Weak Convergence and Fluid Limits in Optimal Time-to-Empty Queueing Control Problems

Martin V. Day*
Department of Mathematics
Virginia Tech
Blacksburg, Virginia 24061
day@math.vt.edu

June 24, 2011

Abstract

We consider a class of controlled queue length processes, in which the control allocates each server's effort among the several classes of customers requiring its service. Served customers are routed through the network according to (prescribed) routing probabilities. In the fluid rescaling, $X^n(t) = \frac{1}{n}X(nt)$, we consider the optimal control problem of minimizing the integral of an undiscounted positive running cost until the first time that $X^n = 0$. Our main result uses weak convergence ideas to show that the optimal value functions V^n of the stochastic control problems for $X^n(t)$ converge (as $n \to \infty$) to the optimal value V of a control problem for the limiting fluid process. This requires certain equicontinuity and boundedness hypotheses on $\{V^n\}$. We observe that these are essentially the same hypotheses that would be needed for the Barles-Perthame approach in terms of semicontinuous viscosity solutions. Sufficient conditions for these equicontinuity and boundedness properties are briefly discussed.

1 Introduction

A theme in recent work on queueing networks is the use of deterministic models as tools to study stability and performance properties of stochastic scheduling and routing strategies. Suppose $X(t)$ is the queue length process for a queueing network and for each positive integer n consider the rescaled process $X^n(t) = \frac{1}{n}X(nt)$. Fluid processes $x(t)$ arise as deterministic (weak) limits of $X^n(t)$ as $n \to \infty$. The fluid process $x(t)$ inherits a control structure from $X(t)$, and the consideration of optimal control problems for it has been a fruitful tool in the design of high-performance stochastic controls for the queueing process $X(\cdot)$. See Chen et. al. [5]; Meyn [18], [19], [17]; and Nazarathy and Weiss [22] for a sampling of recent work along these lines.

The two performance criteria which have most frequently been considered for fluid limits are

- the *time-to-empty* (or clearing time): τ_0,
- the *holding cost* until empty $\int_0^{\tau_0} c \cdot x(t)\, dt$, where $c = (c_1, \ldots, c_d)$, $c_i \geq 0$.

Here τ_0 is the first time the fluid system is empty: $x(\tau_0) = 0$. These are instances of a general class of control problems in which the objective is to minimize a cost functional of the form

$$\int_0^{\tau_0} L(x(t), u(t))\, dt, \tag{1}$$

where $u(t)$ is the deterministic control governing the fluid process $x(t)$. We would expect the fluid cost (1) to agree with the limit as $n \to \infty$ of an analogous mean cost for the rescaled queueing processes:

$$E\left[\int_0^{\tau_0^n} L(X^n(t), U^n(t))\, dt\right], \tag{2}$$

*This paper is an exposition of the results first presented in Fluid Limits of Stochastic Optimal Queueing Network Control Problems, MTNS 2008, Blacksburg, VA, August 1, 2008.

$U^n(t)$ being the stochastic control governing $X^n(\cdot)$, and τ_0^n the first time the state $X^n(t)$ reaches 0. (In the future we will drop the superscript n because we will view τ_0 as a functional on path space, so that $\tau_0^n = \tau_0(X^n(\cdot))$.) We refer to both (1) and (2) as *time-to-empty* problems.

Our purpose in the present paper is to establish convergence of the *optimal* value functions for the control problems (2) to that of (1), as $n \to \infty$:

$$\lim_{n\to\infty} \inf_{U^n(\cdot)} E\left[\int_0^{\tau_0^n} L(X^n(t), U^n(t))\, dt\right] = \inf_{u(\cdot)} \int_0^{\tau_0} L(x(t), u(t))\, dt. \tag{3}$$

We will do this for a moderately general class of processes X^n (see Section 2) and under hypotheses on the running cost L that includes the two particular choices above. (Allowing the cost to depend on the control is relatively rare in the queueing literature. Chen et. al. [5] is one of the few references that consider a cost of that type.) We will not consider techniques for constructing optimal controls $u(\cdot)$ or $U^n(\cdot)$; we focus exclusively on the mathematical issues associated with the convergence (3).

This convergence question has been answered previously for cost functionals using a fixed finite time horizon ($T < \infty$) or an infinite horizon with discounting ($\gamma > 0$):

$$E_x\left[\int_0^T L(X^n(t), U^n(t))\, dt\right] \quad \text{or} \quad E_x\left[\int_0^\infty e^{-\gamma t} L(X^n(t), U^n(t))\, dt\right]. \tag{4}$$

See Kushner [16], Bäuerle [2], Nazarathy and Weiss [22], and also Pang and Day [23]. But so far as we know the queueing literature has not considered the asymptotic behavior of the minimal value of (2), in which there is no discounting and the upper limit is the stopping time τ_0^n rather than a fixed T. Additional difficulties arise for such time-to-empty problems. For one thing, in the absence of discounting it is not enough to establish weak convergence of the (optimally controlled) $X^n(t)$ on each finite time interval $[0, T]$. Some way to control the size of τ_0^n is also needed. Secondly, τ_0^n has very poor continuity properties as a function of the paths $X^n(\cdot)$. Since the ideas of weak convergence are all based on the expected values of *continuous* functionals, this is a difficulty for the analysis of $\int_0^{\tau_0^n} L$ which is not present for the problems (4).

We note, however, that there is a body of work in the literature that connects the value function for the fluid problem (1) to a *different* optimization problem for the queueing system X. We refer to several studies concerned with minimizing the mean of a linear cost, $c \cdot X$, with respect to the system's *steady-state distribution*. The steady-state mean of $c \cdot X$ generally has no dependence on initial condition, but is usually analyzed using a so-called *relative* value function $h(x)$. Two papers of S. Meyn, [20] and [21], consider the sequence of relative value functions h^n produced by policy iteration for the steady-state problem for X and show that in a suitable renormalization they converge to a function $h(x)$. It is shown that for large x, $h(x)/|x|^2$ approximates the fluid value function for our problem (1) above. (The minimum of $\int_0^{\tau_0} c \cdot x(t)\, dt$ typically agrees with the minimum of $\int_0^\infty c \cdot x(t)\, dt$, since once the fluid process reaches 0 it can typically be held there permanently by the control: $x(t) = 0$ all $t > \tau_0$. (See the notion of "weak stability" as defined in [7].) Our fluid value function appears in those references in the $\int_0^\infty c \cdot x(t)\, dt$ form.) We might also mention Meyn's paper [19], in which the purpose is to provide theoretical results (such as regularity and structural properties) for the fluid value function to support various applications in the study of stochastic models, again with reference to steady state performance criteria. In contrast to those studies, our concern here is with the direct comparison of the optimal value functions for the stochastic and fluid versions of *the same* time-to-empty cost, rather than connections of the fluid value function to steady-state optimization.

While motivated by queueing applications, we concentrate on (3) as a mathematical problem. We treat it using general ideas of weak convergence of probability measures. For that reason we work in a somewhat general stochastic control formulation, without using the workload and server allocation formulations that are more specialized to queueing applications. The process X^n is defined using controlled point processes in conjunction with a Skorokhod problem; see (12) below. The interpretation of $X^n(t) = \frac{1}{n}X(nt)$ as a rescaled version of $X(\cdot)$ is retained *only* through the factors of n appearing in (5) and the rates for the controlled point processes. The controls $U^n(t)$ can be any adapted (progressively measurable) processes. Because in (3) the intent is to optimize separately for each value of n, there is no presumption that the $U^n(\cdot)$ for different n are related to each other as different rescalings of some common control for the original queueing process $X(\cdot)$. Moreover, we do not limit ourselves to controls which are functions of the current state or any other particular structural form. We define the fluid process $x(t)$ independently with its own control process $u(t)$. A benefit of the Skorokhod problem formulation (the Γ in (12)) is that nonnegativity of the process components is *not* a constraint on the controls. Thus our formulation allows the controls $U^n(\cdot)$ and $u(\cdot)$ to be viewed as comparable objects in the common metric space of relaxed controls. This is the space in which we utilize weak convergence results.

Section 2 describes the class of processes we consider, and develops their representation in terms of a Skorokhod problem and fluid processes with a martingale perturbation. Section 3 describes the control problems, including hypotheses on the running cost L, and formulates the space of relaxed controls.

With these preparations our convergence problem reduces to questions of continuity and weak convergence in the space of relaxed controls. Half of the convergence we are after, Theorem 1, is essentially the manifestation of a lower semi-continuity property. This is developed in Section 4. The full convergence result, Theorem 2, is developed in Section 5. It depends on certain equicontinuity and boundedness hypotheses for the sequence of value functions for the control problems (2).

Section 6 concludes with brief discussions of two issues. We consider the so-called Barles-Perthame procedure based on the theory of viscosity solutions as an alternative to our weak convergence approach. We observe that both the hypotheses and conclusions of Theorem 2 are essentially the same as what the Barles-Perthame approach would require. Finally we consider how one might establish the equicontinuity and boundedness hypotheses of Theorem 2. Lemma 11 presents sufficient conditions in terms of the original queueing process $X(t) = X^1(t)$. We propose (but do not develop) an alternate approach which would establish sufficiency in terms of the fluid limit process $x(\cdot)$ instead of $X(\cdot)$.

2 Process Representation

The class of queueing processes we consider is what might be called "controlled multi-class Jackson networks." There are d queues (or customer classes) indexed by $i = 1, \ldots, d$ and a number ($K \leq d$) of servers. Server m attends to the needs of a designated set $S_m \subseteq \{1, \ldots, d\}$ of the queues. Taken together $S_m,\ m = 1, \ldots, K$ partition the set $\{1, \ldots, d\}$ of queues. New customers arrive exogenously in queue i according to independent Poisson processes with rates $\lambda_i \geq 0$. The customers in queue $i \in S_m$ each require (independent) exponentially distributed amounts of service with mean $1/\mu_i$, but the server must allocate its effort among its constituent queues. This allocation is indicated by control variables $u_i \geq 0$, subject to the constraint that $\sum_{i \in S_m} u_i \leq 1$ for each m. Thus the set of admissible control values is the compact, convex set

$$\mathcal{U} = \{u \in [0,1]^d : \sum_{i \in S_m} u_i \leq 1 \text{ for each } m\}.$$

If u was held constant, a customer in queue i (once arriving at the head of the queue) would wait an exponentially distributed amount of time with mean $(u_i \mu_i)^{-1}$ for their service to be complete. Once complete, the customer moves to another queue j with probability p_{ij}, or exits the network with probability

$$p_{i0} = 1 - \sum_{j=1}^{d} p_{ij}.$$

In general u is not held constant but is replaced by a $\mathcal{U}$-valued stochastic process $U(t)$. The resulting queue length process is $X(t)$.

We want to consider a sequence ($n = 1, 2, \ldots$) of such queueing processes resulting from the usual fluid rescaling: $X^n(t) = \frac{1}{n} X(nt)$, each with its own control process $U^n(t)$. The goal is to choose each control process to minimize a time-to-empty mean cost criterion of the type described above for $X^n(\cdot)$, and then consider the convergence of the sequence of optimal costs in the limit as $n \to \infty$. Our analysis is based in expressing $X^n(\cdot)$ in terms of a Skorokhod Problem (12) applied to a martingale perturbation of a controlled fluid process (9), and then use appropriate notions of weak convergence of the control processes. This is the same approach as in Pang and Day [23]. However since random routing was not considered there, we provide in this section a summary description of the Skorokhod representation (12) of $X^n(\cdot)$.

2.1 Point Process Construction

Exogenous arrivals are given by independent Poisson point processes $A_i^n(t)$ with intensities $n\lambda_i$, each of which counts the number of arrivals in queue i occurring over $(0,t]$. Given a control process $U^n(t) \in \mathcal{U}$ (progressively measurable), the service events are generated by counting processes $N_{ij}^n(t)$ ($i = 1,\ldots,d$; $j = 0,1,\ldots d$) with intensities given by $np_{ij}\mu_j U_j^n(t)$. The interpretation is that $N_{ij}^n(t)$ counts the *potential* number of services to queue i over $(0,t]$ for which the served customer is routed $i \to j$ ($j = 0$ corresponding to exit). To be proper we should view the collection of $(A_i^n,\ N_{ij}^n(t))$ as a marked point process, with mark space $\{1,\ldots,d\} \cup (\{1,\ldots,d\} \times \{0,1,\ldots,d\})$. A mark of i corresponds to a new arrival in queue i and a mark (i,j) to a service completion in queue i followed by an $i \to j$ transition. Since the mark space is finite we can view them as a collection of individual point processes, the only dependency being the correlation among their intensities induced by the control process. The essential features are that there are no simultaneous increments among the $A_i^n(t),\ N_{ij}^n(t)$, and for any bounded previsible processes $C_i(t),\ C_{ij}(t)$ the following are martingales:

$$\int_0^t C_i(s)\,dA_i^n(s) - \int_0^t C_i(s)n\lambda_i\,ds, \quad \int_0^t C_{ij}(s)\,dN_{ij}^n(s) - \int_0^t C_{ij}(s)np_{ij}\mu_j U_i^n(s)\,ds; \tag{5}$$

see Brémaud [4].

We emphasize that $N_{ij}^n(t)$ is *only* the potential number of $i \to j$ service transitions since those services which occur when $X_i^n(t) = 0$ need to be disregarded. The construction of the true $X^n(t)$ from the $A_i^n(t)$ and $N_{ij}^n(t)$ is the origin of the Skorokhod problem representation. Let

$$e_i = (\cdots 0, \overset{i}{1}, 0 \cdots)$$

be the standard unit vectors in $\mathbb{R}^d$, and define the *service event vectors* by

$$\begin{aligned} \delta_{ij} &= e_j - e_i, \text{ for } j \neq 0, \\ \delta_{i0} &= -e_i. \end{aligned}$$

When an $i \to j$ service transition occurs, the state X^n is incremented by $\frac{1}{n}\delta_{ij}$, and an exogenous arrival in queue i increments the state by $\frac{1}{n}e_i$. We can construct the resulting (scaled) queue length process as follows.

$$X^n(t) = x_0 + \sum_{i=1}^d \frac{1}{n} e_i A_i^n(t) + \sum_{i=1}^d \sum_{j=0}^d \frac{1}{n}\delta_{ij}(N_{ij}^n(t) - \widetilde{N}_{ij}^n(t)), \tag{6}$$

where $\widetilde{N}_{ij}^n$ are the counting processes of suppressed spurious services:

$$\widetilde{N}_{ij}^n(t) = \int_0^t 1_{\{0\}}(X_i^n(s-))\,dN_{ij}^n(s).$$

If we are given the initial state x_0, $X^n(\cdot)$ *is* is determined constructively from $A_i^n(\cdot)$ and $N_{ij}^n(\cdot)$. Suppose we know X^n on $[0,t)$. Then we know $\widetilde{N}_{ij}^n$ and consequently $X^n(\cdot)$ on $[0,t]$. These values remain unchanged on $[0, t{+}T^*)$ where T^* is the next arrival or service time after t. Thus we know $X^n(\cdot)$ on $[0, t{+}T^*)$. We continue this construction through the sequence of arrival or service times. Since the intensities are all bounded there is no finite accumulation point for the sequence of arrival or service times, and so X^n is determined for all $0 \leq t < \infty$. This construction implies that $X^n(t)$ is a (progressive) function of $A_i^n(t)$ and $N_{ij}^n(t)$. We would typically want the control process $U^n(t)$ to be a (progressive) function of $X^n(t)$, and so considering $U^n(t)$ to be a progressive process on the underlying space for $A_i^n(t)$ and $N_{ij}^n(t)$ includes such state-dependent controls in particular. If we take Ω to be the canonical space of paths for our point processes and $\{\mathcal{F}_t\}$ the filtration generated by $A_i^n(t)$ and $N_{ij}^n(t)$, the existence and uniqueness result of Jacod [15, Theorem 3.6] implies that there exists a unique probability measure P on Ω with respect to which $A_i^n(t)$ and $N_{ij}^n(t)$ have the prescribed intensities. In this way any progressively measurable $\mathcal{U}$-valued control process $U^n(t)$ can be considered.

Next we want to "center" the point processes in $X^n(t)$. Using (5) we can write

$$\begin{aligned}
A_i^n(t) &= n\lambda_i t + M_i^n(t)\\
N_{ij}^n(t) &= \int_0^t n\mu_i p_{ij} U_i^n(s)\,ds + M_{ij}^n(t)\\
\widetilde{N}_{ij}^n(t) &= \int_0^t n\mu_i p_{ij} U_i^n(s) 1_{\{0\}}(X_i^n(s-))\,ds + \widetilde{M}_{ij}^n(t),
\end{aligned}$$

where $M_i^n(t)$, $M_{ij}^n(t)$, and $\widetilde{M}_{ij}^n(t)$ are all martingales. We substitute these expressions in in (6). Observe that if we define the the *mean service event vector*

$$\bar{\delta}_i = \sum_{j=0}^d p_{ij}\delta_{ij},$$

then $\sum_{j=0}^d \delta_{ij} p_{ij} = \bar{\delta}_i \sum_{j=0}^d p_{ij}$. Using this fact we find that

$$\sum_{i=1}^d \sum_{j=0}^d \delta_{ij}\widetilde{N}_{ij}^n = \sum_{i=1}^d \bar{\delta}_i \sum_{j=0}^d \widetilde{N}_{ij}^n + \sum_{i=1}^d \sum_{j=0}^d (\delta_{ij} - \bar{\delta}_i)\widetilde{M}_{ij}^n.$$

After substitution and rearrangement (6) takes the following form.

$$\begin{aligned}
X^n(t) = x_0 &+ \int_0^t \sum_{i=1}^d \left(\lambda_i e_i + \mu_i \bar{\delta}_i U_i^n(s)\right)\,ds\\
&+ \sum_{i=1}^d \frac{1}{n} e_i M_i^n(t) + \sum_{i=1}^d \sum_{j=0}^d \frac{1}{n}(\bar{\delta}_i - \delta_{ij})\widetilde{M}_{ij}^n(t) + \sum_{i=1}^d \sum_{j=0}^d \frac{1}{n}\delta_{ij} M_{ij}^n(t)\\
&- \sum_{i=1}^d \bar{\delta}_i \sum_{j=0}^d \widetilde{N}_{ij}^n(t).
\end{aligned}$$

To express this more succinctly we define the *controled velocity* $v : \mathcal{U} \to \mathbb{R}^d$ by

$$v(u) = \sum_{i=1}^d \left(\lambda_i e_i + \mu_i \bar{\delta}_i u_i\right), \tag{7}$$

and take

$$\begin{aligned}
y^n(t) &= x_0 + \int_0^t v(U^n(s))\,ds,\\
M^n(t) &= \sum_{i=1}^d \frac{1}{n} e_i M_i^n(t) + \sum_{i=1}^d \sum_{j=0}^d \frac{1}{n}(\bar{\delta}_i - \delta_{ij})\widetilde{M}_{ij}^n(t) + \sum_{i=1}^d \sum_{j=0}^d \frac{1}{n}\delta_{ij} M_{ij}^n(t), \text{ and} \qquad (8)\\
Y^n(t) &= y^n(t) + M^n(t). \qquad (9)
\end{aligned}$$

In these terms we have

$$X^n(t) = Y^n(t) - \sum_{i=1}^d \bar{\delta}_i \sum_{j=0}^d \widetilde{N}_{ij}^n(t). \tag{10}$$

We view $Y^n(t)$ as the result of perturbing $y^n(t)$ by the martingale $M^n(t)$, and the subtracted term in (10) wil be exactly the constraining process that connects $X^n(t)$ and $Y^n(t)$ in terms of a Skorokhod Problem. The next lemma is key for the fluid approximation of Lemma 2 below.

Lemma 1. *There exists a constant C_q so that for all control processes $U^n(\cdot)$, all $T > 0$, and all $\epsilon > 0$,*

$$P\left(\sup_{0\le t\le T} \|M^n(t)\| > \epsilon\right) \le \frac{C_qT}{n\epsilon^2}$$

Proof. Let $M^{*n}(T) = \sup_{0\le t\le T} \|M^n(t)\|$. By Doob's L^2 martingale inequality [24, Theorem 70.2] we have

$$P(M^{*n}(T) > \epsilon) \le \frac{1}{\epsilon^2}E[M^{*n}(T)^2] \le \frac{4}{\epsilon^2}E[M^n(T)^2].$$

By [4, Ch. III, T15] we can work out this second moment. Rather than writing it out in full, consider a typical term from (8). For instance, since $|U_i^n(s)| \le 1$ we find that

$$\frac{1}{n^2}E[M_{ij}^n(T)^2] = \frac{1}{n^2}E\left[\int_0^T np_{ij}\mu_jU_i^n(s)\,ds\right] \le \frac{\mu_iT}{n}.$$

All other terms are similar in that they result in a bound of the form of a constant times T/n. We find that $E[M^{*n}(T)^2]$ is bounded by a constant C_q times T/n. This establishes the lemma. $\square$

2.2 The Skorokhod Problem

A Skorokhod problem on $G = \mathbb{R}_+^d$, as formulated in [11], is determined by the selection of a set of *constraint vectors* d_i, one for each coordinate face. The appropriate choices for us are the negated mean service event vectors:

$$d_i = -\bar{\delta}_i. \tag{11}$$

These extend to the set-valued function on ∂G defined by

$$d(x) = \{\gamma = \sum_{i:\,x_i=0} \alpha_i d_i :\ \alpha_i \ge 0,\ \|\gamma\| = 1\}.$$

For an "input" process or function $\psi(t) \in \mathbb{R}^d$ with $\psi(0) \in G$, the Skorokhod Problem asks for a pair of functions $\phi(t)$ and $\eta(t)$ satisfying the following:

- $\phi(t) = \psi(t) + \eta(t)$;
- $\phi(t) \in G$;
- $|\eta|(t) < \infty$ (total variation);
- $|\eta|(t) = \int_{[0,t]} 1_{\partial G}(\phi(s))\,d|\eta|(s)$;
- $\eta(t) = \int_{[0,t]} \gamma(s)\,d|\eta|(s)$ for some (measurable) $\gamma(t) \in d(\phi(t))$.

When well-defined, we denote $\phi(\cdot) = \Gamma(\psi(\cdot))$. For our $\psi(t) = Y^n(t)$ we observe that $\phi(t) = X^n(t)$ solves the Skorokhod problem with $\eta(t) = -\sum_{i=1}^d \bar{\delta}_i \sum_{j=0}^d \widetilde{N}_{ij}^n(t)$. Thus we can write

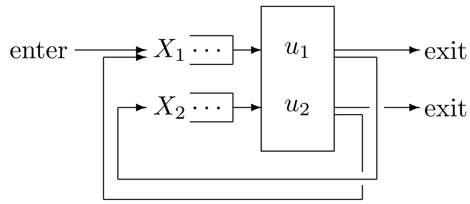
$$X^n(\cdot) = \Gamma(Y^n(\cdot)). \tag{12}$$

When the d_i satisfy certain technical conditions [11] it is known that the problem is well-posed for a large class of $\psi(\cdot)$ including piecewise constant (like our $Y^n(\cdot)$) and absolutely continuous (like our $y^n(\cdot)$). We simply assume that these technical conditions *are* satisfied by our (11). For application of our results to any specific example, the conditions would need to be verified, as we illustrate in the example below. Those conditions also imply that Γ is Lipschitz with respect to the uniform norm on the space of paths: there exists a constant C_Γ so that, for any $0 < T < \infty$,

$$\sup_{0\le t\le T} |\phi(t) - \tilde{\phi}(t)| \le C_\Gamma \sup_{0\le t\le T} |\psi(t) - \tilde{\psi}(t)| \tag{13}$$

whenever $\phi(\cdot) = \Gamma(\psi(\cdot))$ and $\tilde{\phi}(\cdot) = \Gamma(\tilde{\psi}(\cdot))$. (See [11, Theorem 2.2].)

Example. Consider the single server with two queues illustrated in the following figure.



New arrivals only enter at queue 1. When customers in queue 1 are served they either proceed to queue 2 or exit the system, each with probability 1/2. Served customers from queue 2 either return to queue 1 or exit the system, each with probability 1/2. Thus we have routing probabilities

$$p_{11} = 0,\ p_{12} = p_{10} = 1/2; \quad p_{22} = 0,\ p_{21} = p_{20} = 1/2.$$

The mean service event vectors are

$$\bar{\delta}_1 = \frac{1}{2}(-1,1) + \frac{1}{2}(-1,0) = (-1,1/2); \quad \bar{\delta}_2 = \frac{1}{2}(1,-1) + \frac{1}{2}(0,-1) = (1/2,-1).$$

The associated Skorokhod problem uses constraint directions $d_1 = -\bar{\delta}_1$, $d_2 = -\bar{\delta}_2$. The technical conditions of [11] for well-posedness of the Skorokhod problem involve two properties of the matrix with columns d_i:

$$R = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}.$$

First is Dupuis & Ishii's Lipschitz Condition 2.1. For this it is sufficient to check the Harrison-Reimann condition that $Q = I - R$ have spectral radius less than one; see [12, II,Theorem 2.1]. For our example the spectral radius is 1/2. Second is Dupuis & Ishii's Existence Condition 3.1. For this it is sufficient that R be positive definite ([11, Theorem 2.1]), which is true for our example.

2.3 Fluid Processes

Given a control process $u(t)$ (stochastic or deterministic) we can form the associated *fluid process*:

$$\begin{aligned} x(t) &= \Gamma(y(t)),\ \text{where} \\ y(t) &= x_0 + \int_0^t v(u(s))\,ds. \end{aligned} \tag{14}$$

The simplest type of control is a *standard* control. This refers to a (measurable) deterministic function $u : [0,\infty) \to \mathcal{U}$. But the same construction can be used for a stochastic control. If U^n is the stochastic control for X^n, this construction produces the *fluid approximation* to X^n:

$$\begin{aligned} X^n_{\text{fl}}(t) &= \Gamma(y^n(t)),\ \text{where} \\ y^n(t) &= x_0 + \int_0^t v(U^n(s))\,ds. \end{aligned} \tag{15}$$

X^n_{fl} is still a stochastic process, because U^n is, but has absolutely continuous paths. It is the result of dropping the martingale term in (9). The deterministic fluid process $x(t)$ differs from X^n_{fl} only in the choice of control process; otherwise their constructions are the same.

The asymptotic analysis of $X^n(\cdot)$ as $n \to \infty$ is based on two fundamental observations. The second is the continuity of Lemma 4 below. The first is that $X^n \approx X^n_{\text{fl}}$ in the limit, regardless of the control sequence $U^n(\cdot)$. To state this precisely we need to identify the appropriate topology on the space of paths. We use D_G to denote the set of all $\psi : [0,\infty) \to G$ which are right continuous and have left limits. The subset of continuous paths is C_G. On D_G we use the metric topology using the *Skorokhod metric* $\rho(\cdot,\cdot)$, defined in Ethier and Kurtz [13]. This makes D_G a complete separable metric space. Uniform convergence on compact time intervals implies convergence with respect to ρ (but not conversely). However on C_G the relative topology induced by ρ is the same as the topology of uniform convergence on compacts. Here is a statement of the asymptotic equivalence of $X^n(\cdot)$ and $X^n_{\text{fl}}(\cdot)$.

Lemma 2. *For any sequence $U^n(\cdot)$ of stochastic controls and any $T<\infty$, as $n\to\infty$ we have both*

$$\sup_{0\le t\le T}\|X^n(t)-X^n_{\text{fl}}(t)\|\to 0 \text{ and } \rho(X^n(\cdot),X^n_{\text{fl}}(\cdot))\to 0$$

in probability.

Proof. From the semimartingale decomposition (9) we have

$$\sup_{0\le t\le T}\|Y^n(t)-y^n(t)\|=M^{*n}(T),$$

where $M^{*n}(T)=\sup_{0\le t\le T}\|M^n(t)\|$, as in Lemma 1. Since $X^n=\Gamma(Y^n)$ and $X^n_{\text{fl}}=\Gamma(y^n)$, the Lipschitz property (13) gives us

$$\sup_{0\le t\le T}\|X^n(t)-X^n_{\text{fl}}(t)\|\le C_\Gamma M^{*n}(T). \tag{16}$$

By the result of Lemma 1 we have that

$$P(\sup_{[0,T]}\|X^n(t)-X^n_{\text{fl}}(t)\|>\epsilon)\le C_\Gamma^2\frac{C_qT}{\epsilon^2 n},$$

which gives convergence in probability of the uniform norm on each $[0,T]$. This implies convergence of $\rho(X^n,X^n_{\text{fl}})$ to 0 in probability, as claimed. □

3 The Control Problems

We now formulate the time-to-empty control problems whose convergence is our goal. We assume a *running cost function* $L:G\times\mathcal{U}\to[0,\infty)$ satisfying the following properties:

i) $L:G\times\mathcal{U}\to[0,\infty)$ continuous,

ii) $|L(x,u)-L(y,u)|\le C_L\|x-y\|$ for all $x,y\in G$ and $u\in\mathcal{U}$, and some constant C_L,

iii) $|L(x,u)|\le C_L(1+\|x\|)$ for some constant C_L.

iv) $\inf_{u,\|x\|>\epsilon}L(x,u)>0$ for each $\epsilon>0$.

Actually iii) follows from i) and ii) using the compactness of $\mathcal{U}$; it need not be assumed separately. The hypothesis iv) is especially important for our approach; see the details of the proof of Theorem 2.

Since the paths of $X^n(t)$ and $x(t)$ all belong to the space D_G, we are going to view the emptying time τ_0 as a functional on D_G. For $\psi(\cdot)\in D_G$ define

$$\tau_0(\psi(\cdot))=\inf\{t\ge 0:\ \psi(t)=0\}, \tag{17}$$

with the understanding that $\tau_0(\psi(\cdot))=\infty$ if $\psi(t)\ne 0$ for all t. Usually it is clear what we intend for $\psi(\cdot)$, and so we will abbreviate both $\tau_0=\tau_0(X^n(\cdot))$ and $\tau_0=\tau_0(x(\cdot))$, relying on the context to make the distinction.

The control problems we consider are those of minimizing the following time-to-empty cost functionals.

$$J^n(x,U^n(\cdot))=E\left[\int_0^{\tau_0}L(X^n(t),U^n(t))\,dt\right],\quad X^n(0)=x\in G;$$
$$J(x,u(\cdot))=\int_0^{\tau_0}L(x(t),u(t))\,dt,\quad x(0)=x\in G.$$

As above $U^n(\cdot)$ ranges over all progressively measurable $\mathcal{U}$-valued processes and $u(\cdot)$ over all $\mathcal{U}$-valued measurable functions. These functionals are unbounded in general, and could take the value ∞ if for instance $\tau_0 = \infty$. The associated *value functions* are defined on G by

$$V^n(x) = \inf_{U^n(\cdot)} J^n(x, U^n(\cdot)), \quad V(x) = \inf_{u(\cdot)} J(x, u(\cdot)).$$

Our results will be stated in terms of the *upper* and *lower semicontinuous envelopes* of the sequence $\{V^n\}$ of value functions, defined respectively for any $x \in G$ as follows.

$$V^*(x) = \limsup_{\substack{n\to\infty \\ x^n\to x}} V^n(x^n), \quad V_*(x) = \liminf_{\substack{n\to\infty \\ x^n\to x}} V^n(x^n).$$

I.e. $V^*(x)$ is the maximal limit point of all sequences $\{V^n(x^n)\}$ such that $x^n \to x$ in G. From these definitions it is easy to see that V^* is upper semicontinuous, V_* is lower semicontinuous, and $V_* \le V^*$. Since $0 \le V^n$ we also have $0 \le V_*$.

3.1 Relaxed Controls, Fluid Approximation, and Tightness

The merit of our formulation is that both standard and stochastic controls can be considered in a common setting, the class of $\mathcal{R}$ of relaxed controls. A *relaxed control* $\nu \in \mathcal{R}$ is a measure on $[0, \infty) \times \mathcal{U}$ with the property that $\nu([0, T] \times \mathcal{U}) = T$ for all T. Essentially a relaxed control replaces the single value $u(t)$ of a standard control by a probability measure on $\mathcal{U}$ for each t. A standard control $u(\cdot)$ corresponds to the particular relaxed control defined by

$$\nu(A) = \int 1_A(s, u(s))\, ds, \quad A \subseteq [0, \infty) \times \mathcal{U}.$$

For a general relaxed control $\nu \in \mathcal{R}$ the fluid process with initial condition x_0 is the generalization of (14):

$$\begin{aligned} &x(\cdot) = \Gamma(y(\cdot)), \text{ where} \\ &y(t) = x_0 + \int_{[0,t]\times\mathcal{U}} v(u)\, d\nu(s, u). \end{aligned} \tag{18}$$

We use the notation $\chi_{x_0,\nu}(\cdot) = x(\cdot)$ to refer to this particular fluid process. We will sometimes abuse notation to express the integral in (18) succinctly as

$$\int_0^t v(u)\, d\nu \doteq \int_{[0,t]\times\mathcal{U}} v(u)\, d\nu(s, u).$$

Observe that since $\mathcal{U}$ is compact and $v(\cdot)$ is continuous this integral is bounded by Bt for some constant B, independent of the control. By the Lipschitz continuity of the Skorokhod map, it follows that

$$\|\chi_{x_0,\nu}(t)\| \le C_\Gamma(\|x_0\| + Bt). \tag{19}$$

By definition $V(x)$ is the infimum of $\int_0^{\tau_0} L(\chi_{x,\nu}(t), u)\, d\nu$ over standard controls. However the infimum over all relaxed controls in $\mathcal{R}$ amounts to the same thing.

Lemma 3. *$V(x) = \inf_{\nu\in\mathcal{R}} J(x, \nu)$.*

Proof. Consider any relaxed control ν. We can define its mean $\bar{\nu}$ by averaging the $\mathcal{U}$ component:

$$\int_0^t \bar{\nu}(s)\, ds = \int_{[0,t]\times\mathcal{U}} u\, d\nu(s, u).$$

Since $\mathcal{U}$ is compact and convex, $\bar{\nu}(t) \in \mathcal{U}$ almost surely, so that $\bar{\nu}(\cdot)$ is in fact a standard control. Because the controlled velocity (7) is affine in $u \in \mathcal{U}$ the integral of $v(u)$ with respect to a probability measure on

$\mathcal{U}$ is just v evaluted at the mean of the probability measure. Thus, although ν and $\bar{\nu}$ are not equivalent as relaxed controls, they do produce the same fluid process:

$$\int_{[0,t]\times\mathcal{U}} v(u)\, d\nu(s,u) = \int_0^t v(\bar{\nu}(s))\, ds.$$

In other words $\chi_{x,\nu}(t) = \chi_{x,\bar{\nu}}(t)$ and so

$$J(x,\nu) = \int_0^{\tau_0} L(\chi_{x,\nu}(t),u)\, d\nu = \int_0^{\tau_0} L(\chi_{x,\bar{\nu}}(t),u)\, d\bar{\nu} = J(x,\bar{\nu}).$$

The lemma is now clear. □

A stochastic control $U(\cdot)$ corresponds to the $\mathcal{R}$-valued random variable $\mathcal{V}$ defined by

$$\mathcal{V}(A) = \int 1_A(s,U(s))\, ds, \quad A \subseteq [0,\infty)\times\mathcal{U}.$$

Thus $\mathcal{V}$ is defined on the same underlying probability space as $U(\cdot)$. (The dependence on ω in the underlying space Ω is suppressed in the notation.) We can use $\mathcal{V}$ to form a fluid process $\chi_{x_0,\mathcal{V}}(t)$ in the same way as (18), also denoted $\chi_{x_0,U}(t)$. The a-priori bound (19) holds for $\chi_{x_0,\mathcal{V}}(t)$ as well. In this notation our fluid approximation (15) to X^n (with $X^n(0) = x^n$) is denoted

$$X^n_{\text{fl}}(t) = \chi_{x_0,U^n}(t).$$

The second fundamental observation for our analysis is that the map $(x_0,\nu) \mapsto \chi_{x_0,\nu}(\cdot)$ is continuous from $G\times\mathcal{R}$ into D_G. The appropriate topology on $\mathcal{R}$ is what we might call "weak convergence on compact time intervals": $\nu^n \Rrightarrow \nu$ in $\mathcal{R}$ means that for each $0 \le T < \infty$ and bounded continuous $\phi : [0,\infty)\times\mathcal{U} \to \mathbb{R}$,

$$\int_{[0,T]\times\mathcal{U}} \phi(t,u)\, d\nu^n(t,u) \to \int_{[0,T]\times\mathcal{U}} \phi(t,u)\, d\nu(t,u)$$

It turns out that this topology makes $\mathcal{R}$ a compact metric space, and the standard controls $u(\cdot)$ are dense. See [23] for more details. The continuity property is [23, Lemma 3.2], which we simply repeat here.

Lemma 4. *The map $(x,\nu) \mapsto \chi_{x,\nu}$ from $G\times\mathcal{R}$ to D_G is continuous.*

Since stochastic controls are random variables taking values in the metric space $\mathcal{R}$, the usual notion of weak convergence applies to them. To say $\mathcal{V}^n$ converges weakly to some other random relaxed control $\mathcal{V}$ means that

$$E[\Phi(\mathcal{V}^n)] \to E[\Phi(\mathcal{V})] \tag{20}$$

for every $\Phi : \mathcal{R} \to \mathbb{R}$ which is bounded and continuous (with respect to the topology on $\mathcal{R}$ above). This is denoted $\mathcal{V}^n \Rightarrow \mathcal{V}$. (Note that we use double arrow "$\Rightarrow$" for weak convergence of $\mathcal{R}$-valued random variables, and a triple arrow $\nu^n \Rrightarrow \nu$ to indicate convergence of a sequence in $\mathcal{R}$.) We will only be interested in $\mathcal{V}^n$ which are associated with stochastic controls $U^n(\cdot)$. We will write $U^n(\cdot) \Rightarrow \mathcal{V}$ to indicate weak convergence directly in terms of the sequence of stochastic controls. The compactness of $\mathcal{R}$ means that every sequence $\mathcal{V}^n$ is tight. We record this as a lemma for use in the proof of Theorem 1.

Lemma 5. *For any sequence $U^n(t)$ of stochastic controls, there is a random relaxed control $\mathcal{V}$ and a subsequence n' so that*

$$U^{n'}(\cdot) \Rightarrow \mathcal{V}.$$

In light of the proof of Lemma 3 it might seem that we could work in the simpler space of standard controls, with a stochastic control U^n being a random standard control. However the standard controls are not closed under weak convergence, so we would not have tightness and Lemma 5 would fail in that more narrow setting.

We also point out that if $x^n \to x$ in G and $\mathcal{V}^n \Rightarrow \mathcal{V}$ in $\mathcal{R}$, then $(x^n,\mathcal{V}^n) \Rightarrow (x,\mathcal{V})$ in $G\times\mathcal{R}$; see Billingsley [3, Theorem 3.2].

We close this section with a technical result involving random limits of integration.

Lemma 6. *Suppose η_n and η'_n are random variables, all bounded by T, with $|\eta_n - \eta'_n| \to 0$ in probability. Then for any sequece of controls $U^n(\cdot)$, as $n \to \infty$ we have*

$$E\left[\int_0^{\eta_n} L(X^n(t), U^n(t))\, dt\right] - E\left[\int_0^{\eta'_n} L(X^n_{\text{fl}}(t), U^n(t))\, dt\right] \to 0.$$

Proof. By our hypotheses on L,

$$|L(X^n(s), U^n(s)) - L(X^n_{\text{fl}}(s), U^n(s))| \leq C_L \|X^n(s) - X^n_{\text{fl}}(s)\|.$$

Returning to the inequality (16) in the proof of Lemma 2 and the bound on $E[M^{*n}(T)^2]$ from Lemma 1, we obtain

$$E[\sup_{[0,T]} \|X^n(t) - X^n_{\text{fl}}(t)\|\,] \leq C_\Gamma \sqrt{E[M^{*n}(T)^2]} \leq C_\Gamma \sqrt{C_q T/n}.$$

From this we deduce

$$E\left[\left|\int_0^T L(X^n(t), U^n(t))\, dt - \int_0^T L(X^n_{\text{fl}}(t), U^n(t))\, dt\right|\right] \leq C_L C_\Gamma \sqrt{C_q/n}\, T^{3/2}. \tag{21}$$

We can write

$$\left|\int_0^{\eta_n} L(X^n(t), U^n(t))\, dt - \int_0^{\eta'_n} L(X^n_{\text{fl}}(t), U^n(t))\, dt\right|$$
$$\leq \int_0^{\eta_n} |L(X^n(s), U^n(s)) - L(X^n_{\text{fl}}(s), U^n(s))|\, dt + \left|\int_{\eta_n}^{\eta'_n} L(X^n_{\text{fl}}(s), U^n(s))\, dt\right|.$$

The mean of first term vanishes in the limit by (21). In the second, by virtue of (19) and our hypotheses on L, the integrand is bounded independently of the control. So the mean of the second term is bounded by a constant times $E[|\eta_n - \eta|]$, which vanishes in the limit since $|\eta_n - \eta'_n| \to 0$ and the fact that $|\eta_n - \eta| \leq T$. $\square$

4 Lower Semi-Continuity

The first of our main results says that the fluid value function is a lower bound for the limit of the scaled stochastic value functions.

Theorem 1. *$V(x) \leq V_*(x)$ for all $x \in G$.*

This is essentially a consequence of the fact that the map

$$(\psi, \nu) \mapsto \int_0^{\tau_0(\psi)} L(\psi(t), u)\, d\nu$$

is a nonnegative lower semi-continuous function on the space $C_G \times \mathcal{R}$, using continuous paths ψ. However it is *not* lower semicontinuous if we consider ψ in the larger path space D_G. The proof of Theorem 1 must justify the replacement of X^n by X^n_{fl} (asymptotically) in V^n before invoking the above semicontinuity property. We deal with the technicalities in a sequence of lemmas and corollaries leading up to the main proof.

Lemma 7. *The functional $\tau_0 : D_G \to [0, \infty]$ is lower semicontinuous at each $\phi \in C_G$.*

Proof. Suppose $\psi^n(\cdot) \to \phi(\cdot)$ is a convergent sequence in D_G with respect to the Skorokhod metric, and that $\phi(\cdot)$ is continuous. Let $s^n = \tau_0(\psi^n(\cdot))$ and $s^* = \liminf s^n$. We want to show that $\tau_0(\phi(\cdot)) \leq s^*$. It suffices to assume $s^* < \infty$ and, by passing to a subsequence we can assume $s^n \to s^*$. By definition of the Skorokhod metric [13] there exists a sequence of strictly increasing continuous functions μ^n mapping $[0, \infty)$ onto itself for which

$$\sup_{0 \leq t \leq T} |\mu^n(t) - t| \to 0 \text{ and } \sup_{0 \leq t \leq T} |\psi^n(t) - \phi(\mu^n(t))| \to 0 \text{ for every } T < \infty.$$

Since $\psi^n(s^n) = 0$ we find that $\phi(\mu^n(s^n)) \to 0$. Since $\lim \mu^n(s^n) = \lim s^n = s^*$ and ϕ is continuous it follows that $\phi(s^*) = 0$ and, therefore, $\tau_0(\phi(\cdot)) \leq s^*$. $\square$

One way to characterize a lower semicontinuous function is as the pointwise limit of an increasing sequence of continuous functions. Here is such a result for τ_0.

Lemma 8. *There exists a monotone increasing sequence of bounded nonnegative continuous functions $\sigma_k : D_G \to [0,\infty)$ with $\sigma_k \uparrow \tau_0$ on C_G.*

Proof. For each positive integer k we use an "inf-convolution" construction to define

$$\sigma_k(\psi) = \inf_{\phi \in D_G} \left\{ \min(\tau_0(\phi), k) + k\rho(\phi, \psi) \right\}.$$

It is elementary that $0 \leq \sigma_k \leq \min(\tau_0, k)$ and that $\sigma_k \leq \sigma_{k+1}$. Since

$$\left| (\min(\tau_0(\phi), k) + k\rho(\phi, \psi)) - (\min(\tau_0(\phi), k) + k\rho(\phi, \tilde{\psi})) \right| = k|\rho(\phi, \psi) - \rho(\phi, \tilde{\psi})| \leq k\rho(\psi, \tilde{\psi}),$$

it follows that

$$|\sigma_k(\psi) - \sigma_k(\tilde{\psi})| \leq k\rho(\psi, \tilde{\psi}),$$

establishing continuity of σ_k.

Consider a fixed $\psi \in C_G$. We know that τ_0 is lower semicontinuous at ψ so, given $\epsilon > 0$, there is $\delta > 0$ such that

$$\tau_0(\phi) > \tau_0(\psi) - \epsilon \text{ whenever } \rho(\psi, \phi) < \delta.$$

For these ϕ, we can say

$$\min(\tau_0(\phi), k) + k\rho(\phi, \psi) \geq \min(\tau_0(\psi) - \epsilon, k).$$

For those ϕ with $\rho(\psi, \phi) \geq \delta$, we have

$$\min(\tau_0(\phi), k) + k\rho(\phi, \psi) \geq k\delta.$$

Consequently, $\sigma_k(\psi) \geq \min(\tau_0(\psi) - \epsilon, k, k\delta)$ and so we have

$$\tau_0(\psi) - \epsilon \leq \liminf_{k\to\infty} \sigma_k(\psi) \leq \limsup_{k\to\infty} \sigma_k(\psi) \leq \tau_0(\psi).$$

Since this holds for every $\epsilon > 0$, we deduce that $\sigma_k(\psi) \to \tau_0(\psi)$, completing the proof. $\square$

The following generalizes [23, Lemma 4.1] to continuous upper limits of integration.

Lemma 9. *Suppose $\sigma : D_G \to [0,\infty)$ is a bounded continuous function. The following defines a bounded continuous function on $G \times \mathcal{R}$:*

$$(x, \nu) \mapsto \int_0^{\sigma(\chi_{x,\nu})} L(\chi_{x,\nu}(t), u)\, d\nu. \tag{22}$$

Proof. Suppose $x^n \to x$ in G and $\nu^n \Rrightarrow \nu$ in $\mathcal{R}$. From Lemma 4 we know that $\sigma(\chi_{x^n,\nu^n}) \to \sigma(\chi_{x,\nu})$. Thus for any $s_1 < \sigma(\chi_{x,\nu}) < s_2$, we have

$$s_1 < \sigma(\chi_{x^n,\nu^n}) < s_2 \text{ for all large } n. \tag{23}$$

We know from [23, Lemma 4.1] that for all $s \geq 0$

$$\lim_{n\to\infty} \int_0^s L(\chi_{x^n,\nu^n}(t), u)\, d\nu^n = \int_0^s L(\chi_{x,\nu})\, d\nu$$

and therefore, using $L \geq 0$ and (23),

$$\begin{aligned}
\int_0^{s_1} L(\chi_{x,\nu})(t), u)\, d\nu &\leq \liminf_{n\to\infty} \int_0^{\sigma(\chi_{x^n,\nu^n})} L(\chi_{x^n,\nu^n}(t), u)\, d\nu^n \\
&\leq \limsup_{n\to\infty} \int_0^{\sigma(\chi_{x^n,\nu^n})} L(\chi_{x^n,\nu^n}(t), u)\, d\nu^n \\
&\leq \int_0^{s_2} L(\chi_{x,\nu})(t), u)\, d\nu
\end{aligned}$$

Letting $s_1 \to \sigma(\chi_{x,\nu}) \leftarrow s_2$, we conclude that

$$\lim_{n\to\infty} \int_0^{\sigma(\chi_{x^n,\nu^n})} L(\chi_{x^n,\nu^n})(t), u)\, d\nu^n = \int_0^{\sigma(\chi_{x,\nu})} L(\chi_{x,\nu})(t), u)\, d\nu.$$

This establishes the continuity of (22). □

Lemma 10. *The map* $(x, \nu) \mapsto \int_0^{\tau_0} L(\chi_{x,\nu}(t), u)\, d\nu$ *is lower semicontinuous on* $G \times \mathcal{R}$.

This is an easy corollary of Lemmas 8 and 9.

We are ready now to prove the theorem of this section.

Proof of Theorem 1. Consider any sequence $x^n \in G$ with $x^n \to x \in G$. Our goal is to show that $\liminf_{n\to\infty} V^n(x^n) \geq V(x)$. There exists a sequence of stochastic controls $U^n(\cdot)$ with $J(x^n, U^n(\cdot)) \leq V^n(x^n) + \frac{1}{n}$. By passing to a subsequence (Lemma 5) we can assume that $J(x^n, U^n(\cdot)) \to \liminf_n V^n(x^n)$ and that $U^n \Rightarrow \mathcal{V}$ for some $\mathcal{R}$-valued random variable $\mathcal{V}$.

Let σ_k be the sequence of bounded continuous functions of Lemma 8. As per that construction we can assume $\sigma_k \leq k$. Since $\sigma_k(X^n) \leq \tau_0(X^n)$ we have

$$\lim_{n\to\infty} J(x^n, U^n) \geq \liminf_{n\to\infty} E\left[\int_0^{\sigma_k(X^n)} L(X^n(t), U^n(t))\, dt\right].$$

It follows from Lemma 6 that

$$\begin{aligned}
\liminf_{n\to\infty} E\left[\int_0^{\sigma_k(X^n)} L(X^n(t), U^n(t))\, dt\right] &= \liminf_{n\to\infty} E\left[\int_0^{\sigma_k(X^n_{\text{fl}})} L(X^n_{\text{fl}}(t), U^n(t))\, dt\right] \\
&= E\left[\int_0^{\sigma_k(\chi_{x,\mathcal{V}})} L(\chi_{x,\mathcal{V}}, u)\, d\mathcal{V}\right].
\end{aligned}$$

The second equality is a consequence of the weak convergence $(x^n, U^n) \Rightarrow (x, \mathcal{V})$ and the continuity from Lemma 9. Then, since $\chi_{x,\mathcal{V}} \in C_G$, we know that $\sigma_k(\chi_{x,\mathcal{V}}) \uparrow \tau_0(\chi_{x,\mathcal{V}})$ as $k \to \infty$. Therefore we deduce that

$$\begin{aligned}
\liminf V^n(x^n) &\geq E\left[\int_0^{\tau_0(\chi_{x,\mathcal{V}})} L(\chi_{x,\mathcal{V}}(t), u)\, d\mathcal{V}\right] \\
&= E[J(x, \mathcal{V})].
\end{aligned}$$

The right side is bounded below by the minimum over $\nu \in \mathcal{R}$. Using Lemma 3 we conclude that

$$\liminf V^n(x^n) \geq V(x).$$

□

5 The Convergence Theorem

Now we turn to the reverse inequality: $V^* \leq V$. This requires two additional hypotheses on the sequence $\{V^n\}$.

Definition. *We say that* $\{V^n\}$ *is* equicontinuous at 0 *if for each* $\epsilon > 0$ *there is a* $\delta_\epsilon > 0$ *(independent of* n*) so that*

$$V^n(x) < \epsilon \text{ whenever } n \in \mathbb{N},\ x \in G, \quad \text{and } \|0 - x\| < \delta_\epsilon.$$

We say that $\{V^n\}$ *is* locally bounded *if for each* $x \in G$ *there is a* $\delta_x > 0$ *and a finite* B_x *(independent of* n*) so that*

$$V^n(y) \leq B_x \text{ whenever } n \in \mathbb{N},\ y \in G, \quad \text{and } \|x - y\| < \delta_x.$$

Equicontinuity at 0 is equivalent to $V^*(0) = 0$. Local boundedness is equivalent to saying that $V^*(x) < \infty$ for each $x \in G$.

Theorem 2. *Suppose $\{V^n\}$ is equicontinuous at 0 and locally bounded. Then V is continuous in G and $V^n \to V$ uniformly on compacts.*

Proof of Theorem 2. The argument below will show that $V^* \le V$. Since we already know $V \le V_* \le V^*$, this will imply that $V_* = V = V^*$. Since V^* and V_* are upper and lower semicontinuous respectively, it will follow that V is continuous. If $K \subset G$ is compact and V^n did not converge to V uniformly on K, there would exist a sequence $x^n \in K \cap G$ with $x^n \to x$ but $V^n(x^n) \not\to V(x)$. But since

$$\limsup V^n(x^n) \le V^*(x) = V(x) = V_*(x) \le \liminf V^n(x^n),$$

we know that $V^n(x^n) \to V(x)$, a contradiction. Hence everything will follow from $V^* \le V$. Our task is thus to show that

$$\limsup V^n(x^n) \le V(x_0), \tag{24}$$

whenever $x^n \in G$ is a sequence with $x^n \to x_0 \in G$.

The basic idea of the proof is this. The equicontinuity hypothesis means that V^n is small (uniformly in n) in a neighborhood of the origin, so that once X^n reaches such a neighborhood the remaining cost is negligible, provided the control is well-chosen. The uniform boundedness hypothesis implies that $V^n(x^n)$ is bounded. Because of our hypothesis iv) on L, a bound on $V^n(x^n)$ implies a bound on the time T is takes for X^n to reach a neighborhood of the origin. The upshot is that $V^n(x^n)$ is approximated by $\int_0^T L$, which will converge by the lemmas above. The proof consists of the careful development of these ideas.

Consider a sequence $x^n \to x_0$ in G and any $\epsilon > 0$. By equicontinuity at 0 there is $\delta_\epsilon > 0$ so that

$$V^n(x) < \epsilon \text{ whenever } x \in G \text{ and } \|x\| < \delta_\epsilon.$$

This implies that $V^*(x) \le \epsilon$ for all $\|x\| < \delta_\epsilon$. Since $V_* \le V^*$, as a consequence of Theorem 1 have

$$V(x) \le \epsilon \text{ whenever } \|x\| < \delta_\epsilon. \tag{25}$$

From the locally bounded hypothesis it follows that $V^*(x_0) < \infty$, from which we deduce $V(x_0) < \infty$ in the same way. There exists a nearly optimal control $u(\cdot)$ for x_0:

$$J(x_0, u(\cdot)) < V(x_0) + \epsilon.$$

Let $x_u(t) = \chi_{x_0,u(\cdot)}(t)$ be the fluid path starting at x_0 using control $u(\cdot)$. We have

$$\int_0^{\tau_0} L(x_u(t), u(t))\, dt = J(x, u(\cdot)) < \infty.$$

Because of our strict positivity hypothesis iv) on L, there is a finite T at which $\|x_u(T)\| \le \delta_\epsilon/2$. (If no such T existed $J(x, u(\cdot))$ would be infinite.) Let

$$A = 1 + \sup_{0 \le t \le T} \|x_u(t)\|.$$

Define a stochastic control $U^n(\cdot)$ for X^n as follows. Starting from $X^n(0) = x^n$ use the deterministic control $u(\cdot)$ (above) until the first time τ_A that either $\tau_A = T$ or $\|X^n(\tau_A)\| = A$, whichever happens first. Then for $t > \tau_A$ use a nearly optimal control from $X^n(\tau_A)$:

$$J(X^n(\tau_A), U^n(\tau_A + \cdot)) < V(X^n(\tau_A)) + \epsilon. \tag{26}$$

We make the following observations.

(a) $V^n(X^n(\tau_A))$ is bounded in n. This is because the possible values of $X^n(\tau_A)$ are limited to the compact set $\{x \in G : \|x\| \le A\}$ in conjunction with the local boundedness hypothesis.

(b) $P(\tau_A < T) \to 0$. In other words the probability that $\|X^n(\cdot)\|$ reaches A before time T vanishes in the limit. We deduce this from Lemma 2 as follows. For $t < \tau_A$ we have $U^n(t) = u(t)$, and therefore $X^n_{\text{fl}}(\tau_A) = x_u(\tau_A)$. Suppose $\tau_A < T$. Then $\|X^n(\tau_A)\| \geq A$ while $\|X^n_{\text{fl}}(\tau_A)\| = \|x_u(\tau_A)\| \leq A - 1$ by definition of A. It follows that $\|X^n(\tau_A) - X^n_{\text{fl}}(\tau_A)\| \geq 1$. This means that

$$P(\tau_A < T) \leq P(\sup_{0\leq t\leq T} \|X^n(t) - X^n_{\text{fl}}(t)\| \geq 1).$$

Lemma 2 tells us that this probability vanishes in the limit as $n \to \infty$.

(c) $X^n(\cdot) \to x_u(\cdot)$ uniformly in probability on $[0, T]$. As in item b), we know this holds on the interval $[0, \tau_A]$ by Lemma 2 because X^n_{fl} and x_u agree there. Since $P(\tau_A < T) \to 0$, it holds on $[0, T]$ as well.

Now by a standard dynamic programming argument, using (26) we have

$$V^n(x^n) \leq J^n(x^n, U^n) < E\left[\int_0^{\tau_A} L(X^n(t), U^n(t))\, dt\right] + E[V^n(X^n(\tau_A))] + \epsilon. \tag{27}$$

From Lemma 6 and observation (b) we deduce that

$$E\left[\int_0^{\tau_A} L(X^n(t), U^n(t))\, dt\right] \to \int_0^T L(x_u(t), u(t))\, dt.$$

From (b) and (c) it follows that $X^n(\tau_A) \to x_u(T)$ in probability. Since $\|x_u(T)\| \leq \delta_\epsilon/2$, we know that $V(x_u(T)) \leq \epsilon$ by (25). Using (a) for dominated convergence, it follows that $\limsup E[V^n(X^n(\tau_A))] \leq \epsilon$. From (27) we can now conclude that

$$\limsup V^n(x^n) \leq \int_0^T L(x_u(t), u(t))\, dt + 2\epsilon.$$

But by our choices above, we know

$$\int_0^T L(x_u(t), u(t))\, dt \leq J(x_0, u(\cdot)) < V(x_0) + \epsilon.$$

Thus

$$\limsup V^n(x^n) \leq V(x_0) + 3\epsilon.$$

Since $\epsilon > 0$ was arbitrary, this proves (24). □

6 Further Remarks

We close by discussing the Barles-Perthame procedure as an alternative approach to proving Theorem 2, and then how we might establish the boundedness and equicontinuity hypotheses of the theorem.

6.1 Comparison with the Barles-Perthame Procedure

An alternative to the weak convergence approach above is the so-called Barles-Perthame procedure, which is based on semicontinuous viscosity solutions. General discussions of it can be found in the books of Bardi and Capuzzo-Dolcetta [1] and Fleming and Soner [14]. In rough outline, the Barles-Perthame procedure consists of the following steps.

1. Show that V is a viscosity solution to the appropriate Hamilton-Jacobi-Bellman equation with boundary conditions on ∂G.

2. Show that V^* $[V_*]$ is a sub- [super-] solution to the Hamilton-Jacobi-Bellman equation for V with boundary conditions on ∂G.

3. Apply a comparison theorem for semicontinuous viscosity solutions to deduce that $V^* \leq V_*$.

4. Since $V_* \leq V^*$, conclude that $V^* = V_*$ is the unique solution to the Hamilton-Jacobi-Bellman equation for V in the first step and, therefore, agrees with V. Consequently, V is continuous and $V^n \to V$ uniformly on compacts (by the same argument as at the beginning of the proof of Theorem 2 above).

We want to compare the requirements of this procedure with the hypotheses of our Theorem 2 above. For that purpose we refer to the development of the Barles-Perthame procedure in Dupuis, Atar, and Shwartz [10]. They apply the procedure to study the problem of maximizing the first time that a queueing process exits a specified region G. Although there are many differences between their problem and ours, their paper does develop the Barles-Perthame procedure in a queueing process setting, and provides a reasonable indication of how that approach would work if developed for our problem. One prerequisite of the approach is to know that $V^*(x) < \infty$, which is our local boundedness hypothesis. This requires some argument based on the particulars of the problem before either approach can be applied. (In the case of [10], such an upper bound is provided by their Lemma 2.) Similar remarks apply to V_*, but this is trivial for our problem since $V^n \geq 0$.

The appropriate boundary conditions for our time-to-empty control problem consist of a Dirichlet condition $V(0) = 0$ and oblique-derivative boundary conditions on $\partial G \setminus \{0\}$. The subsolution interpretation of the Dirichlet condition for V^* is simply that $V^*(0) \leq 0$. Since $V_*(0) \geq 0$ this is the same as our hypothesis of equicontinuity at 0: $V^*(0) = 0$. This too must be argued based on particulars of the problem at hand. (In [10] Lemma 2 establishes Lipschitz continuity of the V^n with a constant independent of n, which provides equicontinuity everywhere.) The supersolution interpretation is that $V_*(0) \geq 0$, which is again trivial in our context.

The appropriate Hamilton-Jacobi-Bellman equation for our problem would be

$$H(x, DV(x)) = 0,$$

where the Hamiltonian $H(x,p)$ is defined by

$$H(x,p) = \sup_{u \in \mathcal{U}} \{-p \cdot v(u) - L(x,u)\}.$$

For Hamiltonians of this type (lacking direct dependence on the solution value $V(x)$), viscosity solution comparison results (as needed in step 3 above) depend on additional special features of H. Developing an appropriate viscosity comparison result would be the primary task in treating the problem of the present paper by the Barles-Perthame procedure.

Our point then is that the equicontinuity and uniform boundedness hypotheses of Theorem 2 would also need some problem-specific verification for the Barles-Perthame approach, just as they do here, and the conclusion of that approach would be the same as our Theorem 2.

6.2 Sufficient Conditions for Equicontinuity and Local Boundedness

The following lemma provides a sufficient condition for the hypotheses of equicontinuity at 0 and local boundedness required by Theorem 2. Consider the original (unrescaled) queueing process $X(t) = X^1(t)$ in G. The idea is that if there exist controls for which the means of τ_0 and $\int_0^{\tau_0} X(t)\,dt$ satisfy appropriate polynomial bounds in the initial condition $x = X(0)$, then when passed through the fluid rescaling we obtain upper bounds on $V^n(\cdot)$ which imply the desired equicontinuity and boundedness properties. The proof uses the fact that the Skorokhod map Γ is invariant with respect to rescaling: for any constant $c > 0$, if $\phi(\cdot) = \Gamma(\psi(\cdot))$ and $\psi^c(t) = \frac{1}{c}\psi(ct)$, then

$$\phi^c(\cdot) = \Gamma(\psi^c(\cdot)) \text{ where } \phi^c(t) = \frac{1}{c}\phi(ct). \tag{28}$$

(The fact that G is a cone is important here. One can check (28) from the definition of the Skorokhod problem in Section 2.2.)

Lemma 11. *Suppose there are constants c_0, c_1 so that for each nonzero initial position $x \in G$ there exists a stochastic control $U_x(t)$ for which the resulting queueing process $X(t) = X^1(t)$ satisfies the following moment bounds:*

$$E_x[\tau_0] \leq c_0\|x\|, \quad E_x\left[\int_0^{\tau_0} \|X(t)\|\,dt\right] \leq c_1\|x\|^2.$$

Then $\{V^n\}$ *is equicontinuous at* 0 *and locally bounded.*

In general the hypotheses require two separate bounds. However, inspection of the proof shows that if the running cost is bounded, $L(x,u) \leq C$, then just the bound on $E[\tau_0]$ would be enough. Likewise, if $L(x,u) \leq C\|x\|$ then just the quadratic bound would do.

Proof. Given $x^n \in G$, let $X^n(t) = \frac{1}{n}X(nt)$ with $X(0) = x_0 = nx^n$, where $X^1(\cdot)$ uses the control U_{x_0} of the hypotheses. We have $\tau_0(X^n) = \frac{1}{n}\tau_0(X)$, and $X^n(t)$ is subject to the stochastic control $U^n(t) = U_{x_0}(nt)$. We have

$$\begin{aligned}
V^n(x^n) &\leq E_{x^n}\left[\int_0^{\tau_0(X^n)} L(X^n(t), U^n(t))\,dt\right] \\
&\leq C_L E_{x^n}\left[\int_0^{\tau_0(X^n)} 1 + \|X^n(t)\|\,dt\right] \\
&= C_L E_{x_0}\left[\int_0^{\tau_0(X)} (1 + \frac{1}{n}\|X(s)\|)\,\frac{1}{n}ds\right] \\
&= C_L\left(\frac{1}{n}E_{x_0}[\tau_0(X)] + \frac{1}{n^2}E_{x_0}[\int_0^{\tau_0} \|X(s)\|\,ds\right) \\
&\leq C_L\left(\frac{1}{n}c_0\|nx^n\| + \frac{1}{n^2}c_1\|nx^n\|^2\right) \\
&= C_L(c_0\|x^n\| + c_1\|x^n\|^2).
\end{aligned}$$

This estimate implies both equicontinuity at 0 and local boundedness. □

In simple cases it is possible to simply exhibit a control for which the moment bounds required by the lemma can be verified directly. In the spirit of using the fluid process as a tool for the study of the stochastic process $X(t)$ we would like to be able to use $x(\cdot)$ to verify the applicability of Theorem 2. It appears that such a result will be possible, based on the moment estimates of Dai and Meyn [8]. For brevity we only describe the general ideas and leave it to interested readers to explore the details more fully.

Our general process (Section 2) is a special case of the multiclass network considered in [8]. Their results provide moment bounds for the original queueing process $X(t)$ under the hypothesis that the corresponding fluid limit process $x(t)$ is stable, meaning that $\tau_0 < \infty$ for each initial condition x_0. (See their Definition 3.3.) This connection requires some sort of agreement between controls used for $X(\cdot)$ and for $x(\cdot)$. In their formulation that is assured by hypotheses on the structure of the controls. Their analysis applies to several general types of control, including *preemptive-resume priority rules* (see [6] for more description). These are simply state feedback controls of a particular form. Suppose that there does exist such a policy for which the resulting fluid process is stable. (In some cases it is known that the traffic intensity condition implies the existence of such a stabilizing priority policy. For instance Dai and Weiss [9] have shown that for reentrant lines both LBFS and FBFS will do this, both of which are preemptive-resume priority rules.) Proposition 5.3 of [8] then implies upper bounds

$$E_x\left[\int_0^{\tau_C(\delta)} 1 + \|X(t)\|^p\,dt\right] \leq c_{p+1}(1 + \|x\|^{p+1}), \tag{29}$$

where $\tau_C(\delta)$ is the stopping time $\inf\{t \geq \delta :\ X(t) \in C\}$. The proposition from [8] guarantees the existence of C and $\delta > 0$ for which (29) holds for all $p \geq 1$. This bound is not exactly what we need to invoke our Lemma 11, but it is close. We suggest that, by using the preemptive-resume priority rule of (29) as a starting point, controls U_x can be constructed which satisfies the hypotheses of Lemma 11. If this is indeed possible, then it will show that Theorem 2 holds whenever the fluid process $x(t)$ is stabilizable by a preemptive-resume priority rule.

References

[1] M. Bardi and I. Cappuzzo-Dolcetta, Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations, Birkhäuser, Boston, 1997.

[2] N. Bäuerle, *Asymptotic optimality of tracking policies in stochastic networks*, Ann. Appl. Prob. **10** (2000), pp. 1065–1083.

[3] P. Billingsley, Convergence of Probability Measures, J. Wiley & Sons, New York, 1968.

[4] P. Brémaud, Point Processes and Queues: Martingale Dynamics, Springer-Verlag, New York, 1981.

[5] W. Chen, D. Huang, A. Kulkarni, J. Unnikrishnan, Q. Zhu, P. Mehta, S. Meyn, and A. Wierman, *Approximate dynamic programming using fluid and diffusion approximations with applications to power management*, Proceedings of the 48th IEEE Conf. on Decision and Control, December 2009, pp. 3575–3580.

[6] J. G. Dai. *On positive Harris recurrence of multiclass queueing networks: a unified approach via fluid limit models*, Ann. Appl. Prob. **5** (1995), pp. 49–77.

[7] J. G. Dai and W. Lin, *Maximum pressure policies in stochastic processing networks*, Oper. Res. **53** (2005), pp. 197–218.

[8] J. G. Dai and S. P. Meyn, *Stability and convergence of moments for multiclass queueing networks via fluid limit models*, IEEE Trans. Auto. Cont. **40** (1995), pp. 1889–1904.

[9] J. G. Dai and G. Weiss, *Stability and instability of fluid models for certain reentrant lines*, Math. Oper. Res. **21** (1996), pp. 115–134.

[10] P. Dupuis, R. Atar and A. Shwartz, *An escape-time criterion for queueing networks: asymptotic risk-sensitive control via differential games*, Math. Oper. Res. **28** (2003), pp. 801–835

[11] P. Dupuis and H. Ishii, *On Lipschitz continuity of the solution mapping to the Skorokhod problem, with applications*, Stoch. and Stoch. Reports **35** (1991), pp. 31–62.

[12] P. Dupuis and K. Ramanan, *Convex duality and the Skorokhod problem, I and II*, Prob. Th. and Rel. F. **115** (1999), pp. 153–195 and pp. 197–236.

[13] S. N. Ethier and T. G. Kurtz, Markov Processes: Characterization and Convergence, John Wiley & Sons, New York, NY, 1986.

[14] W. H. Fleming and H. M. Soner, Controlled Markov Processes and Viscosity Solutions, Springer-Verlag, New-York, 1993.

[15] J. Jacod, *Multivariate point processes: predictable projection, Radon-Nikodym derivatives, representation of martingales*, Z. Wahr. **31** (1975), pp. 235–253.

[16] H. J. Kushner, Heavy Traffic Analysis of Controlled Queueing and Communication Networks, Springer-Verlag, New York, 2001.

[17] S. P. Meyn, *Stability and asymptotic optimality of generalized MaxWeight policies*, SIAM J. Cont. Opt. **47** (2009), pp. 3259–3294.

[18] S. P. Meyn, Control Techniques for Complex Networks, Cambridge Univ. Press, New York, 2008.

[19] S. P. Meyn, *Workload models for stochastic networks: value functions and performance evaluation*, IEEE Trans. Aut. Cont. **50** (2005), pp. 1106–1122.

[20] S. P. Meyn, *The policy iteration algorithm for average reward Markov decision processes with general state space*, IEEE Trans. Aut. Cont. **42** (1997), pp. 1663–1680.

[21] S. P. Meyn, *Stability and optimization of queueing networks and their fluid models*, in Mathematics of Stochastic Manufacturing Systems pp. 175–199, AMS, Providence, RI (1997).

[22] Y. Nazarathy and G. Weiss, *Near optimal control of queueing networks over a finite time horizon*, Ann. Op. Res. **170** (2005), pp. 233-249.

[23] G. Pang and M. V. Day, *Fluid limits of optimally controlled queueing networks*, J. Appl. Math. and Stoch. Anal. **2007** (2007), Article ID 68958. (Available at `http://www.hindawi.com/journals/ijsa/2007/068958.abs.html`.)

[24] L. C. G. Rogers, David Williams , Diffusions, Markov Processes and Martingales, Volume 1: Foundations (2nd Edition), Cambridge Univ. Press, Cambridge, UK, 2000.