Robust optimal stopping-time control for nonlinear systems

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Abstract. We formulate a robust optimal stopping-time problem for a state-space system and give the connection between various notions of lower value function for the associated games (and storage function for the associated dissipative system) with solutions of the appropriate variational inequality (VI) (the analogue of the Hamilton-Jacobi-Bellman-Isaacs equation for this setting). We show that the stopping-time rule can be obtained by solving the VI in the viscosity sense and a positive definite supersolution of the VI can be used for stability analysis.

Key Words. variational inequality, viscosity solution, worst-case disturbance attenuation, differential game, value function, storage function, nonanticipating strategy, state-feedback control, stopping-time rule

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Abbreviated title. Stopping-time control.

1 Introduction

We consider state space systems of the form

$$\Sigma_{st} \begin{cases} \dot{y}(t) = f(y(t), b(t)) \\ z(t) = h(y(t), b(t)), \end{cases}$$

where $y(\cdot) \in \mathbb{R}^n$ denotes the state, $b(\cdot) \in B \subseteq \mathbb{R}^m$ denotes the deterministic unknown disturbance on the system, and $z(\cdot) \in \mathbb{R}$ is the cost function. In addition we assume that we are given a positive real-valued *stopping cost* function $y \to \Phi(y)$ defined for states $y \in \mathbb{R}^n$. We consider the cost of running the system up to time T with initial condition x, disturbance b and stopping time τ to be the quantity

$$C_T(x, \tau, b) = \int_0^{T\wedge\tau} h(y_x(s, b), b(s))\ ds + 1_{[0,T]}(\tau)\Phi(y_x(\tau, b)).$$

We have used the notations $y_x(s, b)$ for the solution of $\dot{y} = f(y, b)$ with $y_x(0, b) = x$, $1_{[0,T]}$ for the indicator function with value 1 if $\tau \in [0, T]$ and 0 otherwise, and $T \wedge \tau$ for $\min\{T, \tau\}$. (In the sequel we will often abbreviate $y_x(s, b)$ to $y_x(s)$; the precise meaning should be clear from the context.) For a prescribed tolerance level $\gamma > 0$, we seek a *stopping-time rule* $\tau \in [0, \infty]$ so that

$$C_T(x, \tau, b) \leq \gamma^2 \int_0^{T\wedge\tau} |b(s)|^2\ ds + U(x) \tag{1.1}$$

for all locally L^2 disturbances b, all nonnegative real numbers T and some *bias* function U, i.e., a nonnegative real-valued function U with $U(0) = 0$. In the *open loop* version of the problem, τ is simply a nonnegative extended real number. In the *state-feedback* version of the problem, τ is a function of the current state in the sense that one decides on whether to stop or continue at a given point in time t as a function of the state vector $y(t)$ at time t. In the standard game-theoretic formulation of the problem, τ is taken to be a nonanticipating function of the disturbance b, i.e., one decides whether $\tau \leq t$ based solely on the information consisting of the initial state x and the past of the disturbance $b|_{[0,t]}$. The dissipation inequality (1.1) can then be viewed as the analogue of the closed-loop system (with L^2-norm of output signal being taken to be $C_T(x, \tau, b)$ for each finite-time horizon $[0, T]$) having L^2-gain of at most γ. A refinement of the problem then asks for the control τ which gives the best system performance, in the sense that the nonnegative function $U(x)$ is as small as possible. A closely related formulation is to view the stopping-time system as a game with payoff function

$$J_T(x, \tau, b) = \int_0^{T\wedge\tau} [h(y_x(s, b), b(s)) - \gamma^2|b(s)|^2]\ ds + 1_{[0,T]}(\tau)\Phi(y_x(\tau, b))$$

where the disturbance player tries to use $b(\cdot)$ and T to maximize the payoff, while the control player tries to use the stopping time τ to minimize the payoff. The control decision at each moment of time is whether to stop and cut one's losses (with penalty $\Phi(y_x(\tau))$ in addition to the accumulated running cost up to time τ), or to continue running the system (including the possibility of never stopping the system before the disturbance stops it). As we shall introduce a variation on this game below, we shall refer to this game as Game I.

We define a *lower-value* function for Game I as

$$W(x) = \inf_{\tau} \sup_{b,\, T} \left\{ 1_{[0,T]}(\tau_x[b]) \Phi(y_x(\tau_x[b])) + \int_0^{T \wedge \tau_x[b]} [h(y_x(s), b(s)) - \gamma^2 |b(s)|^2] \, ds \right\} \tag{1.2}$$

where the supremum is over all nonnegative real numbers T and L^2-disturbance signals b, while the infimum is over all nonanticipating control strategies $(x, b) \to \tau_x[b]$ satisfying $0 \leq \tau_x[b] \leq \infty$. Then by construction $W(x)$ gives the smallest possible value which can satisfy (1.1) (with W in place of U) for some strategy τ.

We now introduce a variation on Game I which we shall call Game II. For the rules of Game II, the maximizing player no longer controls a cutoff time T but rather only the disturbance b, while the minimizing player is constrained to play only nonanticipating stopping-time rules $(x, b) \to \nu_x[b]$ with finite values ($\nu_x[b] < \infty$ for all (x, b)), and the payoff function is taken to be $J_\infty(x, \nu, b)$. In this formulation, the payoff is guaranteed to be finite due to $\nu_x[b] < \infty$ rather than from $T < \infty$. The lower value function for Game II is then given by

$$V(x) := \inf_{\nu} \sup_{b} \left\{ \Phi(y_x(\nu_x[b])) + \int_0^{\nu_x[b]} [h(y_x(s), b(s)) - \gamma^2 |b(s)|^2] \, ds \right\}, \tag{1.3}$$

where the infimum is over all finite-valued nonanticipating control strategies. From the L^2-gain perspective, $V(x)$ is associated with the desire to optimize the performance bound

$$\int_0^{\nu} h(y_x(s), b(s)) \, ds + \Phi(y_x(\nu)) \leq \gamma^2 \int_0^{\nu} |b(s)|^2 \, ds + U(x),$$

(over finite-valued stopping-time rules ν).

For the purposes of comparison, we also introduce the *available storage function* $S_a(x)$ associated with a disturbance-input to cost-output system (with stopping-time options ignored)

$$S_a(x) := \sup_{b,\, T} \left\{ \int_0^T [h(y_x(s), b(s)) - \gamma^2 |b(s)|^2] \, ds \right\}. \tag{1.4}$$

The function $S_a(x)$ is associated with the desire to optimize the standard performance bound associated with L^2-gain attenuation level γ for an input-output system (with all stopping options ignored)

$$\int_0^T h(y_x(s), b(s))\, ds \le \gamma^2 \int_0^T |b(s)|^2\, ds + U(x).$$

Under some technical assumptions, this available storage function S_a is a viscosity solution in $\mathbb{R}^n$ of the Hamilton-Jacobi-Bellman equation (HJBE) $H(x, DS_a(x)) = 0$, where

$$H(x,p) := \inf_b \{-p \cdot f(x,b) - h(x,b) + \gamma^2 |b|^2\}.$$

Moreover if S_a is continuous, then it is characterized as the minimal, nonnegative, continuous viscosity supersolution of the HJBE [19] (see [15] and [6] for earlier versions and [7, Appendix B], [20] and [13] for further refinements).

In addition we introduce the notion of a *stopping-time storage function* S for a closed-loop stopping-time system (with some particular stopping-time rule $(x,b) \to \tau_x[b]$ already implemented) with disturbance-input b, namely, a nonnegative function $x \to S(x)$ such that

$$\begin{aligned} &1_{(T,\infty)}(\tau_x[b]) S(y_x(T,b)) - S(x) \\ &\quad \le \int_0^{T\wedge\tau_x[b]} [\gamma^2 |b(s)|^2 - h(y_x(s), b(s))]\, ds - 1_{[0,T]}(\tau_x[b]) \Phi(y_x(\tau_x[b], b)) \end{aligned} \tag{1.5}$$

for all $b \in \mathcal{B}$ and $T \ge 0$. If we set $\tau_x[b] = \infty$ for all $x \in \mathbb{R}^n$ and $b \in \mathcal{B}$, we recover the notion of storage function (associated with L^2-gain supply rate) introduced by Willems (see [21] and [17]). The control problem then is to find the stopping-time rule $(x,b) \to \tau_x[b]$ which gives the best performance, as measured by obtaining the minimal possible $S(x)$ as the associated closed-loop storage function. This suggests that the stopping-time available storage function $S_{st,a}$ (i.e., the minimal possible stopping-time storage function over all possible stopping-time rules) should be equal to the lower-value function W for Game I; we shall see that this is indeed the case with appropriate hypotheses imposed.

Our main results concerning the robust stopping-time problems are as follows: *under minimal smoothness assumptions on the problem data,*

1. *If the lower value function W for Game I is upper semicontinuous, then W is a viscosity subsolution in $\mathbb{R}^n$ of the variational inequality (VI) given by*

$$\max\{H(x, DV(x)), V(x) - \Phi(x)\} = 0, \; x \in \mathbb{R}^n.$$

   If W is lower semicontinuous, then W is a viscosity supersolution of the VI. Thus if W is continuous, W is a viscosity solution of the VI. In fact, if W is continuous, then W can be characterized as the the minimal, nonnegative, continuous viscosity supersolution of the VI. (The precise definition of viscosity-sense supersolutions, subsolutions and solutions will be given in Section 2.)

2. *If continuous, the lower value function V for Game II is a viscosity solution of the VI. Moreover in certain cases V is characterized as the maximal viscosity subsolution of the VI.*

3. *Any locally bounded stopping-time storage function (for some stopping-time strategy τ) is a viscosity supersolution of the VI; conversely, if U is any nonnegative continuous viscosity supersolution of the VI, then U is a stopping-time storage function with stopping-time rule of state-feedback form given by $\tau_{U,x}[b] = \inf\{t \geq 0 : U(y_x(t,b)) \geq \Phi(y_x(t,b))\}$, and $U \geq W$.*

It also happens that a positive definite supersolution U of the VI can be used to prove stability of the equilibrium point 0 for the system with zero disturbance $\dot{y} = f(y, 0)$. We also obtain the lower-value function $W(x)$ explicitly for a prototype problem with one-dimensional state space by a simple, direct, geometric construction.

The robust stopping-time control problem as formulated here can be viewed as a stopping-time analogue of the nonlinear version of the standard problem of H_∞-control (see [14], [17]). For the nonlinear H_∞-control problem, one is given a system of the form

$$\dot{y}(t) = f(y(t), a(t), b(t)), \quad y(0) = x. \tag{1.6}$$

where $y(t)$ is the state-variable and $b(t)$ is the disturbance signal as in the stopping-time problem, but the control is a locally L^2 input signal $a(t)$ with values in some real Euclidean space $\mathbb{R}^p$. The cost of running the system up to time T with initial condition x, disturbance b and control a is taken to be the quantity

$$C_T^{H_\infty}(x, a, b) = \int_0^T h(y_x(s, a, b), b(s))\ ds$$

(where $y(t, a, b)$ is the solution of (1.6) with control input $a(t)$, disturbance input $b(t)$ and initial condition $y(0) = x$). The goal of the H_∞-control problem then is to select a control signal $a(t)$ (in state-feedback or nonanticipating strategy form) so as to guarantee that the closed loop system has L^2-gain at most equal to γ, i.e., that the dissipation inequality

$$C_T^{H_\infty}(x, a, b) \leq \gamma^2 \int_0^T |b(s)|^2\ ds + U(x)$$

holds for all locally L^2-disturbances b and finite times T for some nonnegative bias (or storage) function $U(x)$ with $U(0) = 0$. In addition, under appropriate hypotheses, one can use the storage function $U(x)$ as a Lyapunov function to prove that the trajectories of the closed-loop disturbance-free ($b = 0$) system tend asymptotically to the equilibrium point 0 as time tends to infinity. The theory and results sketched above for our stopping-time problem parallel the standard results concerning the nonlinear H_∞-control problem found, e.g., in [17] and [19].

Optimal stopping-time problems have a long history in probability theory. There is an enjoyable introductory exposition in [11] and a more thorough treatment in [8], [12] and [18]. Just as (deterministic) robust control has many analogies with classical stochastic control, our idea here can be viewed as developing a deterministic robust analogue of optimal stopping. A stochastic stopping-time game is formulated in [8, Section 2.9], but with both players having only the option to stop the system (as opposed to our setup with one player having an input-signal control and the other player having a stopping-time option). A deterministic formulation of an optimal stopping-time problem is discussed in Section III.4.2 of [7], but with a discounted cost rather than dissipation inequality (e.g. (1.1)) and with no disturbance competing with the control as in the robust approach.

Optimal stopping-time problems have a superficial resemblance to problems with restricted state space in which a cost like our $\Phi(y_x(\tau))$ is imposed at the exit time τ from some prescribed domain Ω. See Chapter IV of [7] for instance. In optimal stopping problems no such domain is prescribed. The analogous role is played by $\Omega = \{x : W(x) < \Phi(x)\}$, which is only known implicitly in terms of the value function $W(x)$. The point of using some sort of variational inequality (VI), as we do here, is to avoid any explicit reference to this domain and work in the full state space $\mathbb{R}^n$ without regard to any prescribed domain of allowable states.

The derivation of the VI for the stopping-time problem is a direct application of the method of dynamic programming standard in control theory. The technical contribution here to the optimal stopping-time problem can be seen as parallel to that of Soravia in [19] for the nonlinear H_∞-control problem: to extend the game-theoretic, dynamic-programming approach to the infinite-horizon setting where, due to a lack of discount factor in the running cost, the running cost is not guaranteed to be integrable over the infinite interval $[0, \infty)$. This forces the introduction of the extra "disturbance player" T in (1.2) and (1.4) in the formulation of Game I (or of the finiteness restriction for admissible stopping-time rules in the formulation of Game II) and complicates many of the proofs.

Our original motivation for this study of robust optimal stopping-time problems was as a simpler prototype of a robust control problem with switching costs; the robust switching-cost problem is discussed in a separate publication [3]. The switching-cost problem, in turn, was motivated by an application to robust, optimal feedback-control of traffic signals (see [4], [5]), where the imposition of switching costs for change of traffic-signal settings can be used as a tuning parameter for control of the traffic signalization which also eliminates chattering in the optimal control.

The paper is organized as follows. Following the present Introduction, Section

2 presents assumptions and definitions. Section 3 presents the main results on the connection between value functions and solutions of variational inequalities; Section 3.1 handles the lower-value function W for Game I, Section 3.2 gives the results concerning stopping-time storage functions, and Section 3.3 sketches the results for the lower-value function V for Game II. Section 4 presents the state feedback robust stopping control, together with an illustrative example with one-dimensional state space where the value function and the associated feedback control are explicitly computable.

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2 Formulations and Definitions

We make the following assumptions on the problem data:

(A0) $0 \in B \subseteq \mathbb{R}^m$ and B is closed;
$f : \mathbb{R}^n \times B \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times B \to \mathbb{R}$ are continuous;

(A1) f and h are bounded on $B(0,R) \times B$ for all $R > 0$;

(A2) there are moduli ω_f and ω_h such that

$$\begin{aligned} |f(x,b) - f(y,b)| &\leq \omega_f(|x-y|, R) \\ |h(x,b) - h(y,b)| &\leq \omega_h(|x-y|, R), \end{aligned}$$

for all $x, y \in B(0,R)$ and $R > 0$, where a *modulus* is a function $\omega : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that, for all $R > 0$, $\omega(\cdot, R)$ is continuous, nondecreasing and $\omega(0,R) = 0$;

(A3) $(f(x,b) - f(y,b)) \cdot (x-y) \leq L|x-y|^2$ for all $x, y \in \mathbb{R}^n$ and $b \in B$;

(A4) $\Phi : \mathbb{R}^n \to \mathbb{R}^+$ is continuous with positive values;

(A5) $h(x,0) \geq 0$ for all $x \in \mathbb{R}^n$.

Remark 2.1 Note that assumption (A1) eliminates the linear-quadratic case (where f is linear in y and b and h is quadratic in the components of y and b) if B is taken to be an entire Euclidean space $\mathbb{R}^m$. If B is restricted to a compact subset, e.g., a large closed ball $B(0,R) \subset \mathbb{R}^m$, then the assumptions (A0)–(A5) do apply in the linear-quadratic case; this is sufficient for many applications.

For a specified gain parameter $\gamma > 0$ we define the *running cost* function

$$l(x, b) := h(x, b) - \gamma^2 |b|^2,$$

and the *Hamiltonian*

$$H(x, p) := \inf_{b \in B} \{-p \cdot f(x, b) - l(x, b)\}. \tag{2.1}$$

Note that $H(x, p) < +\infty$ for all $x, p \in \mathbb{R}^n$ by (A1). Under assumptions (A0)-(A3), we can show that H is continuous. (The proof is similar to that in [7, page 106].) Let $\mathcal{B}$ denote the set of locally square integrable functions $b : [0, \infty) \to B$. We consider $\mathcal{B}$ to be the set of admissible disturbances. We look at trajectories of the nonlinear dynamical system

$$\dot{y}(s) = f(y(s), b(s)), \quad y(0) = x \in \mathbb{R}^n. \tag{2.2}$$

Under the assumptions (A0), (A1) and (A3), for each $b \in \mathcal{B}$ and $x \in \mathbb{R}^n$ the solution of (2.2) exists and is unique for all $s \geq 0$. (The proof of this result is in III.5 of [7].) The solution of (2.2) will be denoted by $y_x(s, b)$, or briefly by $y_x(s)$ if there is no confusion. The basic estimates on y_x are the following (for the proofs see Section III.5 of [7]):

$$\begin{aligned}
|y_x(t, b) - y_z(t, b)| &\leq e^{Lt} |x - z|, \quad t > 0 & (2.3)\\
|y_x(t, b) - x| &\leq M_x t, \quad t \in [0, 1/M_x], & (2.4)\\
|y_x(t, b)| &\leq (|x| + \sqrt{2Kt}) e^{Kt}, \quad t > 0, & (2.5)
\end{aligned}$$

for all $b \in \mathcal{B}$, where

$$\begin{aligned}
M_x &:= \sup\{|f(z, b)| : |x - z| \leq 1, b \in B\}\\
K &:= L + \sup\{|f(0, b)| : b \in B\}.
\end{aligned}$$

For each $b \in \mathcal{B}$ and $x \in \mathbb{R}^n$, a *stopping-time rule* τ associates a single time: $0 \leq \tau_x[b] \leq +\infty$. The essential *nonanticipating property* of a stopping-time rule τ is that, for every $t \geq 0$, whenever two disturbances b and $\tilde{b}$ agree up to t,

$$b(s) = \tilde{b}(s) \text{ for all } s \leq t$$

then

$$1_{[0,t]}(\tau_x[b]) = 1_{[0,t]}(\tau_x[\tilde{b}]) \text{ for all } x.$$

In other words, knowing the history of $b(s)$ for $s \leq t$ is enough to answer the question of whether or not $\tau_x[b] \leq t$. We denote the set of stopping-time rules which have the nonanticipating property by Γ, i.e.,

$$\Gamma := \{\tau : \mathbb{R}^n \times \mathcal{B} \to [0, +\infty] \ : \ \tau \text{ is nonanticipating}\}.$$

If $b \in \mathcal{B}$ is a disturbance, $x \in \mathbb{R}^n$ is an initial state and $t > 0$, then we may consider $y_x(t, b)$ as a new initial state imposed at the time t. If $\tau \in \Gamma$ has the additional property

$$\tau_x[b] = t + \tau_{y_x(t,b)}[b_t] \text{ for all } b \in \mathcal{B} \text{ and all } x \in \mathbb{R}^n \text{ with } \tau_x[b] \geq t \tag{2.6}$$

(where we have set $b_t(s) = b(t+s)$ for all $b \in \mathcal{B}$), we shall refer to τ as a *state-feedback stopping-time rule*. In this case, given that the system has continued running up to time t, the decision of whether to stop immediately at time t or to continue can be read off from the current value of the state $y(t)$.

Let us introduce the notion of *upper* and *lower semicontinuous envelope* of a function $U : \mathbb{R}^n \to [-\infty, +\infty]$. These two new functions are, respectively,

$$\begin{aligned} U^*(x) &:= \limsup_{r \to 0^+} \{U(z) : |z - x| \leq r\} \\ U_*(x) &:= \liminf_{r \to 0^+} \{U(z) : |z - x| \leq r\}. \end{aligned}$$

It is well-known that if U is locally bounded, then $U_* \in LSC(\mathbb{R}^n)$ and $U^* \in USC(\mathbb{R}^n)$. Now we are ready to give the definition of a viscosity solution of a variational inequality

$$\max\{H(x, DU(x)),\ U(x) - \Phi(x)\} = 0,\ x \in \mathbb{R}^n. \tag{VI}$$

Definition 2.2 *A locally bounded function $U : \mathbb{R}^n \to [-\infty, +\infty]$ is a viscosity subsolution of the VI in $\mathbb{R}^n$ if for any $\Psi \in C^1(\mathbb{R}^n)$*

$$\max\{H(x_0, D\Psi(x_0)),\ U(x_0) - \Phi(x_0)\} \leq 0 \tag{2.7}$$

at any local maximum point $x_0 \in \mathbb{R}^n$ of $U^* - \Psi$. Similarly, U is a viscosity supersolution of the VI in $\mathbb{R}^n$ if for any $\Psi \in C^1(\mathbb{R}^n)$

$$\max\{H(x_1, D\Psi(x_1)),\ U(x_1) - \Phi(x_1)\} \geq 0 \tag{2.8}$$

at any local minimum point $x_1 \in \mathbb{R}^n$ of $U_* - \Psi$. Finally, U is a viscosity solution of the VI if it is simultaneously a viscosity subsolution and supersolution.

When U is continuous, the set of values $D\Psi(x_0)$ occurring in (2.7) is usually called the *superdifferential* of U at x_0:

$$D^+U(x_0) = \{D\Psi(x_0) : \ \Psi \in C^1(\mathbb{R}^n) \text{ such that } U - \Psi \text{ has a local maximum at } x_0\}.$$

The *subdifferential* $D^-U(x_1)$ of values of $D\Psi(x_1)$ occurring in (2.8) is defined analogously. We note however that continuity of U is *not* assumed in the above definition. This is the general definition of viscosity solution for functions U that might be discontinuous.

We shall need the following theorem in the exposition below. The proof is similar to that of Theorem 3.6 in [13], and hence will be omitted. For this result we need a strengthened local version of assumption (A3):

(A3′) For each $x_0 \in \mathbb{R}^n$ and $\epsilon > 0$, there is a constant $K > 0$ so that $|f(x,b) - f(y,b)| \leq K|x-y|$ for all $x, y \in B(x_0, \epsilon)$.

Theorem 2.3 *Let Ω be an open subset of $\mathbb{R}^n$ and $U \in C(\Omega)$. Assume (A0), (A1), (A2) and (A3′). If $H(x,p) \geq 0$ for all $x \in \Omega$ and all $p \in D^-U(x)$, then*

$$U(y_x(\nu, b)) - U(y_x(t,b)) \geq \int_\nu^t l(y_x(s), b(s))\, ds,$$

for all $b \in \mathcal{B}$, $x \in \Omega$, $0 \leq \nu \leq t < \tau_x[b]$, where $\tau_x[b] := \inf\{t \geq 0 : y_x(t,b) \notin \Omega\}$.

Remark 2.4 The suggested proof of Theorem 2.3 from [13, Theorem 3.6] uses additional geometric notions and results from nonsmooth analysis (other than that of subdifferential $D^-U(x)$ introduced here), such as the contingent epiderivative $D_\uparrow U(x)$ of U at x, the contingent tangent cone $T_{\mathrm{Epi}(U)}(x, U(x))$ to the epigraph $\mathrm{Epi}(U)(x, U(x))$ of U at $(x, U(x))$, the equality $\mathrm{Epi}(D_\uparrow U(x)) = T_{Epi(U)}(x, U(x))$ (see Chapter 9 of [2]), and application of a Viability Theorem from [2]. Under the stronger assumption that l is bounded on $\Omega \times B$, the result of Theorem 2.3 follows via a purely test-function approach using only ideas related to viscosity supersolutions as in [7, page 92].

3 Viscosity solutions of variational inequalities

In this section we derive the main results concerning the robust stopping-time problem stated in the Introduction.

3.1 The lower value function for Game I

Some simple inequalities are obvious from the definition of W. By using $T = 0$ (and $\Phi \geq 0$) in the definition of W, we see that

$$W(x) \geq 0. \tag{3.1}$$

Using $\tau \equiv 0$ gives

$$W(x) \leq \Phi(x). \tag{3.2}$$

On the other hand, $\tau \equiv +\infty$ gives

$$W(x) \leq S_a(x), \tag{3.3}$$

where S_a is the available storage function given by (1.4).

Proposition 3.1 *Assume (A0)-(A5). Then for $x \in \mathbb{R}^n$ and $t > 0$*

$$W(x) \leq \sup_{b\in\mathcal{B},\, T>0} \left\{ \int_0^{T\wedge t} l(y_x(s), b(s))\, ds + 1_{[0,T)}(t) W(y_x(t,b)) \right\} \tag{3.4}$$

$$\leq \sup_{b\in\mathcal{B}} \left\{ \int_0^{t} l(y_x(s), b(s))\, ds + W(y_x(t,b)) \right\} \tag{3.5}$$

Proof We begin with (3.4). By the definition of W for each $\epsilon > 0$ there is a stopping-time rule τ^ϵ so that

$$W(z) + \epsilon > \int_0^{T\wedge\tau_z^\epsilon[b]} l(y_z(s), b(s))\, ds + \ 1_{[0,T]}(\tau_z^\epsilon[b])\Phi(y_z(\tau_z^\epsilon[b])) \tag{3.6}$$

for all $z \in \mathbb{R}^n$, $T > 0$ and $b \in \mathcal{B}$. Fix $t > 0$. For each $b \in \mathcal{B}$, we define

$$b_t(s) = b(s+t),\ s \geq 0.$$

For $\tau \in \Gamma$ define $\bar\tau :\ \mathbb{R}^n \times \mathcal{B} \to \mathbb{R}^+ \cup \infty$ by

$$\bar\tau_x[b] = t + \tau_{y_x(t,b)}[b_t].$$

It is easy to check that $\bar\tau$ has the nonanticipating property whenever τ does, and hence $\bar\tau \in \Gamma$ for $\tau \in \Gamma$.

Fix $x \in \mathbb{R}^n$. By the definition of W, for each $\epsilon > 0$ and $\tau \in \Gamma$ we may choose $T_{\tau,\epsilon} > 0$ and $b_{\tau,\epsilon} \in \mathcal{B}$ so that

$$W(x) - \epsilon \leq \int_0^{T_{\tau,\epsilon}\wedge\tau_x[b_{\tau,\epsilon}]} l(y_x(s), b_{\tau,\epsilon}(s))ds + 1_{[0,T]}(\tau_x[b])\Phi(y_x(\tau_x[b_{\tau,\epsilon}])).$$

We may specialize this general inequality to the case where τ is of the form $\bar{\tau}$ for some $\tau \in \Gamma$. In this case $\bar{\tau}_x[b_{\bar{\tau},\epsilon}] \geq t$ and we obtain

$$\begin{array}{l} W(x) - \epsilon \leq \int_0^{T_{\bar{\tau},\epsilon}\wedge t} l(y_x(s), b_{\bar{\tau},\epsilon}(s))\ ds \\ \quad +1_{[0,T_{\bar{\tau},\epsilon})}(t) \left\{ \int_t^{T_{\bar{\tau},\epsilon}\wedge \bar{\tau}_x[b_{\bar{\tau},\epsilon}]} l(y_x(s), b_{\bar{\tau},\epsilon}(s))\ ds + 1_{[t,T_{\bar{\tau},\epsilon}]}(\bar{\tau}_x[b_{\bar{\tau},\epsilon}])\Phi(y_x(\bar{\tau}_x[b_{\bar{\tau},\epsilon}])) \right\} \end{array} \tag{3.7}$$

For $t < T_{\bar{\tau},\epsilon}$, by the change of variable $\nu = s - t$ we have

$$\begin{array}{l} \int_t^{T_{\bar{\tau},\epsilon}\wedge\bar{\tau}_x[b_{\bar{\tau},\epsilon}]} l(y_x(s), b_{\bar{\tau},\epsilon}(s))\ ds + 1_{[t,T_{\bar{\tau},\epsilon}]}(\bar{\tau}_x[b_{\bar{\tau},\epsilon}])\Phi(y_x(\bar{\tau}_x[b_{\bar{\tau},\epsilon}])) \\ \quad = \int_0^{(T_{\bar{\tau},\epsilon}-t)\wedge\tau_{y_x(t,b_{\bar{\tau},\epsilon})}[(b_{\bar{\tau},\epsilon})_t]} l(y_{y_x(t,b_{\bar{\tau},\epsilon})}(\nu), (b_{\bar{\tau},\epsilon})_t(\nu))\ d\nu \\ \qquad +1_{[0,T_{\bar{\tau},\epsilon}-t]}(\tau_{y_x(t,b_{\bar{\tau},\epsilon})}[(b_{\bar{\tau},\epsilon})_t])\Phi(y_{y_x(t,b_{\bar{\tau},\epsilon})}(\tau_{y_x(t,b_{\bar{\tau},\epsilon})}[(b_{\bar{\tau},\epsilon})_t]))\} \end{array} \tag{3.8}$$

Apply (3.6) to the case

$$z = y_x(t, b_{\bar{\tau},\epsilon}), \quad b = (b_{\bar{\tau},\epsilon})_t, \quad T = T_{\bar{\tau},\epsilon} - t.$$

Then (3.6) implies that the right hand side of (3.8) (with τ selected to be the τ^ϵ as in (3.6)) is bounded above by $W(y_x(t, b_{\overline{\tau^\epsilon},\epsilon})) + \epsilon$. From (3.7) we finally conclude that

$$\begin{array}{rcl} W(x) - 2\epsilon & < & \int_0^{T_{\overline{\tau^\epsilon},\epsilon}\wedge t} l(y_x(s), b_{\overline{\tau^\epsilon},\epsilon}(s))\ ds + 1_{[0,T_{\overline{\tau^\epsilon},\epsilon}]}(t)W(y_x(t, b_{\overline{\tau^\epsilon},\epsilon})) \\ & \leq & \sup_{b\in\mathcal{B},\ T>0} \left\{ \int_0^{T\wedge t} l(y_x(s), b(s))\ ds + 1_{[0,T)}(t)W(y_x(t, b)) \right\}. \end{array}$$

Since $\epsilon > 0$ is arbitrary, the result follows.

To establsih the second inequality of the proposition, consider any $b \in \mathcal{B}$ and $0 < T < t$. Define a new $\bar{b} \in \mathcal{B}$ by $\bar{b}(s) = b(s)$ for $s \leq T$ and $\bar{b}(s) = 0$ for $s > T$. It follows that $y_x(s, b(s)) = y_x(s, \bar{b}(s))$ for $0 \leq s \leq T$. For $s > T$ we have

$$l(y_z(s, \bar{b}), \bar{b}(s)) = h(y_z(s, \bar{b}), 0) \geq 0.$$

Together with (3.1) it follows that

$$\int_0^{T\wedge t} l(y_x(s), b(s))\ ds + 1_{[0,T)}(t)W(y_x(t, b)) \leq \int_0^t l(y_x(s), \bar{b}(s))\ ds + W(y_x(t, \bar{b})).$$

This implies (3.5) and completes the proof. $\diamondsuit$

Proposition 3.2 *Assume (A0)-(A4). If $W(x) < \Phi(x)$, then for each $b \in \mathcal{B}$ there exists $\rho = \rho_{x,b} > 0$ such that for all $t \in [0, \rho)$,*

$$W(x) \geq \int_0^t l(y_x(s), b(s))\ ds + W(y_x(t, b)).$$

Proof Fix $x \in \mathbb{R}^n$. By definition of $W(x)$, for each $\epsilon > 0$ there is a choice of strategy $\tau^\epsilon \in \Gamma$ so that

$$W(x) + \epsilon > \int_0^{T \wedge \tau_x^\epsilon[\widetilde{b}]} l(y_x(s), \widetilde{b}(s))\, ds + 1_{[0,t]}(\tau_x^\epsilon[\widetilde{b}])\Phi(y_x(\tau_x^\epsilon[\widetilde{b}])) \tag{3.9}$$

for all $T \geq 0$ and all $\widetilde{b} \in \mathcal{B}$.

We claim that for each $b \in \mathcal{B}$ there is a number $\rho_b > 0$ so that $\tau_x^\epsilon[b] > \rho_b$ for all sufficiently small $\epsilon > 0$. If not, then there is a $b \in \mathcal{B}$ and a sequence of positive numbers $\{\epsilon_n\}$ with limit equal to 0 and with $\tau_x^{\epsilon_n}[b]$ tending to 0. Apply (3.9) with ϵ_n replacing ϵ and with b in place of $\widetilde{b}$, use the continuity of Φ and of $y_x(s) = y_x(s, b)$ along with the assumption that b is locally square-integrable to take the limit in (3.9) as $n \to \infty$ to arrive at $W(x) \geq \Phi(x)$, contrary to assumption. We conclude that for each $b \in \mathcal{B}$ there is a $\rho_b > 0$ so that $\tau_x^\epsilon[b] \geq \rho_b$ for all $\epsilon > 0$ as asserted.

Fix $b \in \mathcal{B}$ and choose any $t \in [0, \rho_b)$. By definition of $W(y_x(t, b))$, for any $\tau \in \Gamma$ and for any $\epsilon > 0$ there is a choice of $T_{\tau,\epsilon} \geq 0$ and of $b_{\tau,\epsilon} \in \mathcal{B}$ so that

$$\begin{aligned} W(y_x(t,b)) - \epsilon \quad \leq \quad & \textstyle\int_0^{T_{\tau,\epsilon} \wedge \tau_{y_x(t,b)}[b_{\tau,\epsilon}]} l(y_{y_x(t,b)}(s), b_{\tau,\epsilon}(s))\, ds \\ & + 1_{[0,T_{\tau,\epsilon}]}(\tau_{y_x(t,b)}[b_{\tau,\epsilon}])\Phi(y_{y_x(t,b)}(\tau_{y_x(t,b)}[b_{\tau,\epsilon}])). \end{aligned} \tag{3.10}$$

In particular, (3.10) holds for all $\tau \in \Gamma$ for which $\tau_x[b] \geq t$. For any $\widehat{b} \in \mathcal{B}$, define $\widehat{b}' \in \mathcal{B}$ by

$$\widehat{b}'(s) = \begin{cases} b(s), & \text{for } 0 \leq s \leq t \\ \widehat{b}(s - t), & \text{for } s > t. \end{cases}$$

For any $\tau \in \Gamma$ with $\tau_x[b] > t$, we may always find another $\overline{\tau} \in \Gamma$ so that

$$\overline{\tau}_x[\widehat{b}'] = \tau_{y_x(t,b)}[\widehat{b}] + t \text{ for all } \widehat{b} \in \mathcal{B}.$$

From (3.10) we then get that, for any $\tau \in \Gamma$ with $\tau_x[b] \geq t$ and any $\epsilon > 0$,

$$\begin{aligned} & \textstyle\int_0^t l(y_x(s), b(s))\, ds + W(y_x(t,b)) - \epsilon \\ & \leq \textstyle\int_0^t l(y_x(s), b(s))\, ds + \int_0^{T_{\tau,\epsilon} \wedge \tau_{y_x(t,b)}[b_{\tau,\epsilon}]} l(y_{y_x(t,b)}(s), b_{\tau,\epsilon}(s))\, ds \\ & \qquad + 1_{[0,T_{\tau,\epsilon}]}(\tau_{y_x(t,b)}[b_{\tau,\epsilon}])\Phi(y_{y_x(t,b)}(\tau_{y_x(t,b)}[b_{\tau,\epsilon}])) \\ & \leq \textstyle\int_0^t l(y_x(s), b(s))\, ds + \int_t^{(T_{\tau,\epsilon}+t) \wedge (\tau_{y_x(t,b)}[b_{\tau,\epsilon}]+t)} l(y_x(s), b'_{\tau,\epsilon}(s))\, ds \\ & \qquad + 1_{[t,T_{\tau,\epsilon}]}(\tau_{y_x(t,b)}[b_{\tau,\epsilon}])\Phi(y_x(\tau_{y_x(t,b)}[b_{\tau,\epsilon}]) \\ & = \textstyle\int_0^{(T_{\tau,\epsilon}+t) \wedge \overline{\tau}_x[b'_{\tau,\epsilon}]} l(y_x(s), b'_{\tau,\epsilon}(s))\, ds + 1_{[0,T_{\tau,\epsilon}+t]}(\overline{\tau}_x[b'_{\tau,\epsilon}])\Phi(y_x(\overline{\tau}_x[b'_{\tau,\epsilon}]). \end{aligned} \tag{3.11}$$

It is important to note that *for any* $\widetilde{\tau} \in \Gamma$ *with* $\widetilde{\tau}_x[b] > t$, *there is a* $\widetilde{\tau}'$ *in* Γ *with* $\widetilde{\tau} = \overline{\widetilde{\tau}'}$. To see this for a given $\widetilde{\tau} \in \Gamma$ we must find a $\widetilde{\tau}' \in \Gamma$ so that

$$\overline{\widetilde{\tau}'_x}[\widehat{b}'] = \begin{cases} \widetilde{\tau}_x[\widehat{b}'] & \text{on the one hand,} \\ \widetilde{\tau}'_{y_x(t,b)}[\widetilde{b}] + t & \text{on the other hand,} \end{cases}$$

or

$$\widetilde{\tau}_x[\widehat{b}'] = \widetilde{\tau}'_{y_x(t,b)}[\widehat{b}] + t$$

for all $\widehat{b} \in \mathcal{B}$. It is always possible to solve for such a $\widetilde{\tau}'$ due to the nonanticipating property of $\widetilde{\tau}$. We apply this observation in particular to the case $\widetilde{\tau} = \tau^\epsilon$ where τ^ϵ is as in (3.9); thus, for each $\epsilon > 0$ there is a $\tau^{\epsilon\prime} \in \Gamma$ so that $\overline{\tau^{\epsilon\prime}} = \tau^\epsilon$.

If we now specialize (3.9) to the case where

$$T = T_{\tau^{\epsilon\prime},\epsilon} + t, \quad \widetilde{b} = b'_{\tau^{\epsilon\prime},\epsilon},$$

we can continue the estimate in (3.11) (with the general τ replaced with $\tau^{\epsilon\prime}$) to get

$$\begin{aligned}
&\textstyle\int_0^t l(y_x(s), b(s))\, ds + W(y_x(t,b)) - \epsilon \\
&\quad \leq \textstyle\int_0^{(T_{\tau^{\epsilon\prime},\epsilon}+t)\wedge\tau^\epsilon_x[b'_{\tau^{\epsilon\prime},\epsilon}]} l(y_x(s), b'_{\tau^{\epsilon\prime},\epsilon}(s))\, ds + 1_{[0,T_{\tau^{\epsilon\prime},\epsilon}+t]}(\tau^\epsilon_x[b'_{\tau^{\epsilon\prime},\epsilon}])\Phi(y_x(\tau^\epsilon_x[b'_{\tau^{\epsilon\prime},\epsilon}])) \\
&\quad < W(x) + \epsilon
\end{aligned}$$

where we used (3.9) for the last step. Since $\epsilon > 0$ is arbitrary, the result follows. $\diamondsuit$

Theorem 3.3 *Assume (A0)-(A5). If W is upper semicontinuous, then W is a viscosity subsolution of the VI in $\mathbb{R}^n$.*

Proof Since $W(x) - \Phi(x) \leq 0$, it is enough to show that W is a viscosity subsolution of $H(x, DW(x)) = 0$. It is generally true that viscosity sub- and super-solutions of a Hamilton-Jacobi equation are (respectively) equivalent to sub- and super-optimality principles in integrated form. Theorem 2.32 in Chapter III of [7] is a nice presentation of such a result. Our (3.5) is the optimality principle naturally associated with viscosity subsolutions of $H(x, DW(x)) = 0$. To apply the theorem of Bardi and Capuzzo-Dolcetta to our $W(x)$, define $u(x) = -W(x)$, $\ell(x,b) = -l(x,b)$, and take $\lambda = 0$. With our $b \in \mathcal{B}$ replacing $\alpha \in \mathcal{A}$, their superoptimality principle is equivalent to our (3.5), and u being a supersolution of their (2.40) is equivalent to W being a subsolution of our $H(x, DW(x)) = 0$, as the reader may check. However we are only assuming that W is upper semicontinuous, which means u is only lower semicontinuous. The stated hypothesis in [7] is that u is continuous. Nevertheless, examining the relevant portion of the proof (which appears on page 105 of [7]), one finds that lower semicontinuity of u is enough for the part of the argument required for our proposition above. Lastly, they assume $\ell(x, \alpha)$ to be bounded, but we want to use $\ell(x,\alpha) = \gamma^2|\alpha|^2 - h(x,\alpha)$, which is unbounded due to the $\gamma^2|\alpha|^2$ term. This does not effect their (2.13), since the common $\gamma^2|\alpha|^2$ terms cancel, and our hypotheses on h provide the needed estimate. Finally the ℓ term in the second integral of their (2.17) vanishes because $\lambda = 0$ for us. Thus their proof remains valid in our context. $\diamondsuit$

Theorem 3.4 *Assume (A0)-(A4). If W is lower semicontinuous, then W is a viscosity supersolution of the VI in $\mathbb{R}^n$.*

Proof Fix $x \in \mathbb{R}^n$. Let $\Psi \in C^1(\mathbb{R}^n)$ be such that x is a local minimum point of $W_* - \Psi$. Since W is lower semicontinuous, we have

$$\Psi(x) - \Psi(z) \geq W_*(x) - W_*(z) = W(x) - W(z), \tag{3.12}$$

for all z in a neighborhood of x. We want to show that

$$\max\{H(x, D\Psi(x)),\ W(x) - \Phi(x)\} \geq 0.$$

If $W(x) = \Phi(x)$, the assertion is trivial. Suppose $W(x) < \Phi(x)$. We want to show that $H(x, D\Psi(x)) \geq 0$. By Proposition 3.2, for each $b \in \mathcal{B}$ choose $t_b > 0$ such that

$$W(x) \geq \int_0^t l(y_x(s), b(s))ds + W(y_x(t, b)),\ \forall t \in [0, t_b) \tag{3.13}$$

Fix an arbitrary $b \in B$ and let $y_x(s)$ be the solution corresponding to the constant disturbance $b(s) = b$ for all s. Under our assumptions on f, there exists $t_1 \in (0, t_b)$ such that $y_x(s)$ is in the neighborhood of x for which (3.12) holds for all $0 < s \leq t_1$. From (3.12) and (3.13), we have

$$\frac{1}{t}[\Psi(x) - \Psi(y_x(t))] \geq \frac{1}{t}\int_0^t l(y_x(s), b)ds,\ \forall t \in (0, t_1).$$

Let $t \to 0$, we have

$$-D\Psi(x) \cdot f(x, b) - h(x, b) + \gamma^2|b| \geq 0.$$

Since $b \in B$ is arbitrary, it follows that

$$H(x, D\Psi(x)) = \inf_{b \in B}\{-D\Psi(x) \cdot f(x, b) - h(x, b) + \gamma^2|b|\} \geq 0. \diamondsuit$$

Corollary 3.5 *Assume (A0)-(A5). If W is continuous, then W is a viscosity solution of the VI in $\mathbb{R}^n$.*

Remark 3.6 Let S_a be the available storage function for the disturbance-to-cost system with stopping options ignored as in (1.4). Since S_a is a viscosity solution of (HJE) in $\mathbb{R}^n$, we have

$$H(x, D^-(S_a)_*(x)) \geq 0 \text{ and } H(x, D^+(S_a)^*(x)) \leq 0,\ x \in \mathbb{R}^n.$$

Thus S_a is a viscosity supersolution of the VI. Moreover if $S_a \leq \Phi$, then S_a is a viscosity solution of the VI.

3.2 The stopping-time storage function

In this section, we collect our results concerning stopping-time storage functions defined as in (1.5).

Theorem 3.7 *Assume (A0)-(A4). Then a locally bounded stopping-time storage function is a viscosity supersolution of the VI.*

Proof Suppose that U is a stopping-time storage function with stopping-time rule τ_U, i.e.

$$\begin{aligned}&\int_0^{T\wedge\tau_{U,x}[b]} h(y_x(s),b(s))\ ds + 1_{[0,T]}(\tau_{U,x}[b])\Phi(y_x(\tau_{U,x}[b],b) + 1_{(T,+\infty]}(\tau_{U,x}[b])U(y_x(T,b))\\ &\leq \gamma^2\int_0^{T\wedge\tau_{U,x}[b]} |b(s)|^2\ ds + U(x) \text{ for all } x\in\mathbb{R}^n,\ b\in\mathcal{B} \text{ and } T\geq 0.\end{aligned} \tag{3.14}$$

Fix $x\in\mathbb{R}^n$. Let $\Psi\in C^1(\mathbb{R}^n)$ be such that x is a local minimum point of $U-\Psi$. We want to show that

$$H(x,D\Psi(x))\geq 0 \quad \text{or} \quad U(x)-\Phi(x)\geq 0.$$

If $U(x)\geq\Phi(x)$, the result is obvious. It remains to show that $H(x,D\Psi(x))\geq 0$ when $U(x)<\Phi(x)$.

Fix an arbitrary $b\in B$. Set $b(s)=b$ for all $s\geq 0$. Choose $x_k\in\mathbb{R}^n$ with $\lim_{k\to\infty}x_k=x$ so that $\lim_{k\to\infty}U(x_k)=U_*(x)$. We claim: *there is a $\delta>0$ so that $\tau_{U,x_k}[b]>\delta$ for all k sufficiently large.* If the claim were not true, dropping down to a subsequence if necessary, we would have $\lim_{k\to\infty}t_k=0$ where $t_k:=\tau_{U,x_k}[b]$. From (3.14) applied with a fixed T sufficiently large, we then would have

$$U(x_k)\geq\int_0^{t_k} l(y_{x_k}(s),b)\ ds+\Phi(y_{x_k}(t_k,b)). \tag{3.15}$$

From the estimates (2.3)–(2.5) together with assumption (A3) and the assumed continuity of Φ, we see that $\lim_{k\to\infty}\Phi(y_{x_k}(t_k,b))=\Phi(x)$ and that $l(y_{x_k}(s),b)$ tends uniformly to $l(y_x(s),b)$ in s on the interval $[0,\delta]$. Hence we can take limits in (3.15) to get

$$U_*(x)\geq\Phi(x).$$

As $U(x)\geq U_*(x)$ by definition of U_*, this contradicts our assumption that $U(x)<\Phi(x)$, and the claim follows.

Hence, for $0<t<\delta$, we may apply (3.14), this time with $T=t$, to get, for each k sufficiently large,

$$U(x_k)\geq\int_0^t l(y_{x_k}(s),b)\ ds+U(y_{x_k}(t,b)).$$

Letting k tend to infinity and again using that $l(y_{x_k}(s), b)$ tends uniformly in s to $l(y_{x_k}(s), b)$ on $[0, t]$ and that $y_{x_k}(t, b)$ tends to $y_x(t, b)$ lead to

$$U_*(x) \geq \int_0^t l(y_x(s), b)\ ds + \liminf_{k\to\infty} U(y_{x_k}(t, b)) \tag{3.16}$$

$$\geq \int_0^t l(y_x(s), b)\ ds + U_*(y_x(t, b)). \tag{3.17}$$

We now can follow the standard procedure as in the proof of Theorem 3.4 to see that $H(x, D\Psi(x)) \geq 0$ as desired. $\diamondsuit$

Remark 3.8 The proof of Theorem 3.7 is adapted from the proof of Proposition 3.2 in [13], where it is shown that, under certain conditions, the lower and upper semicontinuous envelopes of a storage function is again a storage function for the classical case (with no stopping options allowed).

Theorem 3.9 *Let* $U : \mathbb{R}^n \to \mathbb{R}$ *be a nonnegative continuous function and set*

$$\Omega_U := \{x \in \mathbb{R}^n : U(x) < \Phi(x)\}. \tag{3.18}$$

Assume (A0)-(A4) and (A3′). If U *is viscosity supersolution of VI in* $\mathbb{R}^n$ *and the stopping-time rule is given by*

$$\tau_{U,x} := \inf\{t : t \geq 0 \text{ and } y_x(t, b) \notin \Omega_U\}, \tag{3.19}$$

then U *is a stopping-time storage function with stopping-time rule* τ_U *of state-feedback form (see (2.6)), and* $U \geq W$. *Thus if* W *is continuous, then* W *is characterized as the minimal nonnegative continuous viscosity supersolution of VI, as well as the minimal possible continuous closed-loop storage function over all possible stopping-time rules* $\tau \in \Gamma$.

Proof Since $y_x(t + s, b) = y_{y_x(t,b)}(s, b_t)$ (where $b_t(s) = b(t + s)$), it is clear that τ_U as defined in (3.19) satisfies the state-feedback property (2.6).

Suppose that $x \in \mathbb{R}^n \setminus \Omega_U$. Then

$$\begin{aligned} U(x) &\geq \Phi(x) \text{ (by definition of } \Omega_U) \\ &\geq W(x) \text{ (by (3.2))} \end{aligned}$$

and $\tau_{U,x} = 0$ (by definition (3.19) of $\tau_{U,x}$). The condition for U to be a stopping time storage function ((1.5) with U in place of S) collapses to $-U(x) \leq -\Phi(x)$ in case $\tau_{U,x} = 0$, and hence is verified in case $x \in \mathbb{R}^n \setminus \Omega_U$.

Suppose now that $\Omega_U \neq \emptyset$ and let $x \in \Omega_U$. Since U and Φ are continuous, Ω_U is open; hence $\tau_{U,x}[b] > 0$ for all $b \in \mathcal{B}$. By Definition 2.2, $H(x,p) \geq 0$ for all $p \in D^{-}U(x)$. Hence by Theorem 2.3, we have

$$\begin{array}{l} U(y_x(t_1,b)) \geq \int_{t_1}^{t_2}[h(y_x(s),b(s)) - \gamma^2|b(s)|^2]ds + U(y_x(t_2,b)) \\ \text{for all } b \in \mathcal{B}, x \in \Omega_U, 0 \leq t_1 \leq t_2 < \tau_{U,x}[b]. \end{array}$$

Let $T \geq 0$. Take $t_1 = 0$ and replace t_2 by $T \wedge t_2$ to get

$$U(x) \geq \int_0^{T\wedge t_2} l(y_x(s),b(s))ds + U(y_x(T\wedge t_2,b)),\ \forall t_2 \in [0,\tau_{U,x}[b]),\ \forall b \in \mathcal{B}.$$

Letting $t_2 \to \tau_{U,x}[b]$, by continuity of U we get

$$U(x) \geq \int_0^{T\wedge\tau_{U,x}[b]} l(y_x(s),b(s))ds + U(y_x(T\wedge\tau_{U,x}[b])),\ \forall b \in \mathcal{B}.$$

Since $y_x(\tau_{U,x}[b]) \in \partial\Omega_U$ if $\tau_{U,x}[b] < \infty$, we have

$$\begin{array}{rcl} U(y_x(T\wedge\tau_{U,x}[b])) & = & \left\{ \begin{array}{ll} U(y_x(\tau_{U,x}[b])) & \text{for } 0 \leq \tau_{U,x}[b] \leq T, \\ U(y_x(T)) & \text{for } \tau_{U,x}[b] > T \end{array} \right. \\ & = & 1_{[0,T]}(\tau_{U,x}[b])U(y_x(\tau_{U,x}[b])) + 1_{(T,+\infty]}(\tau_{U,x}[b])U(y_x(T)) \\ & = & 1_{[0,T]}(\tau_{U,x}[b])\Phi(y_x(\tau_{U,x}[b])) + 1_{(T,+\infty]}(\tau_{U,x}[b])U(y_x(T)). \end{array}$$

Thus

$$\begin{aligned} U(x) \geq \int_0^{T\wedge\tau_{U,x}[b]} & l(y_x(s),b(s))ds + 1_{(T,+\infty]}(\tau_{U,x}[b])U(y_x(T)) \\ & + 1_{[0,T]}(\tau_{U,x}[b])\Phi(y_x(\tau_{U,x}[b])). \end{aligned} \tag{3.20}$$

This inequality verifies (1.5) (with U in place of S) for the case $x \in \Omega_U$. We conclude that U is a stoping-time storage function with stopping rule τ_U as asserted.

Since the inequality (3.20) holds for all $b \in \mathcal{B}$ and all $T \geq 0$ and U is nonnegative, we have

$$\begin{array}{rcl} U(x) & \geq & \sup_{b\in\mathcal{B},\,T\geq 0}\{\int_0^{T\wedge\tau_{U,x}[b]} l(y_x(s),b(s))\ ds + 1_{[0,T]}(\tau_{U,x}[b])\Phi(y_x(\tau_{U,x}[b]))\} \\ & \geq & \inf_{\tau\in\Gamma}\sup_{b\in\mathcal{B},\,T\geq 0}\{\int_0^{T\wedge\tau_x[b]} l(y_x(s),b(s))ds + 1_{[0,T]}(\tau_x[b])\Phi(y_x(\tau_x[b]))\} \\ & = & W(x). \end{array}$$

We conclude that W, if continuous, is characterized as the minimal nonnegative continuous viscosity supersolution of the VI, as asserted.

Finally, if W is continuous, from Corollary 3.5 we see that W is a viscosity solution of the VI, and hence in particular a viscosity supersolution. The first part of Theorem 3.9 already proved then implies that W is a stopping-time storage function with stopping-time rule τ_W. Moreover, if S is any continuous, stopping-time storage function for some stopping-rule τ, from Theorem 3.7 we see that S is a viscosity supersolution of the VI. Again from the first part of Theorem 3.9 already proved, we then see that $S \geq W$, and hence W is also the minimal, continuous stopping-time storage function, as asserted. $\diamondsuit$

Remark 3.10 The proof of Theorem 3.9 shows that *if U is a stopping-time storage function for some stopping-rule τ, then it is also a stopping-time storage function for the stopping-rule τ_U given by (3.19).* When the stopping rule τ_U is used, then one can easily check that the function U enjoys the following subordination property with respect to Φ along trajectories of the system:

$$\begin{array}{l} U(x) < \Phi(x) \Longrightarrow U(y_x(t,b)) < \Phi(y_x(t,b)) \textit{ for } 0 \leq t < \tau_{U,x}[b]; \\ U(x) \geq \Phi(x) \Longrightarrow \tau_{U,x}[b] = 0 \textit{ for all } b \in \mathcal{B}. \end{array}$$

3.3 The lower value function for Game II

Now we will show some inequalities satisfied by the lower value function $V(x)$ for Game II (see (1.3)), and use these inequalities to show that V is a viscosity solution of the VI in $\mathbb{R}^n$. For convenience, set

$$\begin{aligned} J(x,t,b) &:= \Phi(y_x(t)) + \int_0^t l(y_x(s),b(s))\,ds \\ \Delta &:= \{\nu : \mathbb{R}^n \times \mathcal{B} \to [0,\infty) : \nu \text{ is nonanticipating}\}, \end{aligned}$$

where $l(x,b) = h(x,b) - \gamma^2|b|^2$. Thus $V(x) = \inf_{\nu\in\Delta}\sup_{b\in\mathcal{B}} J(x,\nu_x[b],b)$.

Proposition 3.11 *Assume (A0)-(A4).*

(i) *Then $V \leq \Phi$. If (A5) also holds, then $0 \leq V \leq \Phi$.*

(ii) *Then*

$$V(x) \leq \sup_{b\in\mathcal{B}}\{\int_0^t l(y_x(s),b(s))\,ds + V(y_x(t))\}, \quad \textit{for all } t \geq 0.$$

(iii) *If $V(x) < \Phi(x)$, then for each $b \in \mathcal{B}$ there exists $t_b > 0$ such that*

$$V(x) \geq \int_0^t l(y_x(s),b(s))\,ds + V(y_x(t,b)),\ \forall t \in [0,t_b).$$

Proof **(i)** Using $\nu \equiv 0$ we have

$$V(x) \leq \Phi(x),$$

while using $b \equiv 0$ we have

$$\begin{array}{rcl} V(x) & \geq & \inf_{\nu \in \Delta} \{ \int_0^{\nu[0]} h(y_x(s), 0) ds + \Phi(y_x(\nu[0])) \} \\ & \geq & 0, \end{array}$$

since h is nonnegative by (A5) and Φ are positive by assumption.

(ii) Fix $x \in \mathbb{R}^n$. For each $\nu \in \Delta$ and $\epsilon > 0$ we may choose $b_{\nu,\epsilon} \in \mathcal{B}$ so that

$$V(x) < J(x, \nu_x[b_{\nu,\epsilon}], b_{\nu,\epsilon}) + \epsilon. \tag{3.21}$$

Fix $t > 0$. For each $b \in \mathcal{B}$, define b_t by

$$b_t(s) = b(s + t), \quad s \geq 0$$

Notice that $y_x(t, b) = y_{y_x(t,b)}(0, b_t)$.

On the other hand, we may choose $\nu^\epsilon \in \Delta$ so that

$$V(z) + \epsilon > J(z, \nu^\epsilon_z[b], b) \tag{3.22}$$

for all $z \in \mathbb{R}^n$ and $b \in \mathcal{B}$. For $\nu \in \Delta$ define $\bar{\nu} : \mathbb{R}^n \times \mathcal{B} \to \mathbb{R}^+$ by

$$\bar{\nu}[b] = t + \nu_{y_x(t,b)}[b_t].$$

One can easily check that $\bar{\nu} \in \Delta$ for each $\nu \in \Delta$. From the definition of J, we have

$$\begin{array}{rcl} J(x, \bar{\nu}_x[b], b) & = & \Phi(y_x(\bar{\nu}[b])) + \int_0^{\bar{\nu}[b]} l(y_x(s), b(s)) \; ds \\ & = & \Phi(y_x(\nu_{y_x(t,b)}[b_t] + t)) + \int_0^t l(y_x(s), b(s)) \; ds \\ & & \quad + \int_t^{\nu_{y_x(t,b)}[b_t]+t} l(y_x(s), b(s)) \; ds \\ & = & \int_0^t l(y_x(s), b(s)) \; ds + \Phi(y_{y_x(t,b)}(\nu_{y_x(t,b)}[b_t])) \\ & & + \int_0^{\nu_{y_x(t,b)}[b_t]} l(y_{y_x(t)}(\alpha), b_t(\alpha)) \; d\alpha \\ & = & \int_0^t l(y_x(s), b(s)) \; ds + J(y_x(t,b), \nu_{y_x(t,b)}[b_t], b_t) \end{array} \tag{3.23}$$

If we specialize (3.23) to the case $\nu = \nu^\epsilon$ (where ν^ϵ is as in (3.22)) and specialize (3.22) to the case $z = y_x(t, b)$ and b of the form b_t, then (3.22) provides the estimate on (3.23)

$$J(x, \bar{\nu}_x[b], b) \leq \int_0^t l(y_x(s), b(s)) \; ds + V(y_x(t, b)) + \epsilon \tag{3.24}$$

for all $b \in \mathcal{B}$. If we specialize (3.24) to the case where $b = b_{\bar{\nu},\epsilon}$ and apply (3.21) for the case where ν is of the form $\bar{\nu}$, then (3.21) leads to

$$\begin{aligned} V(x) &< J(x, \bar{\nu}_x[b_{\bar{\nu},\epsilon}], b_{\bar{\nu},\epsilon}) + \epsilon \\ &\leq \int_0^t l(y_x(s), b_{\bar{\nu},\epsilon}(s))\, ds + V(y_x(t, b_{\bar{\nu},\epsilon})) + 2\epsilon \\ &\leq \sup_{b\in\mathcal{B}} \left\{ \int_0^t l(y_x(s), b(s))\, ds + V(y_x(t,b)) \right\} + 2\epsilon \end{aligned}$$

Since ϵ is arbitrary, the result follows.

(iii) The proof of statement (iii) is similar to the proof of Proposition 3.2. $\diamondsuit$

Theorem 3.12 *Assume (A0)-(A4). If upper semicontinuous, V is a viscosity subsolution of the VI. If lower semicontinuous, V is a viscosity supersolution of the VI. Thus if continuous, V is a viscosity solution of the VI.*

Proof First assume that V is upper semicontinuous. We want to show that V is a viscosity subsolution of VI. Since V is upper semicontinuous by assumption, $V^* = V$. Fix $x \in \mathbb{R}^n$. Let $\Psi \in C^1(\mathbb{R}^n)$ and x is a local maximum of $V - \Psi$. We want to show that

$$\max\{H(x, D\Psi(x)),\ V(x) - \Phi(x)\} \leq 0 \tag{3.25}$$

From (i) of Proposition 3.11, $V(x) \leq \Phi(x)$. Thus we want to show that $H(x, D\Psi(x)) \leq 0$. We proceed by contradiction. Suppose that

$$H(x, D\Psi(x)) > \delta > 0.$$

By the definition of H, we therefore have

$$-D\Psi(x) \cdot f(x,b) - h(x,b) + \gamma^2 |b|^2 > \delta,\ \forall b \in B. \tag{3.26}$$

Choose R so that $x \in B(0,R)$ and suppose that z is another point in $B(0,R)$ and $b \in B$. We shall need the general estimate

$$\begin{aligned} &|[-D\Psi(z) \cdot f(z,b) - h(z,b)] - [-D\Psi(x) \cdot f(x,b) - h(x,b)]| \\ &\quad \leq |[-D\Psi(z) + D\Psi(x)] \cdot f(z,b)| + |D\Psi(x) \cdot [f(x,b) - f(z,b)]| \\ &\qquad + |[-h(z,b) + h(x,b)]| \\ &\quad \leq \omega_{D\Psi}(|z-x|, R) M_{f,R} + |D\Psi(x)| \omega_f(|z-x|, R) + \omega_h(|z-x|, R) \end{aligned} \tag{3.27}$$

where $\omega_{D\Psi}(\cdot, R)$ is a modulus of continuity for $D\Psi(\cdot)$ on $B(0,R)$, where $M_{f,R}$ is a bound on $f(z,b)$ for $(z,b) \in B(0,R) \times B$, and where we use (A1) and (A2). By the continuity of the moduli $\omega_{D\Psi}(\cdot, R)$, $\omega_f(\cdot, R)$ and $\omega_h(\cdot, R)$ at the origin, we deduce that there is a $\delta_R > 0$ so that

$$|z - x| < \delta_R \Longrightarrow |[-D\Psi(z) \cdot f(z, b) - h(z, b)] - [-D\Psi(x) \cdot f(x, b) - h(x, b)]| < \delta/2. \tag{3.28}$$

Moreover, by (2.5) we know that there is a $t_x > 0$ so that

$$0 \leq s \leq t_x \Longrightarrow |y_x(s, b) - x| < \delta_R \text{ for all } b \in \mathcal{B}.$$

We conclude that, for $0 \leq s \leq t_x$ and for all $b \in \mathcal{B}$, from (3.26) and (3.27) combined with (3.28) we have

$$\begin{aligned} &-D\Psi(y_x(s, b) \cdot f(y_x(s, b), b(s)) - h(y_x(s, b), b(s)) + \gamma^2|b(s)|^2 \\ &\quad = [-D\Psi(x) \cdot f(x, b(s)) - h(x, b(s)) + \gamma^2|b(s)|^2] \\ &\qquad + \{[-D\Psi(y_x(s, b)) \cdot f(y_x(s), b(s)) - h(y_x(s), b)] \\ &\qquad -[-D\Psi(x) \cdot f(x, b(s)) - h(x, b(s))]\} \\ &\quad \geq \delta - \delta/2 = \delta/2 \text{ for } 0 \leq s \leq t_x. \end{aligned} \tag{3.29}$$

Since x is a local maximum of $V - \Psi$, by (2.4) we may assume that $t_x > 0$ also satisfies

$$V(x) - V(y_{x,b}(s)) \geq \Psi(x) - \Psi(y_x(s, b)) \text{ for } 0 < s < t_x \text{ for all } b \in \mathcal{B}. \tag{3.30}$$

For any t satisfying $0 < t \leq t_x$ we may integrate (3.29) from 0 to t to get

$$\Psi(x) - \Psi(y_{x,b}(t)) > \frac{\delta}{2}t + \int_0^t l(y_x(s), b(s))\, ds. \tag{3.31}$$

As a consequence of (3.30) and (3.31), we have

$$V(x) - V(y_x(t, b)) > \frac{\delta}{2}t + \int_0^t l(y_x(s), b(s))\, ds \text{ for all } b \in \mathcal{B}$$

Thus

$$\begin{aligned} V(x) &\geq \frac{\delta}{2}t + \sup_{b \in \mathcal{B}} \left\{ \int_0^t l(y_x(s), b(s))\, ds + V(y_x(t, b)) \right\} \\ &> \sup_{b \in \mathcal{B}} \left\{ \int_0^t [l(y_x(s), b(s))\, ds + V(y_x(t, b)) \right\} \end{aligned}$$

which contradicts (ii) of Proposition 3.11.

We now assume that V is lower semicontinuous. The proof that V is a viscosity supersolution of VI is similar to the proof of Theorem 3.4. By using (i) and (iii) of Proposition 3.11, one can follow the proof there to show that V is a viscosity supersolution of VI, and the result follows. $\diamondsuit$

Proposition 3.13 *Assume (A0)-(A4). Assume in addition:* there is a continuous B-valued function $(x,p) \to \beta(x,p)$ so that

$$H(x,p) = -p \cdot f(x, \beta(x,p)) - l(x, \beta(x,p)). \tag{3.32}$$

Then, if $\widetilde{V} \in C^1(\mathbb{R}^n)$ *is a subsolution of VI, then* $\widetilde{V} \leq V$. *Hence, under these assumptions, if* $V \in C^1(\mathbb{R}^n)$, *then* V *is the maximal smooth nonnegative subsolution of VI.*

Proof Since $\widetilde{V} \in C^1(\mathbb{R}^n)$ is a subsolution of VI, we have

$$H(x, D\widetilde{V}(x)) \leq 0 \text{ and } \widetilde{V}(x) \leq \Phi(x), \ \forall x \in \mathbb{R}^n. \tag{3.33}$$

Define $\beta_* : \ \mathbb{R}^n \to B$ by

$$\beta_*(z) = \beta(z, D\widetilde{V}(z)) \text{ for } z \in \mathbb{R}^n.$$

Then β_* is continuous in $z \in \mathbb{R}^n$ and from the first part of (3.33) we see that

$$-D\widetilde{V}(z) \cdot f(z, \beta_*(z)) - h(z, \beta_*(z)) + \gamma^2 |\beta_*(z))| = H(z, D\widetilde{V}(z)) \leq 0 \tag{3.34}$$

for all $z \in \mathbb{R}^n$. By assumption β and $D\widetilde{V}$ are continuous; hence $z \to \beta_*(z)$ is continuous on $\mathbb{R}^n$ and there exists a solution $y_x^*(t)$ to the initial-value problem

$$\dot{y}(t) = f(y(t), \beta_*(y(t))), \quad y(0) = x.$$

Note that we may regard $t \to b_x^*(t) := \beta_*(y_x^*(t))$ as an element of $\mathcal{B}$ for each $x \in \mathbb{R}^n$, and then $y_x^*(t) = y_x(t, b_x^*(t))$ for all $t \geq 0$. From (3.34) we deduce

$$-D\widetilde{V}(y_x(t, b_x^*(t)) \cdot f(y_x(t, b_x^*(t)), b_x^*(t)) - h(y_x(t, b_x^*(t))) + \gamma^2 |b_x^*(t)|^2 \leq 0. \tag{3.35}$$

For $\nu \in \Delta$, integrate (3.35) from 0 to $\nu_x[b_x^*]$ and use the second part of (3.33) to get

$$\begin{aligned}
\widetilde{V}(x) &\leq \int_0^{\nu_x[b_x^*]} l(y_x(s), b_x^*(s)) ds + \widetilde{V}(y_x(\nu_x[b_x^*], b_x^*)) \\
&\leq \int_0^{\nu_x[b_x^*]} l(y_x(s), b_x^*(s)) ds + \Phi(y_x(\nu_x[b_x^*], b_x^*)) \\
&\leq \sup_{b \in \mathcal{B}} \{ \int_0^{\nu_x[b]} l(y_x(s), b(s)) \, ds + \Phi(y_x(\nu_x[b], b)) \}.
\end{aligned}$$

Since this holds for each $\nu \in \Delta$, we have

$$\widetilde{V}(x) \leq \inf_{\nu \in \Delta} \sup_{b \in \mathcal{B}} \{ \int_0^{\nu_x[b]} l(y_x(s), b(s)) ds + \Phi(y_x(\nu_x[b], b)) \} = V(x)$$

and the result follows. $\diamondsuit$

Remark 3.14 If one supposes only that $\widetilde{V}$ is a continuous subsolution of the VI, and that, for each $x \in \mathbb{R}^n$ there is a $b_x \in B$ so that

$$D_\uparrow \widetilde{V}(x)(-f(x, b_x)) - h(x, b_x) + \gamma^2 |b_x|^2 = \inf_{b \in \mathcal{B}} \{D_\uparrow \widetilde{V}(x)(-f(x, b)) - h(x, b) + \gamma^2 |b|^2\} \tag{3.36}$$

where $D_\uparrow \widetilde{V}(x)$ is the contingent epiderivative of $\widetilde{V}$ at x (see the discussion in Remark 2.4), then one can follow the proof of Theorem 3.6 in [13] to deduce that

$$D_\uparrow \widetilde{V}(-f(x, b_x)) - h(x, b_x) + \gamma^2 |b_x|^2 \leq 0.$$

Once we have this, by the Viability Theory argument in [13] we get

$$\begin{aligned} \widetilde{V}(x) &\leq \widetilde{V}(y_x(\nu, b_x)) + \int_0^\nu l(y_x(s, b_x), b_x)\ ds \\ &\leq \Phi(y_x(\nu, b_x)) + \int_0^\nu l(y_x(\nu, b_x), b_x)\ ds \end{aligned}$$

for all $\nu \geq 0$. (Here we use that $\widetilde{V}(x) \leq \Phi(x)$ for all x, a consequence of $\widetilde{V}$ being a viscosity subsolution of the VI.) Hence (where we now drop the second argument in y_x),

$$\begin{aligned} \widetilde{V}(x) &\leq \inf_{\nu \geq 0} \left\{ \Phi(y_x(\nu)) + \int_0^\nu l(y_x(s), b_x)\ ds \right\} \\ &\leq \sup_{b \in \mathcal{B}} \inf_{\nu \geq 0} \left\{ \Phi(y_x(\nu)) + \int_0^\nu l(y_x(s), b(s))\ ds \right\} \\ &\leq \inf_{\nu \in \Delta} \sup_{b \in \mathcal{B}} \left\{ \Phi(y_x(\nu_x[b])) + \int_0^{\nu_x[b]} l(y_x(s), b(s))\ ds \right\} \\ &\leq V(x) \end{aligned}$$

where the last inequality follows from the fact that the lower value function of the static game is less than or equal to the lower value function of the differential game (see, e.g., [7]). Thus, it follows that: *under assumptions (A0)–(A4) and (A3′), if for each continuous subsolution $\widetilde{V}$ of the VI and $x \in \mathbb{R}^n$ there is a $b_x \in B$ so that* (3.36) *is satisfied and if V is continuous, then V is the maximal continuous nonnegative subsolution of the VI.* While this modification of Proposition 3.13 removes the smoothness assumption on $\widetilde{V}$, the hypothesis (3.36) is more difficult to verify than (3.32) in Proposition 3.13.

4 State feedback robust stopping-time control

The state feedback robust stopping-time control problem is to find a controller $K : y(\cdot) \to \tau(\cdot)$ and the smallest number $\gamma^* > 0$ so that the closed-loop system with stopping time (Σ_{st}, K) of Figure 1 where $K(y_x(t,b)) = \tau_x[b]$ is γ^*-*stopping dissipative* and *stable*.

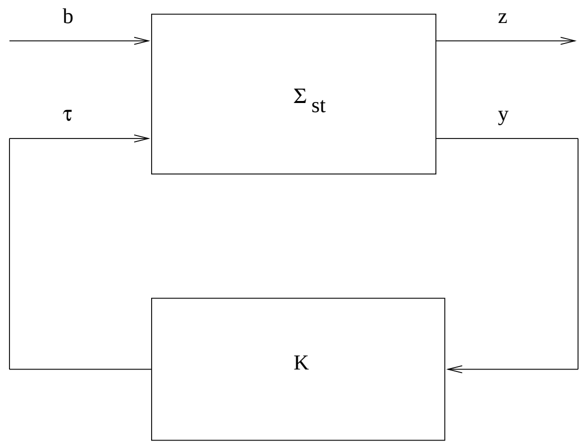

Figure 1: the closed-loop stopping-time system (Σ_{st}, K)

Here, given a $\gamma > 0$, we say that the closed-loop system (Σ_{st}, K) is γ-stopping dissipative if there exists a nonnegative real-valued function U with $U(0) = 0$ so that

$$\int_0^{T\wedge\tau_x[b]} h(y_x(s,b), b(s))ds \leq \gamma^2 \int_0^{T\wedge\tau_x[b]} |b(s)|^2 ds + U(y_x(0,b)),$$

for all $x \in \mathbb{R}^n$, all $b \in \mathcal{B}$ and all $T \geq 0$. In general it is hard to solve for the optimal γ_*, so we instead solve the suboptimal robust stopping-time control problem. The suboptimal robust stopping-time control problem assumes that we are given a $\gamma > \gamma_*$ and seeks to find a controller K with some information structure so that the closed-loop system (Σ_{st}, K) is γ-*stopping dissipative* and *internally stable*, i.e. stable for any initial condition subject to zero disturbance $b = 0$. For the discussion of stability in this section, we mostly specialize the general formalism to the case of $b = 0$; to lighten the notation, we therefore contract the expression $y_x(s,0)$ to $y_x(s)$ and $\tau_x[0]$ to τ_x for the sake of easier reading.

For a fixed $\gamma > 0$, the result from Theorem 3.9 concludes that we can find a controller K so that a closed-loop system (Σ_{st}, K) is γ-stopping dissipative by constructing a continuous, positive-definite, viscosity supersolution of the VI. Thus it remains to show that the closed-loop system is internally stable. As in the classical case (see §3.2 of [17] and §4.5 of [10]), as we now demonstrate, it happens that a continuous, positive-definite, viscosity supersolution of the VI can be used to prove stability for the closed-loop system (Σ_{st}, K).

In general, given a closed-loop stopping-time system (Σ_{st}, K), we say that the origin is a *stable equilibrium* point of the undisturbed stopping-time system $\dot{y} = f(y, 0)$ if

(a) $y_0(s) = 0$ for all $0 \leq s < \tau_0$; and

(b) for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that

$$\begin{cases} |x| < \delta \ \text{ implies } \ |y_x(s)| < \epsilon \ \text{ for all } \ 0 \leq s < \tau_x; \\ \text{furthermore, } |x| < \delta \text{ and } \tau_x = \infty \text{ implies } \lim_{s\to\infty} y_x(s) = 0. \end{cases}$$

We shall need the following Lemma, sometimes referred to as "Barbălat's lemma", in the proof of stability.

Lemma 4.1 *If $\phi(\cdot) : \mathbb{R} \to \mathbb{R}$ is nonnegative, uniformly continuous and $\int_0^\infty \phi(s)\, ds < \infty$, then $\lim_{t\to\infty} \phi(t) = 0$.*

Proof See [1, 16].

We also assume the following conditions on the system Σ_{st}.

$$(A6) \begin{cases} \text{For each } T > 0, \text{ if } b(t) = 0 \text{ and } z(t) = 0 \text{ for all } 0 \leq t \leq T, \\ \text{then } y(t) = 0 \text{ for all } 0 \leq t \leq T. \end{cases}$$
$$(A7) \begin{cases} \text{If } b(t) = 0 \text{ for all } t \geq 0 \text{ and } \lim_{t\to\infty} z(t) = 0, \\ \text{then } \lim_{t\to\infty} y(t) = 0. \end{cases}$$

The conditions (A6) and (A7) are modifications of the usual notions of *zero-state observability* and *zero-state detectability*, respectively, for nonlinear controlled systems (see, e.g., p. 39 of [17]). For the case of linear systems, it is easy to see that these notions correspond to the usual notions of observability and detectability.

Proposition 4.2 *Assume (A0)-(A6). If U be a nonnegative, continuous viscosity supersolution of the VI in $\mathbb{R}^n$, then $U(x) > 0$ for all $x \neq 0$.*

Proof Let $x \in \mathbb{R}^n$. Since U is a nonnegative continuous viscosity supersolution of the VI, we have

$$U(x) \geq \int_0^{T\wedge\tau_x} h(y_x(s), 0)ds + 1_{[0,T]}(\tau_x)\Phi(y_x(\tau_x)) + 1_{(T,+\infty]}(\tau_x)U(y_x(T)), \qquad (4.1)$$

for all $T \geq 0$, where $\tau_x = \inf\{t : t \geq 0 \text{ and } U(y_x(t)) \geq \Phi(y_x(t))\}$. In particular, if $\tau_0 < \infty$, we may take $x = 0$ and $T \geq \tau_0$ in (4.1) to get

$$
\begin{aligned}
U(0) &\geq \int_0^{\tau_0} h(y_0(s), 0)\, ds + \Phi(0) &\quad (4.2)\\
&\geq \Phi(0). &\quad (4.3)
\end{aligned}
$$

Thus $U(0) < \Phi(0)$ forces $\tau_0 = \infty$, and the last assertion of the proposition follows. In particular, $\tau_0 = \infty$ if $U(0) = 0 < \Phi(0)$. For an arbitrary state vector x, from the definition of τ_x and the assumption that $\Phi(x) > 0$, we see that $\tau_x > 0$ whenever $U(x) = 0$.

Since U is nonnegative and Φ is positive, another consequence of (4.1) is

$$U(x) \geq \int_0^{T \wedge \tau_x} h(y_x(s), 0)\, ds. \tag{4.4}$$

We next argue that $U(x) = 0$ forces $x = 0$, from which we get $U(x) > 0$ for all $x \neq 0$ as wanted. Assume therefore that $U(x) = 0$. As noted above, this forces $\tau_x > 0$. Furthermore, from (4.4) we get $h(y_x(s), 0) = 0$ for all $s \in [0, T \wedge \tau_x]$. By (A6), $y_x(s) = 0$ for all $s \in [0, T \wedge \tau_x]$. Thus $x = y_x(0) = 0$ by (A6). $\diamondsuit$

The following is our main result on stability for the case of a stopping-time problem; for the parallel statement and proof for the standard nonlinear H^∞-control problem (formulated, however, only for classical supersolutions of the associated Hamilton-Jacobi inequality), see Lemma 3.2.1 in [17].

Theorem 4.3 *Assume (A1)-(A5), (A3′) and (A7). If U is a nonnegative, continuous, viscosity supersolution of the VI , $U(x) > 0$ for $x \neq 0$ and $U(0) = 0$, then $x = 0$ is the stable equilibrium point of the undisturbed stopping-time system $\dot{y} = f(y, 0)$.*

Proof Set $\Omega := \{x \in \mathbb{R}^n : U(x) < \Phi(x)\}$. By the continuity of U and Φ, Ω is open. Since Φ is positive, $U(0) = 0 < \Phi(0)$ and thus $0 \in \Omega$. Choose $\hat{\delta} > 0$ such that

$$B(0, \hat{\delta}) \subset \Omega. \tag{4.5}$$

Since U is a viscosity supersolution of the VI, we have

$$H(x, DU(x)) \geq 0,\ x \in B(0, \hat{\delta}) \ \text{ in viscosity sense.}$$

By Theorem 2.3, we have

$$
\left\{
\begin{array}{l}
U(x) - U(y_x(t)) \geq \int_0^t h(y_x(s), 0))\, ds \\
\ \text{for all } x \in B(0, \hat{\delta}) \text{ and all } 0 \leq t \leq \tau_x, \\
\text{where } \tau_x = \inf\{t \geq 0 \ : \ U(y_x(t)) = \Phi(y_x(t))\}.
\end{array}
\right. \tag{4.6}
$$

Since $h(\cdot, 0) \geq 0$, we have

$$0 \leq U(y_0(t)) \leq U(0) = 0, \quad \text{for all } 0 \leq t \leq \tau_0.$$

Thus $U(y_0(t)) = 0$ for all $0 \leq t \leq \tau_0$. By the positive definite property of U, $y_0(t) = 0$ for all $0 \leq t \leq \tau_0$.

Next we want to show that for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that (1) if $|x| < \delta$ then $|y_x(t)| < \epsilon$ for all $0 \leq t \leq \tau_x$, and (2) if $|x| < \delta$ and $\tau_x = \infty$ then $\lim_{t\to\infty} y_x(t) = 0$. Let $\epsilon > 0$. Define $r_\epsilon = \inf\{U(x) \mid x \in \partial B(0, \epsilon)\}$. Note that $r_\epsilon > 0$ because $U(x) > 0$ for $x \neq 0$, U(0)=0 and $0 \notin \partial B(0, \epsilon)$. By the continuity of U, choose $0 < \delta \leq \min\{\epsilon, \hat{\delta}\}$ (where $\hat{\delta}$ is chosen to satisfy (4.5)) such that if $|x| < \delta$ then $U(x) < r_\epsilon$. Thus for any $x \in B(0, \delta)$, by (4.6) we have

$$U(y_x(t, 0)) + \int_0^t h(y_x(s), 0)\, ds \leq U(x) < r_\epsilon, \text{ for all } 0 \leq t \leq \tau_x.$$

Since $h(\cdot, 0) \geq 0$, we have

$$U(y_x(t)) < r_\epsilon, \quad \text{for all } 0 \leq t \leq \tau_x.$$

We conclude that there is no $\bar{t} \in [0, \tau_x]$ such that $y_x(\bar{t}) \in \partial B(0, \epsilon)$, and hence by connectedness $y_x(t) \in B(0, \epsilon)$ for $0 \leq t \leq \tau_x$ as required. Moreover if $|x| < \delta$ and $\tau_x = \infty$, then we have

$$U(y_x(t)) + \int_0^t h(y_x(s), 0) ds \leq U(x), \quad \text{for all } t \geq 0.$$

Since U is nonnegative, we have

$$\int_0^t h(y_x(s), 0) ds \leq U(x) < \infty, \quad \text{for all } t \geq 0.$$

By the continuity of f and the boundedness of $y_x(\cdot)$, it follows that $y_x(\cdot)$ is uniformly continuous, and so is $h(y_x(\cdot), 0)$. By Lemma 4.1, $\lim_{t\to\infty} h(y_x(t)) = 0$. By (A7), we have $\lim_{t\to\infty} y_x(t) = 0$. $\diamondsuit$

We conclude with discussion of a simple one-dimensional example for which the minimal viscosity supersolution of the VI and associated feedback stopping-time strategy are explicitly computable.

Example. Consider the one-dimensional state space system of the form

$$\begin{aligned} \dot{y} &= -y + b, \\ z &= y^2, \end{aligned} \tag{4.7}$$

with gain rate

$$\gamma = 2. \tag{4.8}$$

In a linear-quadratic robust control system such as this it would be typical to impose no constraints on the disturbance values $b \in B$. However to satisfy our hypothesis (A1) we will take

$$B = [-1, 1]. \tag{4.9}$$

One may check that for $|p| \leq 2\gamma^2 = 8$ the Hamiltonian (2.1) is equal to

$$H(x,p) = \frac{-1}{16}p^2 + px - x^2. \tag{4.10}$$

For $|p| > 8$ the constraint $|b| \leq 1$ influences the infimum in (2.1), so that (4.10) is incorrect. However in our example no values of $|p| > 8$ will occur. We take the stopping cost to be

$$\Phi(x) = \frac{5}{4} + \cos(4x). \tag{4.11}$$

All hypotheses (A0)–(A5) are satisfied.

A plot of the minimal solution $U(x)$ of (VI) (corresponding to the available storage function), together with a plot of the preassigned stopping cost function $\Phi(x)$ (the dashed curve), is given in Figure 2 below. Verification that the construction described below leads to the minimal viscosity solution is straightforward, if one uses the equivalent characterization

$$D^+U^*(x) = \{p \in \mathbb{R}^n : \limsup_{y \to x} \frac{U^*(y) - U^*(x) - p \cdot (y - x)}{|x - y|} \leq 0\}.$$

of the superdifferential $D^+U(x)$ for a continuous, piecewise C^1 function U in terms of the left and right derivatives of U, along with the analogous characterization of the subdifferential $D^-U(x)$ (2.8); for details on these alternate characterizations, see for instance Lemma 1.7 of Chapter II in [7].

We discuss only $x > 0$ in what follows. (By symmetry we will have $U(x) = U(-x)$ for $x < 0$.) First, observe that the minimal solution of $H(x,p) = 0$ is

$$p_-(x) = (8 - 4\sqrt{3})x.$$

The minimal nonnegative solution of $H(x, S'(x)) = 0$ is the available storage function (1.4) for our system (4.7):

$$S_a(x) = \int_0^x p_-(t)\,dt = (4 - 2\sqrt{3})x^2.$$

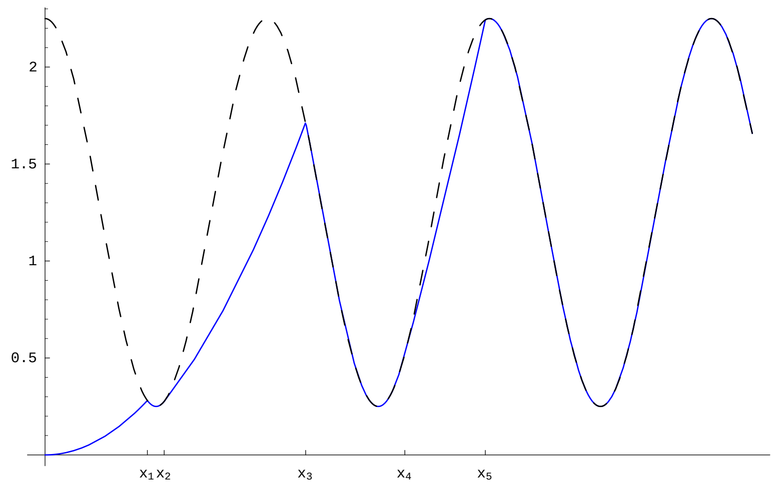

Figure 2: Example solution of variational inequality

The initial segment of the solution is

$$U(x) = S_a(x) \quad \text{for } 0 \leq x < x_1.$$

The value $x_1 = .723487$ is determined by solving $S_a(x_1) = \Phi(x_1)$. Next, Step 2 extends the construction of U to

$$U(x) = \Phi(x) \quad \text{for } x_1 \leq x < x_2.$$

The value of x_2 is the first $x > x_1$ at which $H(x, \Phi'(x)) < 0$ fails. The value $x_2 = .842313$ is located by solving $\Phi'(x) = p_-(x)$.

The construction now proceeds in Step 3, using $U'(x) = p_-(x)$ or

$$U(x) = \Phi(x_2) - S_a(x_2) + S_a(x) \quad \text{on an interval } x_2 \leq x < x_3,$$

where x_3 is maximal such that $U(x) \leq \Phi(x)$ for $x_2 \leq x < x_3$. This turns out to be $x_3 = 1.84258$. Now we repeat Step 2 to find

$$U(x) = \Phi(x) \quad \text{for } x_3 \leq x < x_4,$$

with $x_4 = 2.54367$ the first solution of $\Phi'(x) = p_-(x)$ beyond x_3. Beyond x_4 we take another section with

$$U(x) = \Phi(x_4) - S_a(x_4) + S_a(x) \quad \text{on an interval } x_4 \leq x < x_5,$$

with $x_5 = 3.11278$ determined again by $U(x) = \Phi(x)$. Finally, we find that $\Phi'(x) < p_-(x)$ for all $x > x_5$. This means that the remainder of the definition of U is

$$U(x) = \Phi(x) \quad \text{for } x_5 \leq x.$$

The optimal stopping-time rule, as in Theorem 3.9, for this example is to stop at the first instant the state $y(t)$ enters the set

$$[x_1, x_2] \cup [x_3, x_4] \cup [x_5, \infty)$$

on which $U(x) = \Phi(x)$.

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