Robust optimal switching control for nonlinear systems

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Abstract. We formulate a robust optimal control problem for a general nonlinear system with finitely many admissible control settings and with costs assigned to switching of controls. We formulate the problem both in an L_2-gain/dissipative system framework and in a game-theoretic framework. We show that, under appropriate assumptions, a continuous switching-storage function is characterized as a viscosity supersolution of the appropriate system of quasivariational inequalities (the appropriate generalization of the Hamilton-Jacobi-Bellman-Isaacs equation for this context), and that the minimal such switching-storage function is equal to the continuous switching lower-value function for the game. Finally we show how a prototypical example with one-dimensional state space can be solved by a direct geometric construction.

Key Words. running cost, switching cost, worst-case disturbance attenuation, differential game, state-feedback control, nonanticipating strategy, storage function, lower value function, system of quasivariational inequalities, viscosity solution

AMS Classification. Primary: 49J35; Secondary: 49L20, 49L25, 49J35, 93B36, 93B52

Abbreviated title. Robust optimal switching control.

1 Introduction

We consider a state-space system Σ_{sw}

$$\dot{y} = f(y, a, b) \tag{1.1}$$
$$z = h(y, a, b) \tag{1.2}$$

where $y(t) \in \mathbb{R}^n$ is the state, $a(t) \in A \subset \mathbb{R}^p$ is the control input, $b(t) \in B \subset \mathbb{R}^m$ is the deterministic unknown disturbance, and $z(t) \in \mathbb{R}$ is the cost function. We assume that the set A of admissible control values is a finite set, $A = \{a^1, \dots, a^r\}$. The control signals $a(t)$ are then necessarily piecewise constant with values in A. We normalize control signals $a(t)$ to be right continuous, and refer to the value $a(t)$ as the *new current control* and $a(t^-)$ as the *old current control* at time t. We assume that there is a distinguished input index i_0 for which $f(0, a^{i_0}, 0) = 0$ and $h(0, a^{i_0}, 0) = 0$, so that 0 is an equilibrium point for the autonomous system induced by setting $a(t) = a^{i_0}$ and $b(t) = 0$. In addition we assume that a cost $k(a^i, a^j) \geq 0$ is assigned at each time instant τ_n at which the controller switches from old current control $a(\tau_n^-) = a^i$ to new current control $a(\tau_n) = a^j$. For a given old initial control $a(0^-)$, the associated control decision is to choose switching times

$$0 \leq \tau_1 < \tau_2 < \dots, \qquad \lim_{n \to \infty} \tau_n = \infty$$

and controls

$$a(\tau_1),\ a(\tau_2),\ a(\tau_3), \dots$$

such that the controller switches from the old current control $a(\tau_n^-)$ to the new current control $a(\tau_n) \neq a(\tau_n^-)$ at time τ_n, where we set

$$a(t) = \begin{cases} a(0^-), & t \in [0, \tau_1) \\ a(\tau_n), & t \in [\tau_n, \tau_{n+1}),\ n = 1, 2, \dots, \end{cases}$$

if $\tau_1 > 0$ and

$$a(t) = a(\tau_n), \qquad t \in [\tau_n, \tau_{n+1}),\ n = 1, 2, \dots,$$

otherwise. We assume that the state $y(\cdot)$ of (1.1) does not jump at the switching time τ_n, i.e., the solution $y(\cdot)$ is assumed to be absolutely continuous. The cost of running the system up to time $T \geq 0$ with initial state $y(0) = x$, old initial control $a(0^-) = a^j$, control signal a for $t \geq 0$, and disturbance signal b is given by

$$C_{T^-}(x, a^j, a, b) = \int_0^T h(y_x(t, a, b), a(t), b(t))\ dt + \sum_{\tau:\ 0 \leq \tau < T} k(a(\tau^-), a(\tau)).$$

We have used the notation $y_x(\cdot, a, b)$ for the unique solution of (1.1) corresponding to the choices of the initial condition $y(0) = x$, the control $a(\cdot)$ and the disturbance $b(\cdot)$. In the sequel we will often abbreviate $y_x(\cdot, a, b)$ to $y_x(\cdot)$ or $y(\cdot)$; the precise meaning should be clear from the context.

As the running cost $h(y(t), a(t), b(t)) + k(a(t^-), a(t))$, where $a(t^-) = a^j$ if $t = 0$, involves not only the value $y(t)$ of the state along with the value of the control $a(t)$

and the value of the disturbance $b(t)$ at time t but also the value of the old current control $a(t^-)$, it makes sense to think of the old current control $a(t^-)$ at time t as part of an augmented state vector $y^{aug}(t) = (y(t), a(t^-))$ at time t. This can be done formally by including $a(t^-)$ as part of the state vector, in which case a switching control problem becomes an *impulse* control problem (see [10], where problems of this sort are set in the general framework of hybrid systems). We shall keep the switching-control formalism here; however, in implementing optimization algorithms, we shall see that it is natural to consider augmented state-feedback controls $(x, a^j) \to a(x, a^j)$ rather than merely state-feedback controls $x \to a(x)$ in order to obtain solutions. We shall refer to such augmented state-feedback controls $(x, a^j) \to a(x, a^j) \in A$ as simply *switching state-feedback* controllers. Note that while the augmented-state is required to compute the instantaneous running cost at time t, only the (nonaugmented) state vector $y(t)$ is needed to determine the state trajectory past time t for a given input signal $(a(\cdot), b(\cdot))$ past time t.

The precise formulation of our optimal control problem is as follows. First of all, for a prescribed attenuation level $\gamma > 0$ and given augmented initial state (x, a^j), we seek an admissible control signal $a(\cdot) = a_{x,j}(\cdot)$ with $a(0^-) = a^j$ so that

$$C_{T^-}(x, a^j, a, b) \leq \gamma^2 \int_0^T |b(t)|^2 \, dt + U_\gamma^j(x) \tag{1.3}$$

for all locally L_2 disturbances b, all positive real numbers T and some nonnegative-valued bias function $U_\gamma^j(x)$ with $U_\gamma^{i_0}(0) = 0$. Note that this inequality corresponds to an input-output system having L_2-gain at most γ, where C_{T^-} replaces the L_2-norm of the output signal over the time interval $[0, T]$, and where the equilibrium point is taken to be $(0, a^{i_0})$ in the augmented state space. The dissipation inequality (1.3) then can be viewed as an L_2-gain inequality, and our problem as the analogue of the nonlinear H^∞-control problem for systems with switching costs (see [17]). In the *open loop* version of the problem, the control signal $a(\cdot)$ is simply a piecewise-constant right-continuous function with values in $A = \{a^1, \ldots, a^r\}$. In the *switching state feedback* version of the problem, $a(\cdot)$ is a function of the current state and current old control, i.e., one decides what control to use at time t based on knowledge of the current augmented state $(y(t), a(t^-))$. In the standard game-theoretic formulation of the problem, $a(\cdot)$ is a nonanticipating function $a(\cdot) = \alpha_x^j[b](\cdot)$ (called a *strategy*) of the disturbance b depending also on the initial state x and initial old control value a^j, i.e., for a given augmented initial state (x, a^j), the computation of the control value $\alpha_x^j[b](t)$ at time t uses knowledge only of the past and current values of the disturbance $b(\cdot)$. Secondly, we ask for the admissible control a with $a(0^-) = a^j$ (with whatever information structure) which gives the best system performance, in the sense that the nonnegative functions $U^j(x)$ are as small as possible. A closely related problem formulation is to view the switching-control system as a game with payoff function

$$J_{T^-}(x, a^j, a, b) = \int_{[0,T)} l(y_x(t), a^j, a(t), b(t)), \quad a(0^-) = a^j,\ j = 1, \dots, r,$$

where we view $l(y_x, a^j, a, b)$ as the measure given by

$$l(y(t), a^j, a(t), b(t)) = [h(y(t), a(t), b(t)) - \gamma^2 |b(t)|^2]\, dt + k(a(t^-), a(t))\delta_t, \qquad a(0^-) = a^j,$$

where δ_t is the unit point-mass distribution at the point t. In this game setting, the disturbance player seeks to use $b(t)$ and T to maximize the payoff while the control player seeks to use the choice of piecewise-constant right-continuous function $a(t)$ to minimize the payoff. The *switching lower value* $V_\gamma = (V_\gamma^1, \dots, V_\gamma^r)$ of this game is then given by

$$V_\gamma^j(x) = \inf_{\alpha} \sup_{b,\, T} J_{T^-}(x, a^j, \alpha_x^j[b], b), \quad j = 1, \dots, r \tag{1.4}$$

where the supremum is over all nonnegative real numbers T and all locally L_2-disturbance signals b, while the infimum is over all nonanticipating control strategies $b \to \alpha_x^j[b]$ depending on the initial augmented state (x, a^j). By letting T tend to 0, we see that each component of the switching lower value $V_\gamma(x) = (V_\gamma^1(x), \dots, V_\gamma^r(x))$ is nonnegative. Then by construction $(V_\gamma^1, \dots, V_\gamma^r)$ gives the smallest possible value which can satisfy (1.3) (with V_γ^j in place of U_γ^j) for some nonanticipating strategy $(x, a^j, b) \to \alpha_x^j[b](\cdot) = a(\cdot)$.

In the standard theory of nonlinear H^∞-control, the notion of *storage function* for a dissipative system plays a prominent role (see [17]). For our setting with switching costs, we say that a nonnegative vector function $S_\gamma = (S_\gamma^1, \dots, S_\gamma^r)$ on $\mathbb{R}^n$ is a *switching storage function* for the system (1.1)–(1.2) and given strategy α

$$\begin{aligned}
&S_\gamma^{j(t_2)}(y_x(t_2, \alpha_x^j[b], b)) - S_\gamma^{j(t_1)}(y_x(t_1, \alpha_x^j[b], b)) \\
&\leq \int_{t_1}^{t_2} [\gamma^2 |b(s)|^2 - h(y_x(s), \alpha_x^j[b](s), b(s))]\, ds - \textstyle\sum_{t_1 \leq \tau < t_2} k(\alpha_x^j[b](\tau^-), \alpha_x^j[b](\tau)) \\
&\text{for all } y(0) = x \in \mathbb{R}^n,\ b \text{ measurable with values in } B,\ 0 \leq t_1 < t_2
\end{aligned} \tag{1.5}$$

(where $j(t)$ is specified by $\alpha_x^j[b](t^-) = a^{j(t)}$). The control problem then is to find the switching strategy $\alpha : (x, a^j, b) \to \alpha_x^j[b](\cdot)$ which gives the best performance, as measured by obtaining the minimal possible $S_\gamma(x) = (S_\gamma^1(x), \dots, S_\gamma^r(x))$ as the associated closed-loop switching storage function. Note that any switching storage function may serve as the vector bias function $U_\gamma = (U_\gamma^1, \dots, U_\gamma^r)$ in the L_2-gain inequality (1.3), if in addition $S_\gamma^{i_0}(0) = 0$. This suggests that the *available switching-storage function* (i.e., the minimal possible switching storage function over all possible switching strategies) should equal the switching lower-value V_γ (1.4) for the game described above. We shall see that this is indeed the case with appropriate hypotheses imposed.

Our main results concerning the robust optimal switching-cost problem are as follows: *Under minimal smoothness assumptions on the problem data and compactness of the set B,*

(i) $V_\gamma^j(x) \le \min_{i\neq j}\{V_\gamma^i(x) + k(a^j, a^i)\},\ x \in \mathbb{R}^n,\ j = 1, \dots, r.$

(ii) *If continuous, V_γ is a viscosity solution in $\mathbb{R}^n$ of the system of quasivariational inequalities defined in Section 2 (see* (2.5)*).* (The precise definition of viscosity subsolution, supersolution and solution will be given in Section 2.)

(iii) *If $S_\gamma = (S_\gamma^1, \dots, S_\gamma^r)$ is a continuous switching-storage function for some strategy α, then S_γ is a nonnegative, continuous viscosity supersolution of the SQVI (2.5).*

(iv) *If $U_\gamma = (U_\gamma^1, \dots, U_\gamma^r)$ is a nonnegative, continuous viscosity supersolution of the SQVI (2.5) and U_γ has the property (i), then there is a canonical choice of switching state-feedback control strategy $\alpha_{U_\gamma} : (x, a^j, b) \to \alpha^j_{U_\gamma,x}[b]$ such that U_γ is a switching-storage function for the closed-loop system formed by using the strategy α_{U_γ}; thus,*

$$U_\gamma^j(x) \ge \sup_{b,\,T}\{\int_{[0,T)} l(y_x(s), a^j, \alpha^j_{U_\gamma,x}[b](s), b(s))\} \ge V_\gamma^j(x).$$

The switching lower-value V_γ, if continuous, is characterized as the minimal, nonnegative, continuous viscosity supersolution of (2.5) having property (i) above, as well as the minimal continuous function satisfying property (i) which is a switching storage function for the closed-loop system associated with some nonanticipating strategy α_{V_γ}.

The derivation of this characterization of V_γ presented in this paper is a direct argument which parallels the argument given in [1] for the analogous result for optimal stopping-time problems. An alternative derivation of this characterization relying on a general comparison principle for viscosity super- and subsolutions of SQVI is given in [2].

The usual formulation of the H^∞-control problem also involves a stability constraint. We also prove that, under appropriate conditions, the closed loop system associated with switching strategy α_{U_γ} corresponding to the nonnegative, continuous supersolution U_γ of the SQVI is stable. The main idea is to use the supersolution U_γ

as a Lyapunov function for trajectories of the closed-loop system. Related stability problems for systems with control switching are discussed, e.g., by Branicky in [11].

Infinite-horizon optimal switching-control problems are discussed in [6, Chapter III, Section 4.4] but with a discount factor in the running cost and no disturbance term. Differential games with switching strategies and switching costs for the case of finite horizon problems is discussed in [20] while the case of an infinite horizon with both control and competing disturbance but with a discount factor in the running cost is discussed in [21]. A more general formulation of the finite-horizon optimal control and differential game problems, where a general (not necessarily discrete) measure is allowed to enter both the dynamics and the running cost, is studied in [7]. These authors, under their various assumptions, were able to show that the value function is continuous and is the unique solution of the appropriate system of quasivariational inequalities. However our formulation has no discount factor in the running cost, so the running cost is not guaranteed to be integrable over the infinite interval $[0, \infty]$. This forces the introduction of the extra "disturbance player" T in (1.4). We establish a Dynamic Programming Principle for this setting and derive from it the appropriate system of quasivariational inequalities (SQVI) to be satisfied by V_γ. Due to a lack of positive discount factor and the presence of the extra disturbance player T, our lower-value function V_γ probably in general is not continuous, and moreover cannot be characterized simply as the unique solution of the SQVI as is the case for finite-horizon problems and problems with a positive discount factor. Our contribution is to apply the dynamic-programming method to a robust formulation of the optimal switching-cost problem analogous to the standard nonlinear H^∞-control problem; our results (particularly the characterization of the switching lower value as the minimal viscosity supersolution of the appropriate SQVI) parallel those of Soravia [18] obtained for the standard nonlinear H^∞-control problem (see also [12], [19] and [6, Appendix B] for later, closely related refinements of the nonlinear H_∞ results).

As explained in (iv) above, once we have found a viscosity supersolution (or uniquely determined minimal viscosity supersolution) U_γ of the SQVI, the associated control strategy α_{U_γ} which achieves the L_2-gain inequality (1.3) (or, at the state-space level, the dissipation inequality (1.5)) is easily found. For practical applications there remains the problem of computing the minimal, nonnegative viscosity supersolution of the SQVI. There is an interesting connection between solutions of SQVIs and solutions of variational inequalities (VIs) associated with optimal stopping-time problems, namely: the solution of an SQVI is a fixed point of a map which assigns to a given vector function the collection of solutions of a decoupled system of VIs, or, at the level of value functions, the switching lower-value function for a switching-control problem is a fixed point of the map which assigns to an r-tuple of nonnegative-real valued functions the set of lower value functions for a decoupled collection of stopping-time problems (with different terminal cost functions determined by the input vector function). In principle, it should therefore be possible to find the lower value function for a switching-control problem by iteratively solving for the value functions of a decoupled system of stopping-time problems, and thereby reduce solution of a switching-control problem to the iterative solution of decoupled systems of VIs. This idea is discussed in [9] in the context of stochastic, diffusion problems, and a similar connection existing between impulse-control problems and stopping-time problems has been pointed out in [6, Chapter III, Section 4.3], where, in addition, some convergence results concerning the associated iteration procedure are presented. Thus one can view stopping-time problems as having pedagogical value as stepping stones to the more complicated impulse-control and switching-control problems. In a companion paper [1], we treat a deterministic robust-control version of such optimal stopping-time problems.

For the case of a one-dimensional state space, we solve a simple example of switching-control problem by a direct, geometric construction, and thereby bypass the iterative procedure which uses the connection with stopping-time problems. A related example for the VI associated with an optimal stopping-time problem is treated in [1].

Original motivation for our work arose from the problem of designing a real-time feedback control for traffic signals at a highway intersection (see [3], [4]), where the size of the cost imposed on switching can be used as a tuning parameter to lead to more desirable types of traffic-light signalization. Also a positive switching cost eliminates the chattering present in the solution otherwise.

The paper is organized as follows. In Section 2 we discuss assumptions and definitions. Section 3 presents the main results on the connection between value functions (and storage functions) with systems of quasi-variational inequalities. Section 4 presents stability of the closed-loop switching control system. The final Section 5 discusses computational issues and gives explicit, geometric procedures for computing lower-value functions for a prototypical one-dimensional example.

2 Preliminaries

Let $A = \{a^1, a^2, \cdots, a^r\}$ be a finite set and let B be a compact subset of $\mathbb{R}^m$ containing the origin 0. We consider a general nonlinear system Σ_{sw} (see (1.1)–(1.2)) with a switching-cost function k. We make the following assumptions on problem data $f,\ h,\ k$:

(A1) $f : \mathbb{R}^n \times A \times B \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times A \times B \to \mathbb{R}$ are continuous;

(A2) f and h are bounded on $B(0, R) \times A \times B$ for all $R > 0$;

(A3) there are moduli ω_f and ω_h such that

$$\begin{aligned}|f(x,a,b)-f(y,a,b)| &\le \omega_f(|x-y|,R)\\ |h(x,a,b)-h(y,a,b)| &\le \omega_h(|x-y|,R),\end{aligned}$$

for all $x, y \in B(0,R)$, $R>0$, $a \in A$ and $b \in B$;

(A4) $|f(x,a,b)-f(y,a,b)| \le L|x-y|$ for all $x, y \in \mathbb{R}^n$, $a \in A$ and $b \in B$;

(A5) $k : A \times A \to \mathbb{R}$ and

$$\begin{aligned}k(a^j,a^i) &< k(a^j,a^d)+k(a^d,a^i)\\ k(a^j,a^i) &> 0\\ k(a^j,a^j) &= 0,\end{aligned}$$

for all $a^d,\ a^i,\ a^j \in A,\ d \ne i \ne j$;

(A6) $h(x,a,0) \ge 0$ for all $x \in \mathbb{R}^n,\ a \in A$.

The set of admissible controls for our problem is the set

$$\begin{aligned}\mathcal{A} \;=\; &\{a(\cdot)=\sum_{i\ge 1} a_{i-1}1_{[\tau_{i-1},\tau_i)}(\cdot) : [0,+\infty)\to A|\ a_i \in A;\ a_i \ne a_{i-1} \text{ for } i \ge 1,\\ &0=\tau_0 \le \tau_1 < \tau_2 < \cdots,\ \tau_i \uparrow \infty\}\end{aligned}$$

consisting of piecewise-constant right-continuous functions on $[0,\infty)$ with values in the control set A, where we denote by $\tau_1, \tau_2, \ldots$ the points at which control switchings occur. The set of admissible disturbances is $\mathcal{B}$ which consists of locally L_2-functions on $[0,\infty)$ with values in the set B:

$$\mathcal{B} = \{b : [0,\infty) \to B \mid \int_0^T |b(s)|^2 ds < \infty, \text{ for all } T>0\}.$$

A *strategy* is a map $\alpha\colon \mathbb{R}^n \times A \times \mathcal{B} \to \mathcal{A}$ with value at (x,a^j,b) denoted by $\alpha_x^j[b](\cdot)$. The strategy α assigns control function $a(t)=\alpha_x^j[b](t)$ if the augmented initial condition is (x,a^j) and the disturbance is $b(\cdot)$. Thus, if it happens that $\tau_1 > \tau_0 = 0$, then $a(t)=a_0=a^j$, for $t \in [\tau_0,\tau_1)$. Otherwise $a(t)=a_1 \ne a^j$, for $t \in [0,\tau_2)=[\tau_1,\tau_2)$ and an instantaneous charge of $k(a^j,a(0))$ is incurred at time 0 in the cost function. A strategy α is said to be *nonanticipating* if, for each $x \in \mathbb{R}^n$ and $j \in \{1,\ldots,r\}$, for any $T>0$ and $b,\ \bar{b} \in \mathcal{B}$ with $b(s)=\bar{b}(s)$ for all $s \le T$, it follows that $\alpha_x^j[b](s)=\alpha_x^j[\bar{b}](s)$ for all $s \le T$. We denote by Γ the set of all nonanticipating strategies:

$$\Gamma := \{\alpha\colon \mathbb{R}^n \times A \times \mathcal{B} \to \mathcal{A} \mid \alpha_x^j \text{ is nonanticipating for each } x \in \mathbb{R}^n \text{ and } j=1,\ldots,r\}.$$

We consider trajectories of the nonlinear system

$$\left\{ \begin{array}{l} \dot{y}(t) = f(y(t), a(t), b(t)) \\ y(0) = x. \end{array} \right. \tag{2.1}$$

Under the assumptions (A1), (A2) and (A4), for given $x \in \mathbb{R}^n$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$, the solution of (2.1) exists uniquely for all $t \geq 0$. We denote by $y_x(\cdot, a, b)$ or simply $y_x(\cdot)$ the unique solution of (2.1) corresponding to the choice of the initial condition $x \in \mathbb{R}^n$, the control $a(\cdot) \in \mathcal{A}$ and the disturbance $b(\cdot) \in \mathcal{B}$. We also have the usual estimates on the trajectories (see e.g. [6, pages 97-99]:

$$\begin{aligned} |y_x(t, a, b) - y_z(t, a, b)| &\leq e^{Lt}|x - z|, \quad t > 0 & (2.2)\\ |y_x(t, a, b) - x| &\leq M_x t, \quad t \in [0, 1/M_x], & (2.3)\\ |y_x(t, a, b)| &\leq (|x| + \sqrt{2Kt})e^{Kt} & (2.4) \end{aligned}$$

for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, where

$$\begin{aligned} M_x &:= \max\{|f(z, a, b)| : |x - z| \leq 1, \ a \in A, \ b \in B\} \\ K &:= L + \max\{|f(0, a, b)| : a \in A, \ b \in B\}. \end{aligned}$$

For a specified gain tolerance $\gamma > 0$, we define the Hamiltonian function $H^j : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by setting

$$H^j(y, p) := \min_{b \in B}\{-p \cdot f(y, a^j, b) - h(y, a^j, b) + \gamma^2|b|^2\}, \ j = 1, \dots, r.$$

Note that $H^j(y, p) < +\infty$ for all $y, p \in \mathbb{R}^n$ by (A2). Under assumptions (A1)-(A4), one can show that the Hamiltonian H^j is continuous on $\mathbb{R}^n \times \mathbb{R}^n$ and satisfies

$$\begin{aligned} &|H^j(x, p) - H^j(y, p)| \leq L|x - y||p| + \omega_h(|x - y|, R), \\ &\text{ for all } p \in \mathbb{R}^n, \ x, y \in B(0, R), \ R > 0, \text{ and} \\ &|H^j(x, p) - H^j(x, q)| \leq L(|x| + 1)|p - q|, \text{ for all } x, p, q \in \mathbb{R}^n. \end{aligned}$$

We now introduce the system of quasivariational inequalities (SQVI)

$$\max\{H^j(x, Du^j(x)), \ u^j(x) - \min_{i \neq j}\{u^i(x) + k(a^j, a^i)\}\} = 0, \ x \in \mathbb{R}^n, \ j = 1, 2, \dots, r. \tag{2.5}$$

Definition 2.1 *A vector function $u = (u^1, u^2, \dots, u^r)$, where $u^j \in C(\mathbb{R}^n)$, is a viscosity subsolution of the SQVI (2.5) if, for any $\varphi^j \in C^1(\mathbb{R}^n)$,*

$$\max\{H^j(x_0, D\varphi^j(x_0)), \ u^j(x_0) - \min_{i \neq j}\{u^i(x_0) + k(a^j, a^i)\}\} \leq 0, \ j = 1, 2, \dots, r,$$

at any local maximum point $x_0 \in \mathbb{R}^n$ of $u^j - \varphi^j$. Similarly u is a viscosity supersolution of the SQVI (2.5) if for any $\varphi^j \in C^1(\mathbb{R}^n)$

$$\max\{H^j(x_1, D\varphi^j(x_1)),\ u^j(x_1) - \min_{i \neq j}\{u^i(x_1) + k(a^j, a^i)\}\} \geq 0,\ j = 1, 2, \dots, r,$$

at any local minimum point $x_1 \in \mathbb{R}^n$ of $u^j - \varphi^j$. Finally u is a viscosity solution of the SQVI (2.5) if it is simultaneously a viscosity sub- and supersolution.

3 Main Results

In this section we show the connection of the lower value function $V_\gamma = (V_\gamma^1, \dots, V_\gamma^r)$ (see (1.4)) (and a switching storage function) with the SQVI (2.5).

We begin with the application of the Dynamic Programming to this setting, and then derive some properties of the lower value vector function V_γ (see (1.4)). We then use these properties to show that V_γ, if continuous, is a viscosity solution of the SQVI (2.5). Throughout this section, we assume that V_γ is finite.

Proposition 3.1 *Assume (A1)-(A5). Then for $j = 1, 2, \dots, r$ and $x \in \mathbb{R}^n$, the lower value vector function $V_\gamma = (V_\gamma^1, \dots, V_\gamma^r)$ given by* (1.4) *satisfies*

$$V_\gamma^j(x) \leq \min_{i \neq j}\{V_\gamma^i(x) + k(a^j, a^i)\}.$$

Proof Fix a pair of indices $i, j \in \{1, \dots, r\}$ with $i \neq j$. For a given $x \in \mathbb{R}^n$, $\alpha \in \Gamma$, $b \in \mathcal{B}$ and $T > 0$, we have

$$\begin{aligned}
\int_{[0,T)} \ell(y_x(s), a^j, \alpha_x^j[b](x), b(s)) &= k(a^j, \alpha_x^j[b](0)) + \int_{[0,T)} \ell(y_x(s), \alpha_x^j[b](0), \alpha_x^j[b](s), b(s)) \\
&= k(a^j, \alpha_x^j[b](0)) - k(a^i, \alpha_x^j[b](0)) \\
&\quad + k(a^i, \alpha_x^j[b](0)) + \int_{[0,T)} \ell(y_x(s), \alpha_x^j[b](0), \alpha_x^j[b](s), b(s)) \\
&= k(a^j, \alpha_x^j[b](0)) - k(a^i, \alpha_x^j[b](0)) + \int_{[0,T)} \ell(y_x(s), a^i, \alpha_x^j[b](s), b(s)) \\
&\leq k(a^j, a^i) + \int_{[0,T)} \ell(y_x(s), a^i, \alpha_x^j[b](s), b(s)) \qquad (3.1)
\end{aligned}$$

where the last inequality follows from (A5). By the definition of $V_\gamma^j(x)$, we have

$$V_\gamma^j(x) \leq \sup_{b \in \mathcal{B}, T \geq 0} \int_{[0,T)} \ell(y_x(s), a^j, \alpha_x^j[b](s), b(s))$$

for all $\alpha \in \Gamma$. Taking the supremum over $b \in \mathcal{B}$ and $T \geq 0$ on the right-hand side of (3.1) therefore gives

$$V_\gamma^j(x) \leq k(a^j, a^i) + \sup_{b \in \mathcal{B}, T \geq 0} \int_{[0,T)} \ell(y_x(s), a^i, \alpha_x^j[b](s), b(s)). \tag{3.2}$$

Given any strategy $\alpha \in \Gamma$, we can always find another $\widetilde{\alpha} \in \Gamma$ with $\widetilde{\alpha}_x^i[b] = \alpha_x^j[b]$ for each $b \in \mathcal{B}$, and, conversely, for any $\widetilde{\alpha} \in \Gamma$ there is a $\alpha \in \Gamma$ so that $\widetilde{\alpha}_x^i$ is determined by α in this way. Hence, taking the infimum over all $\alpha \in \Gamma$ in the last terms on the right hand side of (3.2) leaves us with $V_\gamma^i(x)$. Thus

$$V_\gamma^j(x) \leq k(a^j, a^i) + V_\gamma^i(x).$$

Since $i \neq j$ is arbitrary, the result follows. $\diamondsuit$

Theorem 3.2 (Dynamic Programming Principle) *Assume (A1), (A2) and (A4). Then, for $j = 1, 2, \dots, r$, $t > 0$ and $x \in \mathbb{R}^n$, we have*

$$\begin{aligned} V_\gamma^j(x) \quad = \quad & \inf_{\alpha \in \Gamma} \sup_{b \in \mathcal{B},\, T > 0} \{ \int_{[0, t \wedge T)} l(y_x(s, a^j, \alpha_x^j[b], b), \alpha_x^j[b](s), b(s)) + \\ & \qquad 1_{[0,T)}(t) V_\gamma^i(y_x(t, \alpha_x^j[b], b)), \qquad \alpha_x^j[b](t^-) = a^i \}. \end{aligned} \tag{3.3}$$

where

$$l(y(s), a^j, a(s), b(s)) := [h(y(s), a(s), b(s)) - \gamma^2 |b(s)|^2] ds + k(a(s^-), a(s)) \delta_s.$$

with $a(0^-) = a^j$.

Proof Fix $x \in \mathbb{R}^n$, $j \in \{1, 2, \dots, r\}$ and $t > 0$. We denote by $\omega(x)$ the right hand side of (3.3). Let $\epsilon > 0$. For any $z \in \mathbb{R}^n$ and any $a^\ell \in A$, we pick $\bar{\alpha} \in \Gamma$ such that

$$V_\gamma^\ell(z) + \epsilon \geq \int_{[0,T)} l(y_z(s), a^\ell, \bar{\alpha}_z^\ell[b](s), b(s)), \ \forall b \in \mathcal{B}, \ \forall T > 0. \tag{3.4}$$

We first want to show that $\omega(x) \geq V_\gamma^j(x)$. Choose $\hat{\alpha} \in \Gamma$ such that

$$\omega(x) + \epsilon \geq \sup_{b \in \mathcal{B},\, T \geq 0} \left\{ \int_{[0, t \wedge T)} l(y_x(s), a^j, \hat{\alpha}_x^j[b](s), b(s)) + 1_{[0,T)}(t) V_\gamma^i(y_x(t)), \ \hat{\alpha}_x^j[b](t^-) = a^i \right\} \tag{3.5}$$

For each $b \in \mathcal{B}$ and $T > 0$, choose $\delta \in \Gamma$ so that

$$\delta_x^j[b](s) = \begin{cases} \hat{\alpha}_x^j[b](s) & s < t \wedge T \\ \bar{\alpha}_z^i[b(\cdot + t \wedge T)](s - (t \wedge T)) & s \geq t \wedge T \end{cases}$$

with $z := y_x(t \wedge T, \hat{\alpha}^j_x[b], b)$ and $a^i := \hat{\alpha}^j_x[b](t \wedge T)$. Clearly, δ^j_x is nonanticipating because $\hat{\alpha}^j_x$ and $\bar{\alpha}^i_z$ are. Note that

$$y_x(s + t \wedge T, \delta^j_x[b], b) = y_z(s, \bar{\alpha}^i_z[b(\cdot + t \wedge T)], b(\cdot + t \wedge T)), \text{ for } s \geq 0$$

Thus by the change of variables $\tau = s + t \wedge T$, we have

$$\int_{[0,T-(t\wedge T))} l(y_z(s), a^i, \bar{\alpha}^i_z[b(\cdot + t \wedge T)](s), b(s + t \wedge T)) = \int_{[t\wedge T,T)} l(y_x(\tau), a^j, \delta^j_x[b](\tau), b(\tau)) \tag{3.6}$$

As a consequence of (3.4), (3.5) and (3.6), we have

$$\begin{aligned}
\omega(x) + 2\epsilon \quad &\geq \quad \sup_{b\in\mathcal{B},\, T>0} \{\int_{[0,t\wedge T)} l(y_x(s), a^j, \hat{\alpha}^j_x[b](s), b(s)) \\
&\qquad + 1_{[0,T)}(t) \int_{[t\wedge T,T)} l(y_z(s), a^i, \bar{\alpha}^i_z[b](s), b(s))\} \\
&= \quad \sup_{b\in\mathcal{B},\, T>0} \{\int_{[0,T)} l(y_x(s), a^j, \delta^j_x[b](s), b(s))\} \\
&\geq \quad \inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B},\, T>0} \{\int_{[0,T)} l(y_x(s), a^j, \alpha^j_x[b](s), b(s))\} \\
&= \quad V^j_\gamma(x)
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we conclude that $\omega(x) \geq V^j_\gamma(x)$.

Next we want to show that $\omega(x) \leq V^j_\gamma(x)$. From the definition of $\omega(x)$, choose $b_1 \in \mathcal{B}$ and $T_1 \geq 0$ such that

$$\omega(x) - \epsilon \leq \int_{[0,T_1\wedge t)} l(y_x(s), a^j, \bar{\alpha}^j_x[b_1](s), b_1(s)) + 1_{[0,T_1)}(t) V^i_\gamma(y_x(t)) \tag{3.7}$$

where $\bar{\alpha}^j_x$ is defined as in (3.4) and $\bar{\alpha}^j_x[b_1](t^-) = a^i$ for some $a^i \in A$. If $t \geq T_1$, we have

$$\begin{aligned}
\omega(x) - \epsilon \quad &\leq \quad \int_{[0,T_1)} l(y_x(s), a^j, \bar{\alpha}^j_x[b_1](s), b_1(s)) \\
&\leq \quad \sup_{b\in\mathcal{B},\, T>0} \{\int_{[0,T)} l(y_x(s), a^j, \bar{\alpha}^j_x[b](s), b(s))\} \\
&\leq \quad V^j_\gamma(x) + \epsilon,
\end{aligned}$$

where the last inequality follows from (3.4). If $t < T_1$, we have

$$\omega(x) - \epsilon \leq \int_{[0,t)} l(y_x(s), \bar{\alpha}^j_x[b_1](s), b_1(s)) + V^i_\gamma(y_x(t)). \tag{3.8}$$

Set $z := y_x(t, \bar{\alpha}_x^j[b_1], b_1)$. For each $b \in \mathcal{B}$, define $\tilde{b} \in \mathcal{B}$ by

$$\tilde{b}(s) = \begin{cases} b_1(s) & s < t \\ b(s-t) & s \geq t \end{cases}$$

and choose $\widehat{\alpha} \in \Gamma$ so that

$$\widehat{\alpha}_z[b](s) = \bar{\alpha}_x^j[\tilde{b}](s+t) \quad \text{for } s \geq 0.$$

By definition of V_γ^i, choose $b_2 \in \mathcal{B}$ and $T_2 > 0$ such that

$$V_\gamma^i(z) - \epsilon \leq \int_{[0,T_2)} l(y_z(s), a^i, \widehat{\alpha}_z[b_2](s), b_2(s)).$$

Then, by change of variable $\tau = s + t$, we have

$$V_\gamma^i(z) - \epsilon \leq \int_{[t,t+T_2)} l(y_x(\tau), a^j, \bar{\alpha}_x^j[\tilde{b}_2](\tau), \tilde{b}_2(\tau)) \tag{3.9}$$

As a consequence of (3.8) and (3.9) we have

$$\begin{aligned}
\omega(x) - 2\epsilon &\leq \int_{[0,t)} l(y_x(s), a^j, \bar{\alpha}_x^j[b_1](s), b_1(s)) + \int_{[t,t+T_2)} l(y_x(\tau), a^j, \bar{\alpha}_x^j[\tilde{b}_2](\tau), \tilde{b}_2(\tau)) \\
&= \int_{[0,t+T_2)} l(y_x(\tau), a^j, \bar{\alpha}_x^j[\tilde{b}_2](\tau), \tilde{b}_2(\tau)) \\
&\leq \sup_{b \in \mathcal{B},\, T>0} \{ \int_{[0,T)} l(y_x(\tau), a^j, \bar{\alpha}_x^j[b](\tau), b(\tau)) \} \\
&\leq V_\gamma^j(x) + \epsilon,
\end{aligned}$$

where the last inequality follows from (3.4). Since $\epsilon > 0$ is arbitrary, for both cases we have $\omega(x) \leq V_\gamma^j(x)$ as required. $\diamondsuit$

Corollary 3.3 *Assume (A1)-(A4). Then for each* $j \in \{1, \dots, r\}$, $x \in \mathbb{R}^n$ *and* $t > 0$, *we have*

$$V_\gamma^j(x) \leq \sup_{b \in \mathcal{B},\, T>0} \{ \int_0^{t \wedge T} [h(y_x(s), a^j, b(s)) - \gamma^2 |b(s)|^2] ds + 1_{[0,T)}(t) V_\gamma^j(y_x(t)) \}.$$

Proof Fix $j \in \{1, \dots, r\}$, $x \in \mathbb{R}^n$ and $t > 0$. Define $\alpha \in \Gamma$ by setting $\alpha_x^j[b](s) = a^j$ for all $s \geq 0$ for each $b \in \mathcal{B}$. By Theorem 3.2, we have

$$V_\gamma^j(x) \leq \sup_{b \in \mathcal{B},\, T>0} \{ \int_0^{t \wedge T} [h(y_x(s), a^j, b(s)) - \gamma^2 |b(s)|^2] ds + 1_{[0,T)}(t) V_\gamma^j(y_x(t)) \}. \ \diamondsuit$$

Proposition 3.4 *Assume (A1)-(A5). Suppose that for each $j \in \{1, \dots, r\}$, V^j is continuous. If $V^j_\gamma(x) < \min_{i \neq j}\{V^i_\gamma(x) + k(a^j, a^i)\}$, then there exists $\tau = \tau_x > 0$ such that for $0 < t < t_x$*

$$V^j_\gamma(x) = \sup_{b \in \mathcal{B},\, T>0} \{ \int_0^{t \wedge T} [h(y_x(s), a^j, b(s)) - \gamma^2 |b(s)|^2] ds + 1_{[0,T)}(t) V^j_\gamma(y_x(t)) \}.$$

Proof We assume $V^j_\gamma(x) < \min_{i \neq j}\{V^i_\gamma(x) + k(a^j, a^i)\}$. From Corollary 3.3, we know that

$$V^j_\gamma(x) \leq \sup_{b \in \mathcal{B},\, T>0} \{ \int_0^{t \wedge T} [h(y_x(s), a^j, b(s)) - \gamma^2 |b(s)|^2] ds + 1_{[0,T)}(t) V^j_\gamma(y_x(t)) \}, \ \forall t > 0.$$

Suppose there is a sequence $\{t_n\}$ with $0 < t_n < \frac{1}{n}$ for $n = 1, 2, \dots$ such that

$$V^j_\gamma(x) < \sup_{b \in \mathcal{B},\, T>0} \{ \int_0^{t_n \wedge T} [h(y_x(s), a^j, b(s)) - \gamma^2 |b(s)|^2] ds + 1_{[0,T)}(t_n) V^j_\gamma(y_x(t_n)) \}. \tag{3.10}$$

Let $w(x, t_n)$ be the right hand side of (3.10). For each t_n, define $\epsilon_n = \frac{1}{3}[w(x, t_n) - V^j_\gamma(x)]$. As $t_n \to 0$ as $n \to \infty$, from (3.10) we see that $w(x, t_n) \to V^j_\gamma(x)$ and hence $\epsilon_n \to 0$ as $n \to \infty$. It follows that

$$V^j_\gamma(x) + \epsilon_n < w(x, t_n) - \epsilon_n \tag{3.11}$$

Choose $b_n \in \mathcal{B}$ and $T_n \geq 0$ such that

$$w(x, t_n) - \epsilon_n \leq \int_0^{t_n \wedge T_n} [h(y_x(s), a^j, b_n(s)) - \gamma^2 |b_n(s)|^2] ds + 1_{[0,T_n)}(t_n) V^j_\gamma(y_x(t_n)) \tag{3.12}$$

By Theorem 3.2 choose $\alpha_n \in \Gamma$ such that

$$V^j_\gamma(x) + \epsilon_n \geq \int_{[0, t_n \wedge T_n]} l(y_x(s), a^j, (\alpha_n)^j_x[b_n](s), b_n(s)) + 1_{[0,T_n)}(t_n) V^{i_n}_\gamma(y_x(t_n)), \tag{3.13}$$

where $(\alpha_n)^j_x[b_n](t_n^-) = a^{i_n} \in A$. From (3.11), (3.12) and (3.13), we have

$$\begin{aligned} &\textstyle\int_{[0, t_n \wedge T_n)} l(y_x(s), a^j, (\alpha_n)^j_x[b_n](s), b_n(s)) + 1_{[0,T_n)}(t_n) V^{i_n}_\gamma(y_x(t_n)) \\ &< \textstyle\int_0^{t_n \wedge T_n} [h(y_x(s), a^j, b_n(s)) - \gamma^2 |b_n(s)|^2] ds + 1_{[0,T_n)}(t_n) V^j_\gamma(y_x(t_n)). \end{aligned} \tag{3.14}$$

This implies that $(\alpha_n)^j_x[b_n]$ jumps in the interval $[0, t_n \wedge T_n]$. Without loss of generality assume the number of switchings is equal to d_n. If $t_n < T_n$ for infinitely many n, by going down to a subsequence we may assume $t_n \leq T_n$ for all n. From (3.13) we have

$$\begin{aligned}
V^j_\gamma(x) &\geq \limsup_{n\to\infty}\{\int_{[0,t_n\wedge T_n)} l(y_x(s), a^j, \alpha^j_{x,n}[b_n](s), b_n(s)) \\
&\quad +1_{[0,T_n)}(t_n)V^{i_n}_\gamma(y_x(t_n)),\ \alpha^j_{x,n}[b_n](t_n^-) = a^{i_n} \in A\} \\
&= \limsup_{n\to\infty}\{\int_0^{t_n}[h(y_x(s), \alpha^j_{x,n}[b_n](s), b_n(s)) - \gamma^2|b_n(s)|^2]ds \\
&\quad +\sum_{m=1}^{d_n} k(a_{m-1}, a_m) + V^{i_n}_\gamma(y_x(t_n)),\ \alpha^j_{x,n}[b_n](t_n) = a^{i_n} \in A\} \\
&= \limsup_{n\to\infty}\left\{\sum_{m=1}^{d_n} k(a_{m-1}, a_m) + V^{i_n}_\gamma(y_x(t_n)),\ \alpha^j_{x,n}[b_n](t_n^-) = a^{i_n} \in A\right\}.
\end{aligned}$$

By using continuity of $V^{i_n}_\gamma$ and $\sum_{m=1}^{d_n} k(a_{m-1}, a_m) > k(a^j, a^{i_n})$, we have

$$V^j_\gamma(x) \geq \min_{i\neq j}\{V^i_\gamma(x) + k(a^j, a^i)\}$$

which contradicts one of the assumptions. If $t_n \geq T_n$ for infinitely many n, again without loss of generality we may assume $t_n \geq T_n$ for all n. From (3.14) we have

$$\begin{aligned}
&\liminf_{n\to\infty}\{\textstyle\int_{[0,T_n]} l(y_x(s), a^j \alpha^j_{x,n}[b_n](s), b_n(s))\} \\
&\leq \limsup_{n\to\infty}\{\textstyle\int_0^{T_n}[h(y_x(s), a^j, b_n(s)) - \gamma^2|b_n(s)|^2]ds\},
\end{aligned}$$

or equivalently,

$$\begin{aligned}
&\liminf_{n\to\infty}\{\textstyle\int_0^{T_n}[h(y_x(s), \alpha^j_{x,n}[b_n](s), b_n(s)) - \gamma^2|b_n(s)|^2]ds + \sum_{m=1}^{d_n} k(a_{m-1}, a_m)\} \\
&\leq \limsup_{n\to\infty}\{\textstyle\int_0^{T_n}[h(y_x(s), a^j, b_n(s)) - \gamma^2|b_n(s)|^2]ds\}.
\end{aligned}$$

Thus

$$\begin{aligned}
\liminf_{n\to\infty}\left\{\sum_{m=1}^{d_n} k(a_{m-1}, a_m)\right\} &\leq \limsup_{n\to\infty}\left\{\int_0^{T_n} h(y_x(s), a^j, b_n(s))ds\right\} - \\
&\quad \liminf_{n\to\infty}\left\{\int_0^{T_n} h(y_x(s), \alpha^j_{x,n}[b_n](s), b_n(s))ds\right\},
\end{aligned}$$

and in this case $T_n \to 0$ as $n \to \infty$. Note that the integral terms tend to 0 uniformly with respect to $b_n \in \mathcal{B}$ as $T_n \to 0$ due to the compactness assumption on B, the uniform estimate (2.3), and the continuity assumption (A1) on h. Thus we have

$$\liminf_{n\to\infty}\left\{\sum_{m=1}^{d_n} k(a_{m-1}, a_m)\right\} \leq 0$$

which contradicts (A5). $\diamondsuit$

Lemma 3.5 *Assume (A1)- (A5) and* $V_\gamma^j \in C(\mathbb{R}^n)$, $j = 1, \ldots, r$.
If $V_\gamma^j(x) < \min_{i\neq j}\{V_\gamma^i(x) + k(a^j, a^i)\}$, *then there exists* $\tau = \tau_x > 0$ *such that* $V_\gamma^j(x) \geq \sup_{b\in\mathcal{B}}\{\int_0^t [h(y_x(s), a^j, b(s)) - \gamma^2|b(s)|^2]ds + V_\gamma^j(y_x(t))\}$, $\forall t \in (0, \tau_x)$.

Proof From Proposition 3.4, choose $\tau = \tau_x > 0$ such that for all $t \in (0, \tau)$

$$V_\gamma^j(x) = \sup_{b\in\mathcal{B},\, T>0}\{\int_0^{t\wedge T}[h(y_x(s), a^j, b(s)) - \gamma^2|b(s)|^2]ds + 1_{[0,T)}(t)V_\gamma^j(y_x(t))\}.$$

Thus

$$\begin{aligned} V_\gamma^j(x) &\geq \sup_{b\in\mathcal{B},\, T>t}\{\int_0^{t\wedge T}[h(y_x(s), a^j, b(s)) - \gamma^2|b(s)|^2]ds + 1_{[0,T)}(t)V_\gamma^j(y_x(t))\} \\ &= \sup_{b\in\mathcal{B}}\{\int_0^{t}[h(y_x(s), a^j, b(s)) - \gamma^2|b(s)|^2]ds + V_\gamma^j(y_x(t))\}. \diamondsuit \end{aligned}$$

Theorem 3.6 *Assume (A1)-(A6) and* $V_\gamma^j \in C(\mathbb{R}^n)$, $j = 1, \ldots, r$. *Then* V_γ *is a viscosity solution of the SQVI (2.5)*

$$\max\{H^j(x, DV_\gamma^j(x)),\ V_\gamma^j(x) - \min_{i\neq j}\{V_\gamma^i(x) + k(a^j, a^i)\}\} = 0, x \in \mathbb{R}^n,\ j = 1, \ldots, r. \tag{3.15}$$

Proof We first show that V_γ^j is a viscosity supersolution of the SQVI (3.15). Fix $x_0 \in \mathbb{R}^n$ and $a^j \in A$. Let $\varphi^j \in C^1(\mathbb{R}^n)$ and x_0 is a local minimum of $V_\gamma^j - \varphi^j$. We want to show that

$$\max\{H^j(x_0, D\varphi^j(x_0)),\ V_\gamma^j(x_0) - \min_{i\neq j}\{V_\gamma^i(x_0) + k(a^j, a^i)\}\} \geq 0 \tag{3.16}$$

We have two cases to consider

case 1 $V_\gamma^j(x_0) = \min_{i\neq j}\{V_\gamma^i(x_0) + k(a^j, a^i)\}$

case 2 $V_\gamma^j(x_0) < \min_{i\neq j}\{V_\gamma^i(x_0) + k(a^j, a^i)\}$.

If case 1 occurs, we have

$$\begin{aligned} &\max\{H^j(x_0, D\varphi^j(x_0)),\ V_\gamma^j(x_0) - \min_{i\neq j}\{V_\gamma^i(x_0) + k(a^j, a^i)\}\} \\ &\geq V_\gamma^j(x_0) - \min_{i\neq j}\{V_\gamma^i(x_0) + k(a^j, a^i)\} \\ &\geq 0. \end{aligned}$$

If case 2 occurs, we want to show that $H^j(x_0, D\varphi^j(x_0)) \geq 0$. Fix $b \in B$ and set $b(s) = b$ for all $s \geq 0$. From Lemma 3.5, choose $\bar{t}_0 > 0$ such that for $t \in (0, \bar{t}_0)$

$$V_\gamma^j(x_0) - V_\gamma^j(y_{x_0}(t)) \geq \int_0^t [h(y_{x_0}(s), a^j, b) - \gamma^2|b|^2]\, ds. \tag{3.17}$$

Since x_0 is a local minimum of $V_\gamma^j - \varphi^j$, by (2.3) there exists $\hat{t}_0 > 0$ such that

$$\varphi^j(x_0) - \varphi^j(y_{x_0}(s), a^j, b(s))) \geq V_\gamma^j(x_0) - V_\gamma^j(y_{x_0}(s), a^j, b(s))),\ 0 < s < \hat{t}_0 \tag{3.18}$$

Set $t_0 = \min\{\bar{t}_0,\ \hat{t}_0\}$. As a consequence of (3.17) and (3.18) , we have

$$\varphi^j(x_0) - \varphi^j(y_{x_0}(t)) \geq \int_0^t [h(y_{x_0}(s), a^j, b) - \gamma^2|b|^2] ds,\ 0 < t < t_0. \tag{3.19}$$

Divide both sides by t and let $t \to 0$ to get

$$-D\varphi^j(x_0) \cdot f(x_0, a^j, b) - h(x_0, a^j, b) + \gamma^2|b|^2 \geq 0.$$

Since $b \in B$ is arbitrary, we have $H^j(x_0, D\varphi^j(x_0)) \geq 0$.

We next show that V_γ^j is a viscosity subsolution of the SQVI (3.15). Fix $x_1 \in \mathbb{R}^n$ and $a^j \in A$. Let $\varphi^j \in C^1(\mathbb{R}^n)$ and x_1 is a local maximum of $V_\gamma^j - \varphi^j$. We want to show that

$$\max\{H^j(x_1, D\varphi^j(x_1)),\ V_\gamma^j(x_1) - \min_{i \neq j}\{V_\gamma^i(x_1) + k(a^j, a^i)\}\} \leq 0 \tag{3.20}$$

From Proposition 3.1, $V_\gamma^j(x_1) \leq \min_{i \neq j}\{V_\gamma^i(x_1) + k(a^j, a^i)\}$. Thus we want to show that $H^j(x_1, D\varphi^j(x_1)) \leq 0$.

We first consider the case $V_\gamma^j(x_1) > 0$. Let $t > 0$ and $\epsilon > 0$. From Corollary 3.3, choose $\hat{b} = \hat{b}_{t,\epsilon} \in \mathcal{B}$ and $\hat{T} = \hat{T}_{t,\epsilon} \geq 0$ such that

$$V_\gamma^j(x_1) \leq \int_0^{\hat{T} \wedge t} [h(y_{x_1}(s), a^j, \hat{b}(s)) - \gamma^2|\hat{b}(s)|^2]\, ds + 1_{[0,\hat{T})}(t) V_\gamma^j(y_{x_1}(t, \hat{b})) + \epsilon t \tag{3.21}$$

In particular,

$$V_\gamma^j(x_1) \ \leq \ \int_0^{\hat{T} \wedge t} [h(y_{x_1}(s), a^j, \hat{b}(s)) - \gamma^2|\hat{b}(s)|^2]\, ds + V_\gamma^j(y_{x_1}(\hat{T} \wedge t, \hat{b})) + \epsilon t$$

and hence

$$V_\gamma^j(x_1) - V_\gamma^j(y_{x_1}(\hat{T} \wedge t, \hat{b})) \leq \int_0^{\hat{T} \wedge t} [h(y_{x_1}(s), a^j, \hat{b}(s)) - \gamma^2|\hat{b}(s)|^2]\, ds + \epsilon t \tag{3.22}$$

Since x_1 is a local maximum of $V_\gamma^j - \varphi^j$, by (2.3) we may assume that

$$\varphi^j(x_1) - \varphi^j(y_{x_1}(s), a^j, \hat{b}(s)) \le V_\gamma^j(x_1) - V_\gamma^j(y_{x_1}(s), a^j, \hat{b}(s)),\ 0 < s \le t \tag{3.23}$$

Combine (3.22) and (3.23) to get

$$\varphi^j(x_1) - \varphi^j(y_{x_1}(\hat{T} \wedge t, a^j, \hat{b}(t)) \le \int_0^{\hat{T}\wedge t} [h(y_{x_1}(s), a^j, \hat{b}(s)) - \gamma^2|\hat{b}(s)|^2]\, ds + \epsilon t. \tag{3.24}$$

We next argue that, under the assumptions on f and h, it follows that (3.24) is equivalent to

$$\inf_{b\in B}\{-D\varphi^j(x_1)\cdot f(x_1, a^j, b) - h(x_1, a^j, b) + \gamma^2|b|^2\}\cdot(\hat{T}\wedge t) \le \epsilon\, t + o(\hat{T}\wedge t) \tag{3.25}$$

and that

$$\limsup_{t\to 0} \frac{t}{t \wedge \hat{T}_{t,\epsilon}} = 1 \text{ (for each } \epsilon > 0). \tag{3.26}$$

A similar point arises in the context of the robust stopping-time problem (see the proof of Theorem 3.3 in [1]; for the sake of completeness we include the full argument here.

Observe first that (2.3) and (A3) imply

$$|f(y_{x_1}(s), a^j, \hat{b}(s)) - f(x_1, a^j, \hat{b}(s))| \le \omega_f(M_x s, |x| + M_x t_0), \quad \text{for } 0 < s < t_0 \tag{3.27}$$

and

$$|h(y_x(s), a^j, \hat{b}(s)) - h(x_1, a^j, \hat{b}(s))| \le \omega_h(M_{x_1} s, |x| + M_{x_1} t_0), \quad \text{for } 0 < s < t_0 \tag{3.28}$$

where t_0 does not depend on ϵ, t or $\hat{b}$. By (3.28), the integral on the right-hand side of (3.24) can be written as

$$\int_0^{\hat{T}\wedge t} [h(x_1, a^j, \hat{b}(s)) - \gamma^2|\hat{b}(s)|^2]\, ds + o(\hat{T}\wedge t) \text{ as } \hat{T}\wedge t \to 0.$$

Thus

$$\varphi^j(x_1) - \varphi^j(y_{x_1}(\hat{T} \wedge t, a^j, \hat{b}(t)) \le \int_0^{\hat{T}\wedge t} [h(x_1, a^j, \hat{b}(s)) - \gamma^2|\hat{b}(s)|^2]\, ds + \epsilon t + o(\hat{T}\wedge t). \tag{3.29}$$

Moreover

$$
\begin{aligned}
\varphi^j(x_1) - \varphi^j(y_{x_1}(\hat{T} \wedge t, a^j, \hat{b}) &= -\int_0^{\hat{T}\wedge t} \frac{d}{ds}\varphi^j(y_{x_1}(s, a^j, \hat{b})\, ds \\
&= -\int_0^{\hat{T}\wedge t} D\varphi^j(y_{x_1}(s, a^j, \hat{b})) \cdot f(y_{x_1}(s), a^j, \hat{b}(s))\, ds \\
&= -\int_0^{\hat{T}\wedge t} D\varphi^j \cdot f(x_1, a^j, \hat{b}(s))\, ds + o(\hat{T} \wedge t) \qquad (3.30)
\end{aligned}
$$

where we used (2.3), (3.27) and $\varphi^j \in C^1$ in the last equality to estimate the difference between $D\varphi^j \cdot f$ computed at $y_{x_1}(s)$ and at x_1, respectively. Plugging (3.30) into (3.29) gives

$$
\int_0^{\hat{T}\wedge t} -D\varphi^j(x_1) \cdot f(x_1, a^j, \hat{b}(s))\, ds \leq \int_0^{\hat{T}\wedge t} [h(x_1, a^j, \hat{b}(s)) - \gamma^2|\hat{b}|^2]\, ds + \epsilon t + o(\hat{T} \wedge t).
$$

Thus

$$
\int_0^{\hat{T}\wedge t} [-D\varphi^j(x_1) \cdot f(x_1, a^j, \hat{b}(s)) - h(x_1, a^j, \hat{b}(s)) + \gamma^2|\hat{b}(s)|^2]\, ds \leq \epsilon t + o(\hat{T} \wedge t). \tag{3.31}
$$

We estimate the left-hand side of this inequality from below to get next

$$
\inf_{b\in B}\{-D\varphi^j(x_1) \cdot f(x_1, a^j, b) - h(x_1, a^j, b) + |\gamma|^2|b|^2\} \cdot (\hat{T} \wedge t) \leq \epsilon t + o(\hat{T} \wedge t). \tag{3.32}
$$

and (3.25) follows.

We now write $\hat{T}_{t,\epsilon}$ in place of $\hat{T}$ to emphasize the dependence of $\hat{T}$ on t and ϵ. Note that $\frac{t}{\hat{T}_{t,\epsilon}\wedge t} \geq 1$ for all $t > 0$ and hence $\limsup_{t\to 0} \frac{t}{\hat{T}_{t,\epsilon}\wedge t} \geq 1$. We claim that in fact (3.26) holds. Indeed, if not, then, for each fixed $\epsilon > 0$, there would be a sequence of positive numbers $\{t_n\}$ tending to 0 such that $\hat{T}_{t_n,\epsilon} < t_n$ and $\lim_{n\to\infty} \hat{T}_{t_n,\epsilon}/t_n = \rho_\epsilon < 1$. In this case, the inequality (3.21) becomes

$$
V_\gamma^j(x_1) \leq \int_0^{\hat{T}_{t_n,\epsilon}} [h(y_{x_1}(s), a^j\hat{b}(s)) - \gamma^2|\hat{b}(s)|^2]\, ds + \epsilon t_n
$$

for all n, from which we get

$$
\frac{V_\gamma^j(x_1)}{t_n} \leq \frac{1}{t_n}\int_0^{\hat{T}_{t_n,\epsilon}} [h(y_{x_1}(s), a^j, \hat{b}(s)) - \gamma^2|\hat{b}(s)|^2]\, ds + \epsilon \tag{3.33}
$$

for all n. From (2.3) and (A2) we have an estimate of the form $h(y_x(s), a^j, \hat{b}(s)) \leq K_x$ for all s in a sufficiently small interval $[0, \delta)$ (independent of t and ϵ), and hence, for n sufficiently large we have

$$\int_0^{\hat{T}_{t_n,\epsilon}} [h(y_{x_1}(s), a^j, \hat{b}(s)) - \gamma^2 |\hat{b}(s)|^2] \, ds \leq K_{x_1} T_{t_n,\epsilon}.$$

Plugging this into (3.33) gives

$$\frac{V_\gamma^j(x_1)}{t_n} \leq K_{x_1} \frac{\hat{T}_{t_n,\epsilon}}{t_n} + \epsilon.$$

Letting n tend to infinity and using the assumption that $V_\gamma^j(x_1) > 0$ leads to the contradiction $\infty \leq K_x \rho_\epsilon + \epsilon \leq K_x + \epsilon < \infty$. Hence $\limsup_{t \to 0} \frac{t}{t \wedge \hat{T}_{t,\epsilon}} = 1$ for each fixed $\epsilon > 0$ and (3.26) follows.

We now can divide (3.32) by $\hat{T} \wedge t > 0$ and pass to the limit to get

$$\inf_{b \in B} \{-D\varphi^j(x) \cdot f(x, a^j, b) - h(x, a^j, b) + \gamma^2 |b|^2\} \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we conclude that $H^j(x, D\varphi^j(x)) \leq 0$.

It remains to handle the case $V_\gamma^j(x_1) = 0$. In this case we take $\hat{b} \equiv 0$ and use (A6) and $V_\gamma^j \geq 0$ to see that

$$\begin{aligned} V_\gamma^j(x_1) = 0 \quad &\leq \quad \int_0^t h(y_{x_1}(s), \hat{b}) ds + V_\gamma^j(y_{x_1}(t)) \\ &= \quad \int_0^t [h(y_{x_1}(s), \hat{b}(s)) - \gamma^2 |\hat{b}(s)|^2] ds + V_\gamma^j(y_{x_1}(t)), \end{aligned}$$

for all $t \geq 0$. Then it is straightforward to follow the procedure in the first part of the proof to arrive at the desired inequality $H^j(x_1, D\varphi^j(x_1)) \leq 0$. $\diamondsuit$

We next give a connection of a switching storage (vector) function with the SQVI (3.15).

Theorem 3.7 *Assume (A1)-(A5) and assume that $S = (S^1, \ldots, S^r)$ is a continuous switching storage function for the closed loop system formed by the nonanticipating strategy $\alpha \in \Gamma$. Then S is a viscosity supersolution of SQVI (3.15).*

Proof. Fix $x \in \mathbb{R}^n$ and $j \in \{1, \ldots, r\}$. Let $\varphi^j \in C^1(\mathbb{R}^n)$ be such that x is a local minimum of $S^j - \varphi^j$. Let $b \in B$. Set $b(s) = b$ for $s \geq 0$. Choose $t_1 > 0$ so that

$$S^j(x) - \varphi^j(x) \leq S^j(y_x(s, \alpha_x^j[b], b)) - \varphi^j(y_x(s, \alpha_x^j[b], b)), \quad \text{for all } 0 \leq s \leq t_1. \quad (3.34)$$

We have two cases to consider:

case 1 : $S^j(x) \geq \min_{i \neq j}\{S^i(x) + k(a^j, a^i)\}$,

case 2 $S^j(x) < \min_{i \neq j}\{S^i(x) + k(a^j, a^i)\}$.

If case 1 occurs, then

$$\begin{aligned}&\max\{H^j(x, D\varphi^j(x)), S^j(x) - \min_{i \neq j}\{S^i(x) + k(a^j, a^i)\}\}\\ &\geq S^j(x) - \min_{i \neq j}\{S^i(x) + k(a^j, a^i)\}\\ &\geq 0.\end{aligned}$$

If case 2 occurs, we claim that for each $\widehat{b} \in B$ there exists a $t_2 = t_2(\widehat{b}) > 0$ such that

$$\alpha_x^j[\widehat{b}](s) = a^j \text{ for } 0 \leq s \leq t_2.$$

Indeed, if not, then, for each $t > 0$ there exists a $\bar{b}_t \in \mathcal{B}$ such that

$$\alpha_x^j[\bar{b}_t](\tau_t) = a^{j(t)} \neq a^j \text{ for some } \tau,\ 0 \leq \tau_t < t. \tag{3.35}$$

Since S is a switching storage function, we have

$$\begin{aligned}&S^j(x) - S^{j(t)}(y_x(t, \alpha_x^j[\bar{b}_t], \bar{b}_t))\\ &\qquad \geq \int_0^t [h(y_x(s), \alpha_x^j[\bar{b}_t](s), \bar{b}_t(s)) - \gamma^2|\bar{b}_t(s)|^2\ ds + \sum_{\tau < t} k(a^{j(\tau^-)}, a^{j(\tau)}).\end{aligned}$$

By letting t tend to 0 along some subsequence if necessary, we get an index $j(0^+)$ so that

$$S^j(x) - S^{j(0^+)}(x) \geq k(a^j, a^{j(0^+)}).$$

From (3.35) we see that this implies that $j(0^+) \neq j$. Thus

$$S^j(x) \geq \min_{i \neq j}\{S^i(x) + k(a^j, a^i)\}$$

which gives a contradiction. Thus the claim is proved.

Since S is a switching storage function, we have

$$\begin{aligned}&S^j(x) - S^{j(t)}(y_x(t, \alpha_x^j[b], b)\\ &\qquad \geq \int_0^t [h(y_x(s), \alpha_x^j[b](s), b(s)) - \gamma^2|b(s)|^2]\ ds \text{ for all } 0 < t \leq t_2.\end{aligned} \tag{3.36}$$

Set $t_3 = \min\{t_1, t_2\}$. Then (3.34) and (3.36) imply that

$$\begin{aligned}&\varphi^j(x) - \varphi^j(y_x(t, \alpha_x^j[b], b))\\ &\qquad \geq \int_0^t [h(y_x(s), \alpha_x^j[b](s), b) - \gamma^2|b|^2]\ ds \text{ for } 0 < t < t_3.\end{aligned} \tag{3.37}$$

Divide (3.37) by t and let t tend to 0 to get

$$-D\varphi^j(x) \cdot f(x, a^j, b) - h(x, a^j, b) + \gamma^2 |b|^2 \geq 0.$$

Since this inequality holds for an arbitrary $b \in B$, we have $H^j(x, D\varphi^j(x)) \geq 0$ as required. $\diamondsuit$

We now proceed to the synthesis of a switching-control strategy achieving the dissipation inequality for a given viscosity supersolution $U = (U^1, \dots, U^r)$ of SQVI (3.15). Given a continuous nonnegative vector function $U = (U^1, \dots, U^r)$ on $\mathbb{R}^n$ satisfying the condition

$$U^j(x) \leq \min_{i \neq j} \{U^i(x) + k(a^j, a^i)\} \text{ for all } x \in \mathbb{R}^n,\ j = 1, \dots, r,$$

we associate a state-feedback switching strategy $\alpha_U : (y(t), a^j) \to \alpha^j(y(t))$ by the rule

$$\alpha^j(y(t)) = \begin{cases} a^j \text{ if } U^j(y(t)) < \min_{i \neq j} \{U^i(y(t)) + k(a^j, a^i)\}; \\ \text{any } a^\ell \neq a^j \text{ such that } U^\ell(y(t)) + k(a^j, a^\ell) = \min_{i \neq j} \{U^i(y(t)) + k(a^j, a^i)\}, \\ \qquad \text{otherwise.} \end{cases} \tag{3.38}$$

In other words, the associated feedback switching strategy is: *if the current state is $y(t)$ and the current old control is $a(t^-) = a^j$, then set $a(t) = \alpha^j(y(t))$.* Such a strategy can also be expressed as a nonanticipating strategy $\alpha_U : (x, a^j, b) \to \alpha^j_{U,x}[b]$; explicitly for this particular case α_U, we have $\alpha^j_{U,x}[b]$ is given by

$$\alpha^j_{U,x}[b](t) = \sum_{n \geq 1} a_{n-1} 1_{[\tau_{n-1}, \tau_n)}(t) \text{ for } t \geq 0 \tag{3.39}$$

and $\alpha^j_{U,x}[b](0^-) = a_0$ where

$$\tau_0 = 0, \quad a_0 = a^{j_0} = a^j$$

and for $n = 1, 2, 3, \dots$

$$\tau_n[b] = \begin{cases} \inf\{t > \tau_{n-1} :\ U^{j_{n-1}}(y_{y(\tau_{n-1})}(t - \tau_{n-1}, a^{j_{n-1}}, b(\cdot - \tau_{n-1})) \\ \quad = \min_{i \neq j_{n-1}} \{U^i(y_{y(\tau_{n-1})}(t - \tau_{n-1}, a^{j_{n-1}}, b(\cdot - \tau_{n-1})) + k(a^{j_{n-1}}, a^i)\}\}, \\ +\infty \text{ if the preceding set is empty,} \end{cases}$$

$$a_n = a^{j_n} = \begin{cases} \text{any } a^l \neq a^{j_{n-1}} \text{ such that} \\ \quad \min_{i \neq j_{n-1}} \{U^i(y_{y(\tau_{n-1})}(\tau_n - \tau_{n-1}), a^{j_{n-1}}, b(\cdot - \tau_{n-1}))) + k(a^{j_{n-1}}, a^i)\} \\ \quad = U^l(y_{y(\tau_{n-1})}(\tau_n - \tau_{n-1}, a^{j_{n-1}}, b(\cdot - \tau_{n-1}))) + k(a^{j_{n-1}}, a^l), \text{ if } \tau_n < \infty; \\ \text{undefined, if } \tau_n = \infty. \end{cases} \tag{3.40}$$

Note that if $\tau_1 = \tau_0 = 0$, there is an immediate switch from a_0 to a_1 at time 0 and the $n = 1$ term in (3.39) is vacuous. Moreover by (A5), $\tau_n > \tau_{n-1}$ for $\tau_{n-1} < \infty$ and $n > 1$. To see this, we assume that $\tau_n = \tau_{n-1} < \infty$ for some $n > 1$. From the definition of τ_{n-1} and τ_n, we would have

$$\begin{aligned} U^{j_{n-2}}(y(\tau_{n-1})) &= U^{j_{n-1}}(y(\tau_{n-1})) + k(a^{j_{n-2}}, a^{j_{n-1}}) \\ &= U^{j_n}(y(\tau_{n-1})) + k(a^{j_{n-1}}, a^{j_n}) + k(a^{j_{n-2}}, a^{j_{n-1}}) \text{ (and hence } j_n \neq j_{n-2}) \\ &> U^{j_n}(y(\tau_{n-1})) + k(a^{j_{n-2}}, a^{j_n}) \\ &\geq \min_{i \neq j_{n-2}} \{U^i(y(\tau_{n-1})) + k(a^{j_{n-2}}, a^i)\}, \end{aligned}$$

which gives a contradiction.

Theorem 3.8 *Assume*
(i) *(A1)-(A5) hold.*
(ii) $U = (U^1, \ldots, U^r)$ *is a nonnegative continuous viscosity supersolution in* $\mathbb{R}^n$ *of the SQVI (3.15)*

$$\max\{H^j(x, DU^j(x)),\ U^j(x) - \min_{i \neq j}\{U^i(x) + k(a^j, a^i)\}\} = 0,\ x \in \mathbb{R}^n,\ j = 1, \ldots, r,$$

(iii) $U^j(x) \leq \min_{i \neq j}\{U^i(x) + k(a^j, a^i)\},\ x \in \mathbb{R}^n,\ j \in \{1, \ldots, r\}$.
Let α_U *be the state-feedback strategy defined by (3.38), or equivalently, the nonanticipating disturbance-feedback strategy* α_U *defined by (3.40). Then* $U = (U^1, \ldots, U^r)$ *is a storage function for the closed-loop system formed by the strategy* α_U*. In particular, we have*

$$U^j(x) \geq \sup_{b \in \mathcal{B},\, T \geq 0} \{\int_{[0,T)} l(y_x(s), a^j, \alpha^j_{U,x}[b](s), b(s))\} \geq V^j_\gamma(x),$$

for each $x \in \mathbb{R}^n$ *and* $a^j \in A$*. Thus* V_γ*, if continuous, is characterized as the minimal, nonnegative, continuous, viscosity supersolution of the SQVI (3.15) satisfying condition (iii), as well as the minimal continuous switching storage function satisfying condition (iii) for the closed-loop system associated with some nonanticipating strategy* α_{V_γ}*.*

Proof Let $\alpha^j_{U,x}[b](t)$ be the switching strategy defined as in (3.40). We claim that

$$\tau_n \to \infty \quad \text{as} \quad n \to \infty.$$

If $\tau_n = \infty$ for some n, then it is trivially true. Otherwise, since we observed just before the statement of Theorem 3.8 that $\{\tau_n\}$ is a nondecreasing sequence, it would follow that

$$\lim_{n \to \infty} \tau_n = T < \infty. \tag{3.41}$$

with $0 \le \tau_n < T$ for all n. From (3.41), we have that $\{\tau_n\}$ is a Cauchy sequence, and hence for all $\nu > 0$ there is some n such that $\tau_n < \tau_{n-1} + \nu$. By the definition of τ_n,

$$U^{j_{n-1}}(y_x(\tau_n)) = U^l(y_x(\tau_n)) + k(a^{j_{n-1}}, a^l) \text{ for some } a^l \neq a^{j_{n-1}} \tag{3.42}$$

(We have written $y_x(t)$ for $y_x(t, \alpha_x^j[b], b)$.) By definition of τ_{n-1}, we have

$$U^{j_{n-2}}(y_x(\tau_{n-1})) = U^{j_{n-1}}(y_x(\tau_{n-1})) + k(a^{j_{n-2}}, a^{j_{n-1}}). \tag{3.43}$$

By (iii), we have

$$\begin{aligned} U^{j_{n-2}}(y_x(\tau_{n-1})) &\le \min_{i \neq j_{n-2}} \{U^i(y_x(\tau_{n-1})) + k(a^{j_{n-2}}, a^i)\} \\ &\le U^l(y_x(\tau_{n-1})) + k(a^{j_{n-2}}, a^l) \text{ if } l \neq j_{n-2} \end{aligned}$$

and hence

$$U^{j_{n-2}}(y_x(\tau_{n-1})) \le U^l(y_x(\tau_{n-1})) + k(a^{j_{n-2}}, a^l) \tag{3.44}$$

if $l \neq j_{n-2}$. If $l = j_{n-2}$, (3.44) holds with equality (by (A5)), and hence (3.44) in fact holds without restriction. From (3.43) and (3.44), we have

$$k(a^{j_{n-2}}, a^{j_{n-1}}) - k(a^{j_{n-2}}, a^l) \le U^l(y_x(\tau_{n-1})) - U^{j_{n-1}}(y_x(\tau_{n-1})) \tag{3.45}$$

As a consequence of (3.42) and (3.45), we have

$$\begin{aligned} 0 &< k(a^{j_{n-2}}, a^{j_{n-1}}) + k(a^{j_{n-1}}, a^l) - k(a^{j_{n-2}}, a^l) \\ &\le U^l(y_x(\tau_{n-1})) - U^l(y_x(\tau_n)) + U^{j_{n-1}}(y_x(\tau_n)) - U^{j_{n-1}}(y_x(\tau_{n-1})) \\ &\le \omega_l(\nu) + \omega_{j_{n-1}}(\nu) \end{aligned}$$

and hence (by the strict triangle inequality in (A5))

$$0 < \min_{i,j,l\colon i \neq j \neq l} \left\{k(a^i, a^j) + k(a^j, a^l) - k(a^i, a^l)\right\} \le \omega_\ell(\nu) + \omega_j(\nu)$$

where in general ω_j is a modulus of continuity for $U^j(y_x(\cdot))$ on the interval $[0, T]$. Letting ν tend to zero now leads to a contradiction, and the claim follows.

Hence $\alpha_x^j[b](t) = \sum a_{n-1} 1_{[\tau_{n-1}, \tau_n)}(t) \in \Gamma$. Since U is a viscosity supersolution of the SQVI (3.15), we have $H^{j_n}(y_x(s), DU^{j_n}(y_x(s))) \ge 0$, in the viscosity-solution sense for $\tau_n < s < \tau_{n+1}$. Thus (see [6, Section II.5.5]

$$U^{j_n}(y_x(s)) - U^{j_n}(y_x(t)) \ge \int_s^t [h(y_x(s), a^{j_n}, b(s)) - \gamma^2 |b(s)|^2] ds,$$

for all $b \in \mathcal{B},\ \tau_n < s \leq t < \tau_{n+1}$. Letting $s \to \tau_n^+$ and $t \to \tau_{n+1}^-$, we get

$$U^{j_n}(y_x(\tau_n)) - U^{j_n}(y_x(\tau_{n+1}) \geq \int_{\tau_n}^{\tau_{n+1}} [h(y_x(s), a^{j_n}, b(s)) - \gamma^2 |b(s)|^2] ds,\ \forall b \in \mathcal{B}. \tag{3.46}$$

We also have

$$U^{j_n}(y_x(\tau_{n+1})) = U^{j_{n+1}}(y_x(\tau_{n+1})) + k(a^{j_n}, a^{j_{n+1}}),\ \text{ for } \tau_{n+1} < \infty. \tag{3.47}$$

Adding (3.46) over $\tau_n \leq T$ and using (3.47), we have

$$\begin{aligned} U^{j_0}(x) &\geq \int_0^T [h(y_x(s), \alpha_x^j[b](s), b(s)) - \gamma^2 |b(s)|^2] ds + \sum_{\tau_n \leq T} k(a_{n-1}, a_n) + U^{j_n}(y_x(T)) \\ &\geq \int_0^T [h(y_x(s), \alpha_x^j[b](s), b(s)) - \gamma^2 |b(s)|^2] ds + \sum_{\tau_n \leq T} k(a_{n-1}, a_n). \end{aligned}$$

Since this inequality holds for arbitrary $b \in \mathcal{B}$ and $T \geq 0$, we have

$$U^j(x) \geq \sup_{b \in \mathcal{B},\, T \geq 0} \left\{ \int_{[0,T]} l(y_x(s), a^j, \alpha_x^j[b](s), b(s)) \right\}.$$

Thus $U^j(x) \geq V_\gamma^j(x)$. By Theorem 3.6, we know that V_γ is a viscosity supersolution of the SQVI (3.15) if it is continuous. (Note that the proof of the viscosity-supersolution property of V_γ in Theorem 3.6 does not use the assumption (A6).) Also V_γ has the property (iii) by Proposition 3.1. Thus we conclude that, if continuous, V_γ is the minimal, nonnegative, continuous, viscosity supersolution of SQVI (3.15) which satisfies condition (iii)

The first part of this Theorem (Theorem 3.8) already proved then implies that V_γ is a switching storage function. Moreover if S is any continuous, switching storage function for some nonanticipating strategy α_{V_γ}, from Theorem 3.7 we see that S is a viscosity supersolution of the SQVI (3.15). Again from the first part of this Theorem already proved, we then see that $S \geq V_\gamma$ if S has the property (iii), and hence V_γ is also the minimal, continuous switching storage function satisfying the condition (iii), as asserted. $\diamondsuit$

4 Stability for switching-control problems

In this section we show how the solution of the SQVI (3.15) can be used for stability analysis.

We consider the system (1.1) – (1.2) with some control strategy α plugged in to get a closed-loop system with the disturbance signal as the only input

$$\Sigma_{sw} \left\{ \begin{array}{lcl} \dot{y} & = & f(y, \alpha_x^j[b], b), \ y(0) = x, \ a(0^-) = a^j \\ z & = & h(y, \alpha_x^j[b], b). \end{array} \right.$$

An example of such a strategy α is the canonical strategy α_U (see (3.38) or (3.40)) determined by a continuous supersolution of the SQVI (3.15). Moreover, if $V_\gamma = (V_\gamma^1, \dots, V_\gamma^r)$ is the vector lower-value function for the associated game as in (1.4) and we assume that 0 is an equilibrium point for the autonomous system formed from (1.1)–(1.2) by taking $a(s) = a^{i_0}$ and $b(s) = 0$ (so $f(0, a^{i_0}, 0) = 0$ and $h(0, a^{i_0}, 0) = 0$), then it is easy to check that $V_\gamma^{i_0}(0) = 0$. Furthermore, the associated strategy $\alpha = \alpha_{V_\gamma}$ has the property that

$$\alpha_0^{i_0}[0] = a^{i_0}, \tag{4.1}$$

so 0 is an equilibrium point of the closed-loop system Σ_{sw} with $\alpha = \alpha_{V_\gamma}$ and $a(0^-) = a^{i_0}$ as well. Our goal is to give conditions which guarantee a sort of converse, starting with any continuous supersolution U of the SQVI (3.15).

We first need a few preliminaries. The following elementary result can be found e.g. in [16].

Lemma 4.1 *If $\phi(\cdot) : \mathbb{R} \to \mathbb{R}$ is a nonnegative, uniformly continuous function such that $\int_0^\infty \phi(s)\, ds < \infty$, then $\lim_{t\to\infty} \phi(t) = 0$.*

We say that the closed-loop switching system Σ_{sw} is *zero-state observable* for initial control setting a^j if, whenever $h(y_x(t), \alpha_x^j[0](t), 0) = 0$ for all $t \geq 0$, then $y_x(t) = y_x(t, \alpha_x^j[0], 0) = 0$ for all $t \geq 0$. We say that the closed-loop system Σ_{sw} is *zero-state detectable* for initial control setting a^j if

$$\lim_{t\to\infty} h(y_x(t), \alpha_x^j[0](t), 0) = 0, \text{ implies that } \lim_{t\to\infty} y_x(t, \alpha_x^j[0], 0) = 0.$$

The following proposition gives conditions which guarantee that a particular component U^j of a viscosity supersolution $U = (U^1, \dots, U^r)$ be positive-definite, a conclusion which will be needed as a hypothesis in the stability theorem to follow.

Proposition 4.2 *Assume*
(i) *(A1)-(A6) hold;*
(ii) *Σ_{sw} is zero-state observable for some initial control setting a^j;*
(iii) *$U = (U^1, \dots, U^r)$ is a nonnegative continuous viscosity supersolution of the SQVI (3.15)*

$$\max\{H^j(x, DU^j(x)), U^j(x) - \min_{i\neq j}\{U^i(x) + k(a^j, a^i)\}\} = 0, \ x \in \mathbb{R}^n, \ j = 1, \dots, r;$$

(iv) *$U^j(x) \le \min_{i\ne j}\{U^i(x)+k(a^j,a^i)\},\ x\in \mathbb{R}^n,\ j=1,\dots,r.$*
Then $U^j(x)>0$ for $x\ne 0$.

Proof Let $x\in\mathbb{R}^n$. By Theorem 3.8, U is a storage function for Σ_{sw} if we use $\alpha=\alpha_U$ given by (3.38) or equivalently, (3.40). Thus

$$\begin{aligned} U^j(x) &\ge \int_{[0,T)} l(y_x(s),a^j,\alpha^j_{U,x}[0](s),0)\ ds + U^{j(T)}(y_x(T,\alpha^j_{U,x}[0],0)) \\ &\ge \int_{[0,T)} l(y_x(s),a^j,\alpha^j_{U,x}[0](s),0)\ ds \ \text{ for all } T>0. \end{aligned}$$

Since k is nonnegative, we have

$$U^j(x) \ge \int_0^T h(y_x(s),\alpha^j_x[0](s),0)\ ds, \text{ for all } T\ge 0.$$

Thus if $U^j(x)=0$, then $h(y_x(s,\alpha^j_x[0],0),\alpha^j_x[0](s),0)=0$ for all $s\ge 0$ because h is nonnegative by assumption (A6). Since Σ_{sw} is zero-state observable for initial control setting a^j, it follows that $y_x(s,\alpha^j_x[0],0)=0$ for all $s\ge 0$. Thus $x=y_x(0,\alpha_x[0],0)=0$. Since U^j is nonnegative, we conclude that if $x\ne 0$ then $U^j(x)>0$. $\diamondsuit$

Proposition 4.3 *Assume*
(i) *(A1)-(A6) hold;*
(ii) *$U=(U^1,\dots,U^r)$ is a nonnegative continuous viscosity supersolution of the SQVI (3.15)*

$$\max\{H^j(x,DU^j(x)),U^j(x)-\min_{i\ne j}\{U^i(x)+k(a^j,a^i)\}\}=0,\ x\in\mathbb{R}^n,\ j=1,\dots,r;$$

(iii) *$U^j(x)\le\min_{i\ne j}\{U^i(x)+k(a^j,a^i)\},\ x\in\mathbb{R}^n,\ j=1,\dots,r;$*
(iv) *there is an $i_0\in\{1,\dots,r\}$ such that $U^{i_0}(0)=0$ and $U^{i_0}(x)>0$ for $x\ne 0$.*
(v) *Σ_{sw} is zero-state detectable for all initial control settings $a^j\in A$.*
Then the strategy α_U associated with U as in (3.38) or (3.40) is such that $\alpha^{i_0}_U[0](s)=a^{i_0}$ for all s and 0 is an equilibrium point for the system $\dot y=f(y,a^i_0,0)$. Moreover, 0 is a globally asymptotically stable equilibrium point for the system Σ_{sw}, in the sense that the solution $y(t)=y^j_x(t,\alpha^j_{U,x}[0],0)$ of

$$\dot y = f(y,\alpha^j_{U,x}[0],0), \quad y(0)=x$$

has the property that

$$\lim_{t\to\infty} y^j_x(t,\alpha^j_{U,x}[0],0)=0$$

for all $x\in\mathbb{R}^n$ and all $a^j\in A$.

Proof Suppose that $U^{i_0}(0) = 0$ and $U^{i_0}(x) > 0$ for $x \neq 0$. Let $T \geq 0$ and $x \in \mathbb{R}^n$. Since U is a storage function for the closed-loop system formed from (1.1)–(1.2) with $\alpha = \alpha_U$, we have

$$U^{i_0}(x) \geq \int_0^T h(y_x(s), \alpha_x^{i_0}[0](s), 0)\ ds + \sum_{\tau < T} k(\alpha_{U,x}^{i_0}(\tau^-), \alpha_{U,x}^{i_0}(\tau)) + U^{j(T)}(y_x(T, \alpha_{U,x}^{i_0}[0], 0)). \tag{4.2}$$

Since $h,\ k,\ U$ are nonnegative and $U^{i_0}(0) = 0$ by our assumptions, substitution of $x = 0$ in (4.2) forces

$$\sum_{\tau < T} k\left(\alpha_{U,0}^{i_0}[0](\tau^-), \alpha_{U,0}^{i_0}[0](\tau)\right) = 0.$$

This implies that $\alpha_{U,0}^{i_0}[0](t) = a^{i_0}$ for all $0 \leq t \leq T$. Thus

$$0 \leq U^{j(T)}(y_0(T, \alpha_{U,0}^{i_0}[0], 0)) = U^{i_0}(y_0(T, \alpha_{U,0}^{i_0}[0], 0)) \leq U^{i_0}(0) = 0.$$

By the positive definite property of U^{i_0}, we have $y_0(T, \alpha_{U,0}^{i_0}[0], 0) = 0$. Since $T \geq 0$ is arbitrary, we conclude that 0 is a equilibrium point of the system $\dot{y} = f(y, a^{i_0}, 0)$.

Next we want to show that 0 is a globally asymptotically stable equilibrium point for the closed-loop switching system Σ_{sw} with $\alpha = \alpha_U$. Again, from the storage-function property of $U = (U^1, \ldots, U^r)$ for the system Σ_{sw} with $\alpha = \alpha_U$, we have

$$\int_0^T h(y_x(s), \alpha_{U,x}^j[0](s), 0)\ ds \leq U^j(x) < \infty \text{ for all } T > 0.$$

Thus $\lim_{t \to \infty} h(y_x(t, \alpha_{U,x}^j[0], 0) = 0$ by Lemma 4.1. By the detectability assumption (v), we have $\lim_{t \to \infty} y_x(t, \alpha_{U,x}^j[0], 0) = 0$ as required. $\diamondsuit$

5 Computational issues

The results of Sections 3 reduce the solution of the robust optimal switching-control problem to the solution of a SQVI (3.15). For these results to be useful, of course, one must be able to compute solutions of such an equation, or more precisely for our situation, the minimal viscosity supersolution of such a system of equations. In this section we make a few general observations concerning these issues and give an explicit, direct solution for a simple example with one-dimensional state space. For examples with higher dimensional state space, more sophisticated numerical methods are needed; this is an ongoing topic for future research.

5.1 A connection between solutions of SQVIs and VIs

Suppose that $U = (U^1, \ldots, U^r)$, where $U^j \in C(\mathbb{R}^n)$ for $j = 1, \ldots, r$, is the minimal viscosity supersolution of SQVI

$$\max\{H^j(x, DU^j(x)),\, U^j(x) - \min_{i \neq j}\{U^i(x) + k(a^j, a^i)\}\} = 0,\ j = 1, \ldots, r. \tag{5.1}$$

Then each U^j can be interpreted as the minimal viscosity supersolution of the variational inequality (VI)

$$\max\{H(x, DU(x)), U(x) - \Phi(x)\} = 0$$

with Hamiltonian H equal to H^j and with stopping cost Φ equal to $\Phi^j = \min_{i \neq j}\{U^i + k(a^j, a^i)\}$. This suggests defining an iteration map F as follows. Given an r-tuple $U = (U^1, \ldots, U^r)$ of nonnegative real-valued functions, define a new r-tuple $F(U) = (F(U)^1, \ldots, F(U)^r)$ of nonnegative real-valued functions by

$$F(U)^j = \text{ the minimal viscosity supersolution of VI with } H = H^j \text{ and } \Phi = \Phi^j.$$

Note that U is the minimal viscosity supersolution of SQVI (5.1) if and only if $F(U) = U$, i.e., if and only if U is a fixed point of F. Formally, one can solve the fixed point problem by guessing a starting point $U_0 = (U_0^1, \ldots, U_0^r)$ and then iterating

$$U_{n+1} = F(U_n),\ n = 0, 1, 2, \ldots .$$

If $U_n \to U_\infty$ and F is continuous, then from $U_{n+1} = F(U_n)$ one can take the limit to get $U_\infty = F(U_\infty)$ from which we see that U_∞ is a fixed point for F. For finite horizon problems, or problems with a positive discount factor in the running cost, the connection is a little cleaner, as in this situation one has a uniqueness theorem for solutions of the relevant SQVI (5.1).

A similar remark giving a connection between the impulsive control problem and the stopping time problem is given in [6, Chapter III Section 4.3], where some convergence results are also given. It would be of interest to develop similar convergence results for the SQVI (5.1) associated with an optimal switching-control problem.

5.2 Optimal switching-control problem with one-dimensional state space

In this subsection we consider an optimal switching cost problem with one-dimensional state space. While in principle it should be possible to solve the problem by using the construction in [1] to perform each iterative step in the procedure outlined in Section

5.1, it turns out that, for the example which we discuss here, one can solve explicitly by a direct, geometric, noniterative procedure which we now describe.

We consider the special case of the general problem where there are only two controls $A = \{a^1, a^2\}$, with respective system dynamics given by

$$f(y, a^1, b) = -y + b; \quad f(y, a^2, b) = -\mu(y - 1) + b.$$

(A value for the parameter $\mu > 1$ will be specified below.) We take the output to be simply the squared state

$$h(y, a, b) = y^2$$

and the switching cost to be given by a parameter $\beta > 0$:

$$k(a^1, a^2) = k(a^2, a^1) = \beta; \quad k(a^1, a^1) = k(a^2, a^2) = 0.$$

All the hypotheses (A1)-(A6) are satisfied. All other assumptions are satisfied with the exception that $B = \mathbb{R}$ is not compact; to alleviate this difficulty, one can restrict B to a large finite interval $[-M, M]$; to live with this restriction, one must adjust the definition of the hamiltonian functions $H^1(x, p)$ and $H^2(x, p)$ in the discussion to follow. We will construct a solution to the SQVI (5.1) for this example via a variation of the algorithm presented in [1]; rather than proving that the solution so constructed is the minimal nonnegative supersolution of SQVI (5.1), we verify directly that it is the lower value function $V_\gamma = (V_\gamma^1, V_\gamma^2)$ of the switching-control differential game (2.1).

Because we will take $\mu > 1$, for large $|y|$ the control a^2 will drive the state toward 0 more strongly than a^1. However the origin is stable only if a^1 used when $|y|$ is small. Thus we would expect an optimal strategy to switch to a^2 for y away from the origin, but then back again to $a = 1$ near the origin. The details of this will be determined by our solution (V^1, V^2) to SQVI (5.1) .

The two Hamiltonian functions work out to be

$$\begin{aligned} H^1(x, p) &= px - x^2 - \frac{1}{4\gamma^2}p^2 \\ H^2(x, p) &= \mu p(1 - x) - x^2 - \frac{1}{4\gamma^2}p^2. \end{aligned}$$

These are both instances of the general formula

$$\begin{aligned} H(x, p) &= \inf_b \{-(g(x) + b) \cdot p - x^2 + \gamma^2 b^2\} \\ &= -pg(x) - x^2 - \frac{1}{4\gamma^2}p^2 \\ &= (\gamma g(x))^2 - x^2 - \left(\frac{1}{2\gamma}p + \gamma g(x)\right)^2 \end{aligned} \tag{5.2}$$

where $g(x) = -x$ for a^1 and $g(x) = -\mu(x-1)$ for a^2. Provided $|x| < \gamma|g(x)|$ the equation $H(x,p) = 0$ has two distinct real solutions:

$$p_\pm(x) = -2\gamma^2 g(x) \pm 2\gamma\sqrt{\gamma^2 g(x)^2 - x^2}.$$

We will use $p^a_\pm(x)$ $(a = 1,2)$ to refer to these specifically for our two choices of $g(x)$. Observe that $H(x,p) \le 0$ if and only if $p \le p_-(x)$, $p \ge p_+(x)$, or $|x| > \gamma|g(x)|$. This will be important for working with (5.8) below. Note also that the infimum in (5.2) is achieved for $b^* = \frac{1}{2\gamma^2}p$. When $p = p_\pm(x)$ in particular we have

$$\begin{aligned} f(x,a,b^*) &= g(x) + \frac{1}{2\gamma^2}p_\pm(x) \\ &= \pm\frac{1}{\gamma}\sqrt{\gamma^2 g(x)^2 - x^2}, \end{aligned}$$

which will be positive (negative) in the case of p_+ (p_-, respectively). Moreover, since $H(x, p_\pm(x)) = 0$, we will have

$$(g(x) + b^*) \cdot p_\pm(x) + x^2 = \gamma^2(b^*)^2.$$

These observations will be important in confirming the optimality of our switching policy below. The expressions for $p^1_\pm(x)$ have a simple composite expression: with

$$\rho = \gamma^2 - \gamma\sqrt{\gamma^2 - 1}$$

we have

$$2\rho x = \begin{cases} p^1_-(x) & \text{if } x \ge 0 \\ p^1_+(x) & \text{if } x \le 0. \end{cases} \tag{5.3}$$

We now exhibit the desired solution of the SQVI (5.1) for the following specific parameter values:

$$\mu = 3, \quad \beta = .4, \quad \gamma = 2. \tag{5.4}$$

Let

$$\begin{aligned} W^1(x) &= \rho x^2 \\ W^2_-(x) &= \int p^2_-(x)\,dx, \text{ for } x \ge 1.2 \\ W^2_+(x) &= \int p^2_+(x)\,dx, \text{ for } x \le \frac{6}{7}. \end{aligned}$$

(One may check that for our parameter values $p^2_\pm(x)$ is undefined for $\frac{6}{7} < x < 1.2$.) Using values $x_2 \approx -1.31775$, $x_1 = 3/2$, $x_3 \approx 2.55389$ we can present the lower value function(s) for our game:

$$V^2(x) = \begin{cases} W^2_+(x) + C_0 & \text{for } x < 0 \\ \beta + W^1(x) & \text{for } 0 \le x \le x_1 \\ W^2_-(x) + C_1 & \text{for } x_1 < x, \end{cases} \tag{5.5}$$

where the constants C_0, C_1 are chosen to make V^2 continuous, and

$$V^1(x) = \begin{cases} \beta + V^2(x) & \text{for } x \le x_2 \\ W^1(x) & \text{for } x_2 < x < x_3 \\ \beta + V^2(x) & \text{for } x_3 \le x. \end{cases} \tag{5.6}$$

Graphs are presented in Figure (1). Our arguments below depend on a number of inequalities involving DV^a as defined by (5.6), (5.5). For brevity, we will verify several of them graphically rather than algebraically.

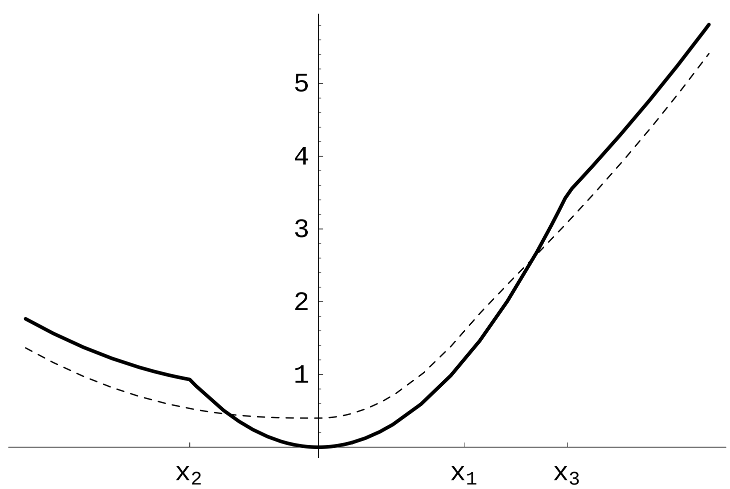

Figure 1: V^1 (solid) and V^2 (dashed)

The procedure for constructing (5.6), (5.5), and the significance of the particular values x_1, x_2, x_3, will become apparent as we now work through the verification of SQVI (5.1). Observe that SQVI (5.1) is equivalent to the following three conditions for each $a \in \{1, 2\}$. (Here a' will generically denote the other value of a: $a' = 3 - a$.)

$$V^a(x) \le \beta + V^{a'}(x), \text{ for all } x, \tag{5.7}$$

$$H^a(x, D^+V^a(x)) \le 0, \text{ for all } x, \tag{5.8}$$

$$H^a(x, D^-V^a(x)) \ge 0, \text{ for those } x \text{ with } V^a(x) < \beta + V^{a'}(x), \tag{5.9}$$

where $D^+V^a(x)$ refers to the *superdifferential* of V^a at x and $D^-V^a(x)$ refers to the *subdifferential* of V^a at x. The superdifferential $D^+V^a(x)$ can be characterized as the set of all possible slopes $\varphi'(x)$ for a smooth test function φ such that $V^a - \varphi$ has a local maximum at x; similarly the subdifferential $D^-V^a(x)$ is characterized as the set of all possible slopes $\varphi'(x)$ for a smooth test function φ such that $V^a(x) - \varphi$ has a local minimum at x (see [6, page 29]). At points x where both V^1 and V^2 are smooth, these conditions can be expressed more explicitly as: *Necessarily* $|V^1(x) - V^2(x)| \le \beta$.

1. *If* $V^1(x) - V^2(x) = \beta$, *then* $(V^1)'(x) = (V^2)'(x) =: q(x)$ *(since* $V^1 - V^2$ *has a maximum at* x*), and*

$$H^1(x, q(x)) \le 0, \quad H^2(x, q(x)) = 0.$$

2. *If* $V^1(x) - V^2(x) = -\beta$, *then similarly,* $(V^1)'(x) = (V^2)'(x) =: q(x)$ *and*

$$H^1(x, q(x)) = 0, \quad H^2(x, q(x)) \le 0.$$

3. *If* $|V^1(x) - V^2(x)| < \beta$, *then both*

$$H^1(x, (V^1)'(x)) = 0, \quad H^2(x, (V^2)'(x)) = 0.$$

There are a number of other cases, depending on whether x is a smooth point for one or both of V^1 and V^2 and on the relative sizes of the one-sided derivatives of V^a at x if x is a nonsmooth point for V^a. We will work these conditions out as they are needed.

We begin the construction by noticing that our choice of $h(y, a, b) = y^2$ makes the system both zero-state observable and detectable (for any control strategy), so that Propositions 4.2 and 4.3 apply. In particular, for the optimal control and the disturbance $b \equiv 0$ the system must converge to 0 for all initial states x and initial control values a. Since a^1 is the only control value which stabilizes the system at 0, it seems clear that, near $x = 0$, $V^1(x)$ must be the available storage function $W^1(x)$ associated with the fixed control a^1. If one starts with the control a^2 and if $x > 0$ is close to 0, it is optimal to switch immediately to control a^1: the system will have to switch to a^1 eventually in order to reach $x = 0$ and will only drive up the cost in $|x|$ by switching later. Hence for $x > 0$ and close to 0, we expect to have $V^2(x) = \beta + W^1(x) = \beta + V^1(x)$. On the other hand, if we start with control a^2 and initial state $x < 0$ and small in magnitude, we do better to use a^2 to drive us to the origin and then switch to a^1 to keep us at the origin. This leads us to conclude that, for small x with $x < 0$, $V^2(x)$ is the minimal solution of $H^2(x, (V^2)'(x)) = 0$ with initialization $V^2(0) = \beta$. By such direct qualitative reasoning we deduce that the form of $(V^1(x), V^2(x))$ for x in a neighborhood of the origin 0 is as asserted.

For x close to the origin and positive, we are in case (2): we need to check $H^1(x, q(x)) = 0$ while $H^2(x, q(x)) \leq 0$, where the $q(x)$ is the common value of $(V^1)'(x)$ and $(V^2)'(x)$, or $p^1_-(x)$. The first equation holds trivially while the second holds as a consequence of $p^1_-(x) < p^2_-(x)$ for $0 \leq x < \delta$ for some $\delta > 0$.

Calculation shows that the first δ for which this latter equality fails is $\delta = x_1 = 3/2$, where $p^1_-(x)$ and $p^2_-(x)$ cross. At this stage, we arrange that $(V^2)'(x)$ be equal to $p^2_-(x)$ instead of $p^1_-(x)$ while $(V^1)'(x)$ continues to equal $p^1_-(x)$ to the immediate right of x_1. Note that the continuation of $V^2(x)$ defined in this way is smooth through x_1. In this way we have arranged that both Hamilton-Jacobi equations are satisfied $(H^a(x, (V^a)'(x)) = 0$ for $a = 1, 2)$. The only catch is to guarantee that we maintain $|V^1(x) - V^2(x)| \leq \beta$. This condition holds for an interval to the right of x_1 since we have $V^1(x_1) - V^2(x_1) = -\beta$ while $(V^1)'(x) - (V^2)'(x) = p^1_-(x) - p^2_-(x) \geq 0$.

Calculation shows that the first point to the right of x_1 at which $|V^1(x) - V^2(x)| < \beta$ fails is the point x_3 where $V^1(x) - V^2(x) = \beta$; if we continue with the same definitions of $V^1(x)$ and $V^2(x)$ to the right of x_3, we get $V^1(x) - V^2(x) > \beta$ for x to the immediate right of x_3. To fix this problem, to the immediate right of x_3 we arrange that $(V^2)'(x)$ still be equal to $p^2_-(x)$ but now set $V^1(x) = V^2(x) + \beta$. Then points to the immediate right of x_3 are smooth for both V^1 and V^2 and the applicable case for the check of a viscosity solution at such points is case (1). Trivially we still have $H^2(x, (V^2)'(x)) = H^2(x, p^2_-(x)) = 0$ while $H^1(x, (V^1)'(x)) = H^1(x, p^2_-(x)) \leq 0$ since necessarily $V^1(x) - V^2(x)$ is increasing at x_3 from which we get $p^1_-(x) - p^2_-(x) > 0$ on an interval containing x_3 in its interior. At the point x_3 itself, we have $D^+V^2(x_3) = \{p^2_-(x)\} = D^-V^2(x_3)$ while $D^-V^1(x_3) = \emptyset$ and $D^+V^1(x_3) = [p^2_-(x_3), p^1_-(x_3)]$. To check that (V^1, V^2) is a viscosity solution of SQVI (5.1) at x_3 one simply checks that (i) $H^2(x_3, (V^2)'(x_3)) = H^2(x_3, p^2_-(x_3)) = 0$ and (ii) $H^1(x, p) \leq 0$ for all $p \in [p^2_-(x_3), p^1_-(x_3)]$.

The discussion for $x < 0$ is quite similar to the above. To the immediate left of 0, $(V^1)'(x)$ is taken equal to $p^1_+(x)$ rather than to $p^1_-(x)$, while $(V^2)'(x)$ is taken equal to $p^2_-(x)$. Thus 0 is a smooth point for $V^2(x)$. For points x to the immediate left of 0, we have $H^a(x, (V^a)'(x)) = 0$ for $a = 1, 2$, so the only remaining issue for (V^1, V^2) to be a viscosity solution at such points is the inequality $|V^1(x) - V^2(x)| \leq \beta$. To verify this, one can check that $V^1(0) - V^2(0) = -\beta$ and $(V^1)'(x) - (V^2)'(x) = p^1_+(x) - p^2_+(x) < 0$ on an interval $-\delta < x < 0$. We maintain these definitions of $V^1(x)$ and $V^2(x)$ as x moves to the left away from the origin until we reach the point x_2 where $V^1(x) - V^2(x) = \beta$ and continuation of these definitions for x to the left of x_2 would lead to the unacceptable inequality $V^1(x) - V^2(x) > \beta$. For x to the left of x_2 we let $V^2(x)$ continue to follow $p^2_+(x)$ while we set $V^1(x) = V^2(x) + \beta$. To the left of x_2 we then have $H^2(x, (V^2)'(x)) = H^2(x, p^2_+(x)) = 0$ while $H^1(x, (V^1)'(x)) = H^1(x, p^2_+(x)) \leq 0$ since we still have $p^2_+(x) > p^1_+(x)$ for $x < 0$; this verifies that (V^1, V^2) is a viscosity solution of SQVI (5.1) for $x < x_3$. At $x = x_3$, one checks the viscosity solution conditions by noting that $H^2(x_3, (V^2)'(x_2)) = H^2(x_2, p_+^2(x_2)) = 0$ and $H^1(x_2, p) \leq 0$ for all $p \in D^+V^1(x_2) = [p_+^1(x_2), p_+^2(x_2)]$.

It should be possible to verify that any deviation from this construction which maintains the property that (V^1, V^2) is a viscosity supersolution leads to a larger (V^1, V^2); Theorem 3.8 (apart from the technical gaps that we have searched only through all piecewise C^1 viscosity supersolutions rather than through all lower semi-continuous viscosity supersolutions and that $B = \mathbb{R}$ is not compact) then implies that (V^1, V^2) constructed as above is the lower-value function for this switching-control game. Instead we now give an alternative direct argument that (V^1, V^2) is indeed the lower value function.

The strategy α^* associated with our solution (5.5), (5.6) is easy to describe in state-feedback terms. Define the *switching sets*

$$\begin{aligned} S_1 &= \{x : \; V^2(x) = \beta + V^1(x)\} = [0, x_1], \\ S_2 &= \{x : \; V^1(x) = \beta + V^2(x)\} = (-\infty, x_2] \cup [x_3, \infty). \end{aligned}$$

The strategy α^* will instantly switch from $a = 1$ to $a = 2$ whenever $y(t) \in S_2$, and instantly switch from $a = a^2$ to $a = a^1$ whenever $y(t) \in S_1$. Otherwise α^* continues using the current control state. Theorem 3.8 would imply that $V_\gamma^a \leq V^a$, where V_γ^a are the lower values. We will prove directly that in fact $V_\gamma^a = V^a$, and that our strategy α^* is optimal. To be precise, we shall show that for any j and any strategy $\alpha \in \Gamma$

$$V^j(y(0)) \leq \sup_{b \in \mathcal{B}} \sup_{T>0} \left\{ \int_0^T [h(y_x(s), \alpha_x^j[b](s), b(s)) - \gamma^2 |b(s)|^2] \, ds + \sum_{\tau_i \leq T} k(a_{i-i}, a_i) \right\}. \tag{5.10}$$

Moreover, for our strategy α^*, (5.10) will be an equality for all x, j. The key to this is the existence of a particular "worst case" disturbance, as described in the following proposition. This proposition is intended only in the context of the particular example and parameter values described above.

Proposition 5.1 *For any $x \in \mathbb{R}^n$, $j \in \{1, 2\}$ and strategy $\alpha \in \Gamma$, there exists a disturbance $b^* = b^*_{\alpha_x^j} \in \mathcal{B}$ with the property that*

$$b^*(t) = \frac{1}{2\gamma^2} (V^{\alpha_x^j[b^*])(t)})'(y_x(t, \alpha_x^j[b], b)),$$

holds for all but finitely many t in every interval $[0, T]$.

Proof Suppose j, $\alpha \in \Gamma$ and an initial point $x \in \mathbb{R}^n$ are given. Begin by considering the solution of

$$\dot{y} = f(y, a^j, \frac{1}{\gamma^2}(V^j)'(y)); \quad y(0) = x. \tag{5.11}$$

For $j = 2$ the right side is C^1, so the solution is uniquely determined. For $j = 1$, the right side has discontinuities at x_2 and x_3, but since $f(x, a^j, \frac{1}{\gamma^2}(V^1)'(x))$ does not change sign across the discontinuities, the solution is again uniquely determined. Graphs of $f(y, a^j, \frac{1}{\gamma^2}(V^j)'(y))$ are provided in Figures 2 and 3 below. (We comment that although the graphs appear piecewise linear, they are not. Figure 2 is linear only for $0 < x < x_1$ and Figure 3 is only linear for $x_2 < x < x_3$, as inspection of the formulas shows.) Since $y\dot{y} < 0$ for sufficiently large $|y|$, it is clear that the solution of (5.11) is defined for all $t \geq 0$. Observe also for $j = 1$ that, for any solution of (5.11), there is at most one value of t for which $y(t)$ is at one of the discontinuities of $(V^1)'$. Thus $(V^j)'(y(t))$ is undefined for at most a single t value.

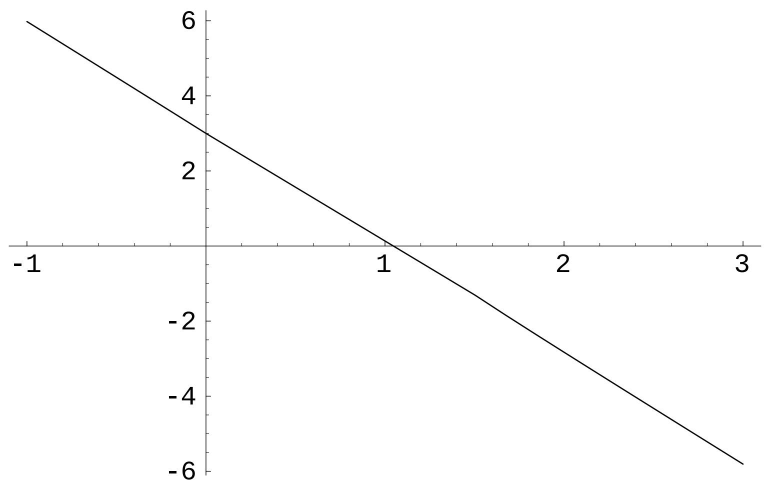

Figure 2: Plot of $f(x, 2, \frac{1}{2\gamma^2}DV^2(x))$.

Now consider the disturbance $b(t) = \frac{1}{\gamma^2}(V^j)'(y(t))$. The control $\alpha_x^j[b](t)$ produced for this disturbance will only take the value j on the initial interval: $0 = \tau_0 \leq t \leq \tau_1$. We define $b^*(t) = b(t) = \frac{1}{\gamma^2}(V^j)'(y(t))$ for these t. At $t = \tau_1$ the control $\alpha_x^j[b]$ will switch from j to j'. We therefore redefine $y(t)$ for $t > \tau_1$ as the solution of

$$\dot{y} = f(y, j', \frac{1}{\gamma^2}(V^{j'})'(y))$$

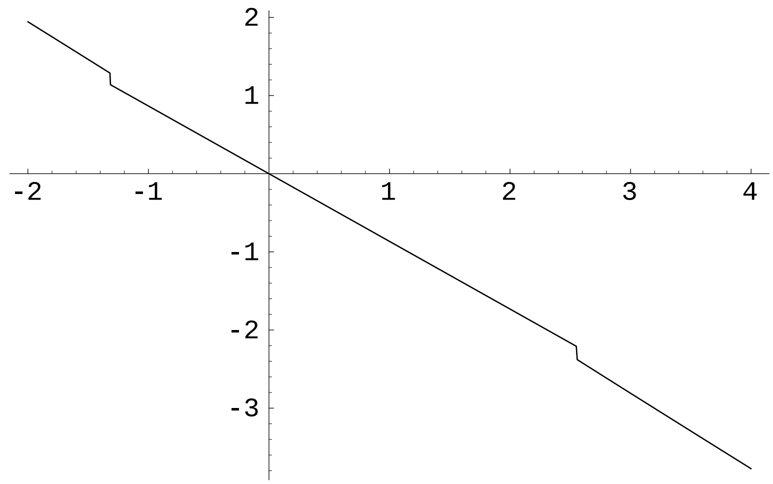

Figure 3: Plot of $f(x, 1, \frac{1}{2\gamma^2} DV^1(x))$.

with initial value $y(\tau_1)$ as already determined. Likewise, redefine $b(t) = \frac{1}{\gamma^2}(V^{j'})'(y(t))$ for $t > \tau_1$. Because we have not changed b on $[0, \tau_1]$, the nonanticipating property of strategies insures that $\alpha_x[b](t)$ for $t \leq \tau_1$ and τ_1 remain the same for this revised b. Using the new b, the control $\alpha_x^j[b](t)$ determines the next switching time τ_2. We know that $\tau_1 < \tau_2 \leq \infty$ and $\alpha_x^j[b](t) = j'$ for $\tau_1 < t \leq \tau_2$. We now extend our definition of b^* with $b^*(t) = b(t)$ for $\tau_1 < t \leq \tau_2$. At τ_2 the control switches again, back to j. So we now redefine $y(t)$ and $b(t)$ for $t > \tau_2$ by taking $y(\tau_1)$ as already determined, solving

$$\dot{y} = f(y, j, \frac{1}{\gamma^2}(V^j)'(y))$$

and redefining $b(t) = \frac{1}{\gamma^2}(V^j)'(y(t))$ for $t > \tau_2$. For $t \leq \tau_2$ the values of $b(t)$, $\alpha_x^j[b](t)$, and $y(t)$ remain unchanged, again by the nonanticipating hypothesis. We now identify the switching time τ_3 associated with $\alpha_x^j[b](t)$, and extend our definition for $\tau_2 < t \leq \tau_3$ using $b^*(t) = b(t)$. At τ_3 the control will switch again to j', so continue our redefinition process again for $t > \tau_3$.

Continuing this redefinition and extension process, we produce the desired disturbance $b^*(t)$ and state trajectory $y(t)$ associated with the control $\alpha_x^j[b^*](t)$ satisfying the requirements of the proposition. The only conceivable failure of this construction would be if the switching times τ_i which are generated in the construction were to have a finite limit: $\lim \tau_i = s < \infty$. Our hypotheses on the strategy α disallow this however, for the following reason. If it were the case that $\lim \tau_i = s < \infty$, then extend our definition of b^* in any way to $t \geq s$, say $b^*(t) = 0$. By hypothesis, $\alpha_x^j[b^*]$ is an admissible control in $\mathcal{A}$, which means in particular that its switching times τ_i do not have a finite accumulation point. But extension of b^* for $t > s$ does not alter the switching times $\tau_i < s$, by the nonanticipating property again. This would mean that $\alpha[b^*]$ does have an infinite number of switching times $\tau_i < s$, a contradiction. Finally, by our comments above, on each interval $[\tau_i, \tau_{i+1}]$ there is at most a single t value at which $(V^{\alpha_x^j[b^*]})'(t)$ is undefined. Thus there are at most a finite number of such t in any $[0, T]$. $\diamondsuit$

Consider now any strategy $\alpha \in \Gamma$, initial position $x = y(0)$ and associated disturbance b^* be as in the proposition. On any time interval $[\tau_i, \tau_{i+1}]$ between consecutive switching times, (5.8) and the fact that $b^*(t)$ achieves the infimum in (5.2) for $x = y(t)$ and $p = (V^{a_i})'(x)$ implies that (for all but finitely many t)

$$\frac{d}{dt} V^{a_i}(y(t)) \geq (\gamma b^*(t))^2 - h(y(t), a_i, b^*(t)).$$

Thus for any $\tau_i < t \leq \tau_{i+1}$ we have

$$V^{a_i}(y(t)) - V^{a_i}(y(\tau_i)) \geq \int_{\tau_i}^{t} \gamma^2 |b^*|^2 - h\, ds.$$

Across a switching time τ_i we have from (5.7)

$$V^{a_i} - V^{a_{i-1}} \geq -\beta = -k(a_{i-1}, a_i).$$

Adding these inequalities over $\tau_i \leq T$ we see that

$$V^{\alpha[b^*](T)}(y(T)) - V^{\alpha[b^*](0)}(y(0)) \geq - \left\{ \int_0^T [h - \gamma^2 |b^*|^2]\, ds + \sum_{\tau_i \leq T} k(a_{i-1}, a_i) \right\}.$$

A rearrangement of this gives

$$V^{\alpha[b^*](T)}(y(T)) + \left\{ \int_0^T [h - \gamma^2 |b^*|^2]\, ds + \sum_{\tau_i \leq T} k(a_{i-1}, a_i) \right\} \geq V^{\alpha[b^*](0)}(y(0)). \quad (5.12)$$

When we consider α^* specifically, we recognize that

$$H^{a_i}(y(t), (V^{a_i})'(y(t))) = 0$$

(where we set in general $H^{a^i} = H^i$ and $V^{a^i} = V^i$ for $i = 1, 2$) for t between the τ_i, and at τ_i

$$V^{a_{i+1}} - V^{a_i} = -\beta = -k(a_{i+1}, a_i).$$

This means that (5.12) is an equality for α^* specifically.

To finish our optimality argument we will show that for α in general above, as $T \to \infty$ we must have either $y(T) \to 0$ and $\alpha[b^*](T) \to 1$, or else

$$\int_0^T [h - \gamma^2 |b^*|^2]\, ds + \sum_{\tau_i \le T} k(a_{i-1}, a_i) \to +\infty. \tag{5.13}$$

In the case of α^* specifically, we will have the former possibility. Since $V^1(0) = 0$ and is continuous, these facts imply (5.10) as claimed. The verification of these asserted limiting properties depends on some particular inequalities for $(V^a)'(x)$ as determined by (5.6), (5.5). First, we assert that, for both $a = 1$ and $a = 2$,

$$h(x, a, b^*) - \gamma^2 |b^*|^2 = |x|^2 - \frac{1}{4\gamma^2}[(V^a)'(x)]^2 > 0, \text{ for } x \neq 0. \tag{5.14}$$

Moreover $|x|^2 - \frac{1}{4\gamma^2}[(V^a)'(x)^2$ has a positive lower bound on $\{x : \ |x| \ge \epsilon\}$ for each $\epsilon > 0$. Instead of what would be a very tedious algebraic demonstration of this, we simply offer the graphical demonstration in Figure 4. For the parameter values (5.4) we have plotted $b^* = \frac{1}{2\gamma}(V^a)'(x)$ (solid lines) and $q = x$ (dashed lines) as functions of x. The validity of (5.14) is apparent.

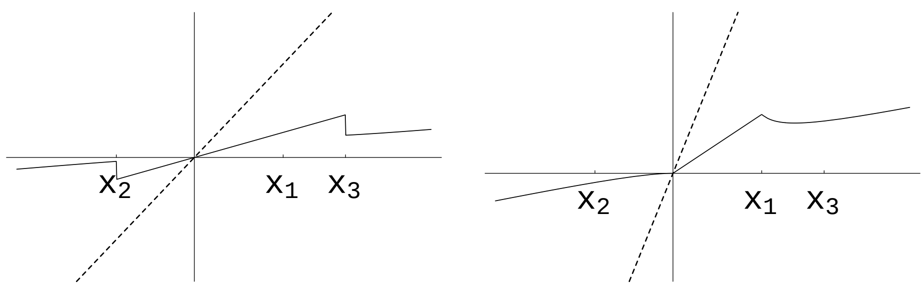

Figure 4: Graphical verification of (5.14) for DV^1 (left) and DV^2 (right)

The other fact we need is that for $a = 2$ and the corresponding disturbance $b^*(t)$, the state-dynamics does not have an equilibrium at 0. This is easy to see, because at $x = 0$ we have $b^* = \frac{1}{2\gamma^2}(V^2)'(0) = 0$, but $f(0, a^2, b^*) = -\mu + b^*$. A graph of $f(x, a^2, b^*) = -\mu(x-1) + \frac{1}{2\gamma^2}(V^2)'(x)$ is provided in Figure 2, where we see a unique equilibrium just beyond $x = 1$.

In the case of $a = 1$ however, $\dot{x} = f(x, a^1, \frac{1}{2\gamma^2}(V^1)'(x))$ has a unique globally asymptotically stable equilibrium at $x = 0$, as is evident in Figure 3.

We turn then to the verification of the assertion of (5.13) or its alternative: assuming (5.13) to be false we claim that $y(T) \to 0$ and $\alpha[b^*](T) \to 1$. By the nonnegativity from (5.14) we must have both

$$\sum_{\tau_i<\infty} k(a_{i-1}, a_i) < \infty, \text{ and } \int_0^\infty [h - \gamma^2 |b^*|^2] \, dx < \infty. \tag{5.15}$$

The first of these implies that there are only a finite number of switches; $\alpha[b^*](t) = a^{i^*}$ is constant from some time on. It is not possible that $i^* = 2$ because in that case $y(t)$ would be converging to the positive equilibrium of Figure 2, which implies by (5.14) that, as $t \to \infty$,

$$h(y(t), a_{i^*}, b^*(t)) - \gamma^* |b^*(t)|^2 \to C > 0.$$

This contradicts the second part of (5.15). Therefore $i^* = 1$, which shows that $\alpha[b^*](T) \to 1$. But since $\alpha[b^*](t) = 1$ from some point on, the stability illustrated in Figure 3 means that $y(t) \to 0$ as claimed. This completes our verification of the optimality of the strategy α^*.

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