (AC), so that Theorem 3 implies the equivalence of all three formulations (locally on the smooth face Γ_2; the corners are outside the scope of our treatment here).

Finally we argue that (18) implies (33), as claimed above. Consider the subsolution assertion for $\xi_0 \in D^+V(x)$, $x \in \Gamma_2$. If $\xi_0 \cdot \gamma(x) \geq 0$, then just as in the proof of Theorem 1 above it follows that $H(x,\xi_0) \leq H^\pi(x,\xi_0)$. Suppose then that $\xi_0 \cdot \gamma(x) < 0$. We claim that $H_{\partial,2}(x,\xi_0) \leq H^\pi(x,\xi_0)$. To see this consider any u_1 and take $u_0 = 1$. This makes $L_0(u_0) = 0$ and $f(x,u) = \beta v_\beta + f_1(u_1)$. Since $n(x) \cdot f_1(u_1) \geq 0$, it follows that for $u = (1, u_1)$

$$\pi(x, f(x,u)) = cv_\beta + f_1(u_1) \text{ for some } 0 \leq c \leq \beta,$$

and therefore

$$-\xi_0 \cdot f_1(u_1) \leq -\xi_0 \cdot (cv_\beta + f_1(u_1)) = -\xi_0 \cdot \pi(x, f(x,u)).$$

Thus for $u = (1, u_1)$ we have

$$-\xi_0 \cdot f_1(u_1) - L_1(u_1) \leq -\xi_0 \cdot f(x,u) - L(x,u).$$

Taking the supremum only with respect to u_1 on the left produces $H_{\partial,2}(x,\xi_0)$ while the supremum over all $u = (u_0, u_1)$ on the right gives $H^\pi(x,\xi_0)$. We find that

$$H_{\partial,2}(x,\xi_0) \leq H^\pi(x,\xi_0),$$

and therefore for all $\xi_0 \in D^+V(x)$ we have

$$\min(H(x,\xi_0), H_{\partial,2}(x,\xi_0)) \leq H^\pi(x,\xi_0).$$

Thus a subsolution of (18) must also be a subsolution of (33). The supersolution assertion being analogous, this concludes our argument.

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