ROBUST OPTIMAL SERVICE ANALYSIS OF SINGLE-SERVER RE-ENTRANT QUEUES

MARTIN V. DAY, J. HALL, J. MENENDEZ, D. POTTER, AND I. ROTHSTEIN

Abstract. We generalize the analysis of Ball, Day and Kachroo [2] to a fluid model of a single server re-entrant queue. The approach is to solve the Hamilton-Jacobi-Isaacs equation associated with optimal robust control of the system. The method of "staged" characteristics is generalized from [2] to construct the solution explicitly. Formulas are developed allowing explicit calculations for the Skorokhod problem involved in the system equations. Such formulas are particularly important for numerical verification of conditions on the boundary of the nonnegative orthant. The optimal control (server) strategy is shown to be of linear-index type. Dai-type stability properties are discussed. A modification of the model in which new "customers" are allowed only at a specified entry queue is considered in 2 dimensions. The same optimal strategy is found in that case as well.

1. Introduction

A robust control approach to the design of service disciplines for queueing systems was initiated by Ball, Day and Kachroo [2]. That paper, largely motived by applications to vehicular traffic systems, considered a fluid model of the single server with no re-entry. In Section 4 of [2] the 2-queue version of that system was modified so that the served output of the first queue becomes input to the second queue, forming the simplest example of what is called a re-entrant line in queueing system literature [11]. In this paper we develop the same general approach for the n-dimensional version of the fully re-entrant single server, illustrated in Figure 1.

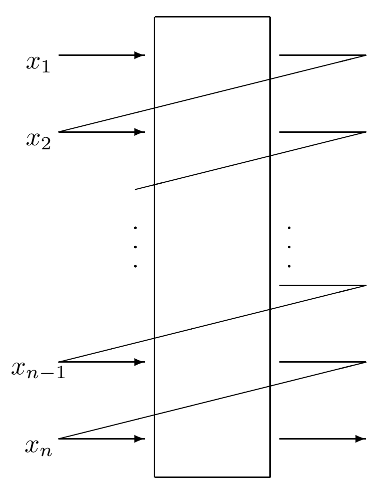

Figure 1. Re-entrant Server

There is much current interest in developing optimal service strategies for queueing systems. The volume by Kelly and Williams [9] includes several articles addressing this. Although queueing models are generally integer-valued and stochastic, Dai [4] and others have developed connections between the stability of

Date: November 30, 2001.
Research supported by the ASPIRES program of Research and Graduate Studies, and the Millennium program of the College of Arts and Sciences, both of Virginia Tech.

stochastic queuing systems and their deterministic fluid limits. Thus optimal strategies for fluid models are recognized as significant for stochastic models. Fluid models for a large class of queueing systems can be described by equations of the general form (5) introduced in Section 2 below. We pursue the same robust control approach as in [2] for such models. Much of what we present here is a further development of ideas from that paper. In particular, Section 2.1 gives explicit representations of the velocity projection map $\pi(x, v)$ of the Skorokhod reflection mechanism which comes into play when one or more queues are empty. Section 3 develops the construction of the value function for our control problem. Here, as in [2], the construction proceeds without regard to the Skorokhod dynamics on the boundary of the nonnegative orthant. (In more general multiple server examples the Skorokhod dynamics will play a more decisive role.) Even so, the solution we construct must be shown to satisfy various inequalities associated with optimality with respect to the Skorokhod dynamics on the boundary. We do not provide a deductive proof of these inequalities, but rely instead on a system of numerical confirmation for individual test cases in Section 4. The explicit representations of $\pi(x, v)$ are important for this, and for the optimality argument of Section 5. The version of that argument given here improves on the one in [2] in that it applies to all admissible strategies, not just those of state feedback form.

Our model allows new arrivals and unserved departures in the form of an exogenous load $q_i(t)$ for *each* x_i; see (1) below. In some queueing applications this feature would be inappropriate. For instance in typical re-entrant lines, new arrivals only occur at a specified entry queue and departures only as service is completed at a designated final queue. In Section 6 we will look at the the 2-dimensional case of our model under the more restrictive assumption that exogenous arrivals are *only* allowed in the entry queue x_1. This requires a number changes in our calculations. But we find that this change to the model does *not* effect the resulting optimal service policy.

2. The Model and Approach of Optimal Control

We describe in this section the general model formulation and performance criteria that we will use. Fluid models for a large class of queueing systems can be described by equations of the nominal form

$$\dot{x}(t) = q(t) - Gu(t). \tag{1}$$

The *state variable* is n-dimensional: $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. For queueing models $x(t)$ must remain in the nonnegative orthant, K in (4) below. For that purpose we will couple (1) with Skorokhod problem dynamics, resulting in (5) below. The term $q(t)$ is the *load* on the system due to new arrivals (or unserved departures if $q_i(t) < 0$). The service allocation is specified by the *control function* $u(t)$ whose values are taken from a finite set $U_0 \subseteq \mathbb{R}^m$ of possible service control settings. For purposes of an adequate existence theory for solutions to (5) we relax this to allow $u(t)$ to be taken from the convex hull

$$\mathcal{U} = \text{conv } U_0. \tag{2}$$

For our single-server examples we will simply take $U_0 = \{e_1, \dots, e_n\}$, the standard unit vectors in $\mathbb{R}^n$. Thus $u \in \mathcal{U}$ has coordinates $u_i \geq 0$ with $\sum u_i = 1$. The matrix G converts $u(t)$ to the appropriate vector of contributions to $\dot{x}$. For the case to be considered here (Figure 1) G will be the lower triangular matrix

$$G = \begin{bmatrix} s_1 & 0 & & \dots & 0 \\ -s_1 & s_2 & 0 & & \vdots \\ 0 & -s_2 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & s_{n-1} & 0 \\ 0 & \dots & 0 & -s_{n-1} & s_n \end{bmatrix}.$$

The $s_i > 0$ are parameters which specify the service rates for the respective queues. Thus when $u(t) = e_k$ ($k < n$), the effect of $-Gu(t)$ in (1) is to drain queue x_k at rate s_k with the served customers entering x_{k+1} at the same rate:

$$\dot{x}_k = q_k(t) - s_k, \quad \dot{x}_{k+1}(t) = q_{k+1}(t) + s_k, \text{ and } \dot{x}_i = q_i \text{ otherwise.}$$

Multiple server examples are easily modeled by (1) and (5) as well. Consider for example the 2-server re-entrant line in Figure 2. It would be natural to use

$$G = \begin{bmatrix} s_1 & 0 & 0 & 0 \\ -s_1 & 0 & s_2 & 0 \\ 0 & s_3 & -s_2 & 0 \\ 0 & -s_3 & 0 & s_4 \end{bmatrix} \quad \text{and} \quad U_0 = \{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \}. \tag{3}$$

The first two columns in G correspond to the service allocation at server A and the second two at server B. The $u \in U_0$ correspond to the 4 different combinations in which server A chooses between x_1 or x_3 and server B chooses between x_2 or x_4.

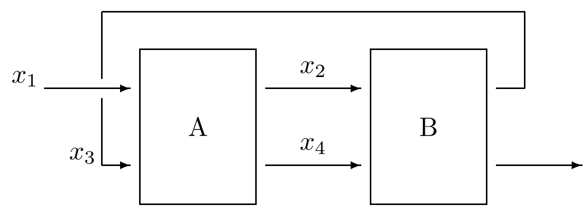

FIGURE 2. 2-Server re-entrant line

2.1. Skorokhod Problem Dynamics.

We denote by K the *nonnegative orthant* of $\mathbb{R}^n$:

$$K = \{x \in \mathbb{R}^n : \; x_i \geq 0 \text{ for all } i\}. \tag{4}$$

The *faces* of K are

$$\partial_i K = \{x \in K : \; x_i = 0\}.$$

The *interior normal* to $\partial_i K$ is the standard unit vector $n_i = e_i$. We will use

$$N = \{1, \ldots, n\}$$

to denote the set of all coordinate indices. For $x \in K$,

$$I(x) = \{i \in N : \; x_i = 0\},$$

will denote the set of indices with zero coordinate values.

An essential feature of queueing models is that $x(t)$ remains in K for all t. One could simply impose this as a constraint the control functions $u(t)$ and loads $q(t)$ which are considered admissible. Although some constraints on the load are reasonable, we find it much more natural in general to couple (1) with the dynamics of a Skorokhod problem. On each face $\partial_i K$ we specify a *constraint vector* d_i. If the solution of (1) attempts to exit K through $\partial_i K$, then the idea is to add some positive multiple of d_i to the right side of (1) to prevent the exit. A precise formulation is the following: given $x(0) \in K$ and $q(t), u(t)$ (which we will assume to be locally integrable), let

$$y(t) = x(0) + \int_0^t q(s) + Gu(s)\, ds.$$

The *Skorokhod Problem* is to find a continuous function $x(t) \in K$, a measurable function $r(t) \in \mathbb{R}^n$ and a nondecreasing function $\ell(t) \geq 0$ which satisfy the following for $t \geq 0$:

- $x(t) = y(t) + \int_{(0,t]} r(s)\, d\ell(s)$;
- for each t, $r(t) = \sum_{i \in I(x(t))} \lambda_i d_i$ for some $\lambda_i \geq 0$;
- $\ell(t) = \int_{(0,t]} 1_{x(s) \in \partial K}\, d\ell(s)$.

By imposing a normalization $\|r(t)\| = 1$, there will be a unique solution, provided K and d_i satisfy certain conditions. Dupuis and Ishii [5] and Dupuis and Ramanan [7] provide a substantial body of theory of Skorokhod problems in general. In particular they show, using a *velocity projection map* $\pi(x, v)$, that a Skorokhod problem can be expressed as a differential system. The velocity projection map is of the form

$$\pi(x, v) = v + \sum_{i \in I(x)} \beta_i d_i$$

for an appropriate choice of $\beta_i \geq 0$. (See (6) below.) The result of coupling our (1) with the appropriate Skorokhod problem is expressed as

$$\dot{x}(t) = \pi(x(t), q(t) - Gu(t)), \tag{5}$$

holding almost surely.

The appropriate constraint vectors d_i are determined by the structure of the system in Figure 1. If $x \in \partial_i K$ and server i is active ($u_i > 0$) but the applied service rate $s_i u_i$ exceeds the inflow q_i to x_i, then according to (1) x would exit K through $\partial_i K$. In an actual network the system could not really use the full service capacity $s_i u_i$ allocated to x_i. Instead, service would take place at a lower level which exactly balances the inflow and outflow of queue x_i. Mathematically this is achieved by adding a positive multiple of the column Ge_i of G to the right side of the system (1), bringing $\dot{x}_i$ to 0 and producing the correct reduction of the throughput $x_i \to x_{i+1}$ to the next queue. So we take $d_i = \frac{1}{s_i} Ge_i$ (the normalization being so that $n_i \cdot d_i = 1$). The same prescription is appropriate for the example of Figure 2: take d_i to be the unique column of G having a positive entry in row i, normalized so that $n_i \cdot d_i = 1$.

At this point we wish to highlight the fact that no restrictions on $u(t)$ and $q(t)$ are needed to keep $x(t)$ in the nonnegative orthant; (5) will determine a state trajectory with $x(t) \in K$ regardless. Thus we always instruct the server to work at full capacity ($\sum u_i(t) = 1$) and the Skorokhod dynamics can be viewed as automatically reducing the service rates to the levels that can actually be implemented. The model allows a separate load term $q_i(t)$ for each queue. For re-entrant queues, one typically would only want to allow $q_i > 0$ for the entry queue in each re-entrant sequence. In Figure 1 for instance it would be natural to assume $q_2 = \ldots = q_n = 0$. A nice feature of single servers with respect to the L_2 performance criteria of Section 2.4 is that the the optimal strategy is the same for all loads, regardless of which coordinates might be zero. In queueing applications one also naturally assumes that $q_i \geq 0$. However it was argued in [2] that, for purposes of vehicular traffic for instance, it is reasonable to consider $q_i < 0$. This would correspond to customers that leave the system without waiting to receive service all the way through. This is a reasonable consideration in some applications. However it is hard to conceive of a realistic interpretation for $q_i < 0$ when $x_i = 0$. Even so, (5) will still yield a mathematical solution. The effect of the additional $+\beta_i d_i$ terms in $\pi(x, q - Gu)$ might then be seen not as reductions to the service rates but as a transference of the reducing influence of $q_i < 0$ from the empty x_i to the queues x_j further along in sequence; fluid at x_j would be drawn backwards through the system to satisfy to external demand due to $q_i < 0$ at previous x_i.

With d_i defined, we face the important technical issue of existence and regularity properties of the Skorokhod problem. This issue is treated in detail in [5] and [7]. Those treatments consider a more general convex polyhedron in place of our K. Our particular choice of the nonnegative orthant falls within the scope of the earlier work [12]. Let $D = [d_i]$ be the matrix with the constraint vectors as columns. In our case, $D = I - Q$ where Q is the subdiagonal matrix with entries

$$q_{ij} = \begin{cases} 1 & \text{if } i = j + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Assuming that Q has nonnegative entries and spectral radius less than 1, both clearly satisfied for us, [12] provided a direct construction of the solution of the Skorokhod problem. In [7] it is shown that these conditions from [12] fall within the scope of a more general set of sufficient conditions for existence and Lipschitz continuity of the Skorokhod map $y(\cdot) \mapsto x(\cdot)$.

Drawing on the ideas of [12], we can give a direct construction of the $\pi(x, v)$ appearing in the differential formulation (5) of the Skorokhod problem. For any $x \in K$ and $v \in \mathbb{R}^n$, we will show that $w = \pi(x, v)$ can be characterized using a linear complementarity problem:

$$w = v + \sum_{I(x)} d_i \beta_i \tag{6}$$

subject to the following constraints for each $i \in I(x)$:

$$\beta_i \geq 0, \tag{7}$$

$$w_i \geq 0, \tag{8}$$

$$w_i \beta_i = 0. \tag{9}$$

For $i \notin I(x)$ there is no constraint on w_i, and we consider $\beta_i = 0$ to be implicit. Let $\beta = [\beta_i]_1^n$. Using $D = I - Q$, and rewriting (6) as

$$Q\beta - v + w = \beta$$

it is easy to see that the complementarity problem is equivalent to saying β is a fixed point $\beta = \Psi_x(\beta)$ of the map $\Psi_x : \mathbb{R}^n \to \mathbb{R}^n$ defined coordinate-wise by

$$(\Psi_x(\beta))_i = \begin{cases} (Q\beta - v)_i^+ & \text{if } i \in I(x) \\ 0 & \text{if } i \notin I(x). \end{cases} \tag{10}$$

The notation y^+ refers to the usual positive part: $y^+ = \max(0, y)$. (This fixed point representation is a particular case of the general fixed point representation of variational inequalities in Chapter 1 of [10].) We first observe the existence of a unique fixed point. The argument of [12] is to observe that (after a linear change of variables) Ψ is a contraction, under the nonnegativity and spectral radius assumption mentioned above. However this is even simpler for our particular Q; $\beta = \Psi_x(\beta)$ reduces to

$$\beta_1 = \begin{cases} (-v_1)^+ & \text{if } 1 \in I(x) \\ 0 & \text{if } 1 \notin I(x) \end{cases}, \quad \text{and } \beta_i = \begin{cases} (\beta_{i-1} - v_i)^+ & \text{if } i \in I(x) \\ 0 & \text{if } i \notin I(x) \end{cases} \quad \text{for } 1 < i,$$

which determine the β_i sequentially. Iteration of Ψ_x from any initial β will converge to the fixed point after at most n steps. This makes it particularly simple to see that $v \mapsto \beta$ and thus $v \mapsto \pi(x, v)$ are continuous, and to evaluate $\pi(x, v)$ numerically. Indeed $\pi(x, v)$ is Lipschitz in v for a fixed x, and is jointly continuous in (x, v) if x is restricted so that $I(x)$ is constant.

We can easily check that $\pi(x, v)$ as defined by the above complementarity problem is indeed the velocity projection map as identified in [5]. First, following an observation of [3], we can check that $\pi(y) = \pi(0, y)$ is the discrete projection map of Assumption 3.1 of [5]. Indeed, if $y \in K$ then clearly $\beta = 0$ and $w = y$ solves the complementarity problem: $\pi(0, y) = y$. For $y \notin K$,

$$y - \pi(0, y) = y - w = - \sum_{i \in I(x),\ w_i > 0} \beta_i d_i$$

which is of the form $\alpha\gamma$ for some $\alpha \leq 0$ and $\gamma \in d(w)$, where $d(x)$ is the set of reflection directions as defined in [5, §3]. Next, suppose $x \in K$, $v \in \mathbb{R}^n$, and let w, β solve the complementarity problem for $w = \pi(x, v)$ above. We claim that

$$\pi(0, x + hv) = x + hw, \quad \text{provided } h > 0 \text{ is sufficiently small.} \tag{11}$$

This will imply that

$$\pi(x, v) = w = \lim_{h \downarrow 0} \frac{\pi(x + hv) - x}{h},$$

which is the characterization of $\pi(x, v)$ in [5, §5.3]. Since w, β solve (6), it follows that

$$x + hw = (x + hv) + \sum_1^n h\beta_i \, d_i.$$

We want to see that $\tilde{w} = x + hw$ and $\tilde{\beta} = h\beta$ satify the complementarity conditions (7) — (9) for $\tilde{v} = x + hv$ associated with $\tilde{x} = 0$, for which $I(\tilde{x}) = N$. First consider $i \in I(x)$. Since $x_i = 0$, we have

$$\tilde{w}_i = (x + hw)_i = hw_i \geq 0.$$

Since $h > 0$ we also know $\tilde{\beta}_i = h\beta_i \geq 0$, For the prodiuct (9) we have

$$\tilde{w}_i\tilde{\beta}_i = (x + hw)_i h\beta_i = h^2 w_i \beta_i = 0.$$

Next consider $i \notin I(x)$. Provided $h > 0$ is sufficiently small we have

$$\tilde{w} = (x + hw)_i > 0.$$

Since $\beta_i = 0$ we have $\tilde{\beta}_i = 0$, which also confirms the product condition. This verifies (11), as desired.

For purposes of our calculations, the characterizations of $\pi(x, v)$ in Lemma 1 below will be useful. If $J \subseteq N$ we will use

$$N_J = [n_j]_{j\in J} \text{ and } D_J = [d_j]_{j\in J}$$

to denote the matrices whose columns are the normal vectors n_j and constraint directions d_j for the $j \in J$. Given v, $w = \pi(x, v)$, and the corresponding β_i as described above, let

$$F_0 = \{i : \ \beta_i > 0\} \text{ and } L = \{i \in I(x) : \ w_i = 0\}.$$

From the complementarity problem we know $F_0 \subseteq L \subseteq I(x)$. Using any $F_0 \subseteq F \subseteq L$ the values of β_i, $i \in F$ are determined by setting $w_i = 0$, $i \in F$ in (6). In other words we can solve for $\beta_F = [\beta_i]_{i\in F}$ directly in $0 = N_F^T(v + D_F\beta_F)$ to obtain $\beta_F = B_F v$ where

$$B_F = -(N_F^T D_F)^{-1} N_F^T, \tag{12}$$

and consequently

$$w = \pi(x, v) = R_F v \tag{13}$$

where R_F is the *reflection matrix*

$$R_F = I + D_F B_F.$$

For $F = \emptyset$ we simply take $R_\emptyset = I$. The fact that $w_i \geq 0$ for $i \notin F$ is equivalent to

$$N^T_{I(x)\setminus F} R_F v \geq 0.$$

More precisely,

$$N^T_{L\setminus F} R_F v = 0 \text{ and } N^T_{I(x)\setminus L} R_F v > 0.$$

Suppose that we don't know F_0 or L at the outset, but just take an arbitrary $F \subseteq I(x)$, calculate $\beta_F = B_F v$, and take $w = R_F v$. By construction $w_i = 0$ for $i \in F$ and $\beta_i = 0$ for $i \notin F$. So item 3 of the complementarity problem is satisfied. If item 1 is satisfied of $i \in F$, which is to say

$$B_F v \geq 0,$$

and item 2 holds for $i \in I(x) \setminus F$, which is to say

$$N^T_{I(x)\setminus F} R_F v \geq 0,$$

then we can say that $w = R_F v$ is in fact $\pi(x, v)$. This discussion proves the following lemma.

Lemma 1. *Given $x \in K$, $v \in \mathbb{R}^n$, and $F \subseteq I(x)$, the following are equivalent:*

1. $\pi(x, v) = R_F v$;
2. *both of the following hold:*
   (a) $B_F v \geq 0$ *when* $F \neq \emptyset$, *and*
   (b) $N^T_{I(x)\setminus F} R_F v \geq 0$ *when* $F \neq I(x)$;
3. *for some L with $F \subseteq L \subseteq I(x)$ all of the following hold:*
   (a) $B_F v > 0$ *when* $F \neq \emptyset$, *and*
   (b) $N^T_{L\setminus F} R_F v = 0$ *when* $F \neq L$, *and*
   (c) $N^T_{I(x)\setminus L} R_F v > 0$ *when* $L \neq I(x)$.

Notice that the strict inequality in 3 (a) simply identifies F as $F_0 = \{i : \ \beta_i > 0\}$.

2.2. **The Optimal Control Policy.** Our goal is to design a feedback control strategy $\alpha_*(x)$, prescribing a value in the extended control set $\mathcal{U}$ for each $x \in K$, so that using $u(t) = \alpha_*(x(t))$ produces optimal performance of the system. The criteria used to determine optimality is based on

$$\int_0^T \frac{1}{2}\|x(t)\|^2 - \frac{\gamma^2}{2}\|q(t)\|^2\, dt, \tag{14}$$

where $\gamma > 0$ is a parameter. Roughly speaking, the control should keep the integrated cost (14) small, so that $x(t)$ remains small compared to the load $q(t)$ in a time-averaged sense. We will give this a more precise formulation in Section 2.4 below. The "running cost" $\frac{1}{2}\|x\|^2 - \frac{\gamma^2}{2}\|q\|^2$ of (14) has it roots in classical H_∞ control, and is attractive for its broad familiarity and success in a wide range of control applications. Other choices might be more appropriate for particular queueing applications, such as those associated with optimal draining and time-to-empty criteria; see [18] and [1]. There are however considerations that favor L^2 in the traffic setting. For a given total customer population, the L^2 norm favors balanced queue lengths over a situation in which some queues are empty and others are full. When each customer is a person who has to wait in a queue, a cost structure that can be minimized by using excessive waits for some small class of customers would be considered unacceptable.

The optimal policy itself is easy to identify by naive considerations at this point. In order to minimize (14) for a given $q(t)$ one would minimize $\frac{1}{2}\|x(t)\|^2$, for which one would naturally try to choose $u(t)$ to minimize $\frac{d}{dt}\frac{1}{2}\|x(t)\|^2 = x(t) \cdot \dot{x}(t)$. In the interior of K, where (1) applies, this suggests that the optimal u is that which maximizes $x \cdot Gu$ over $u \in \mathcal{U}$. On ∂K $\dot{x}$ is given by (5), which makes finding the u to minimize $x \cdot \pi(x, q - Gu)$ potentially more difficult. However if we assume that all $q_i \geq 0$ then the Skorokhod dynamics do not affect the minimizing u. To see this consider $x \in K$ with $x \neq 0$ and suppose $u_* \in \mathcal{U}$ maximizes $x \cdot Gu$. It is easy to see that $\sup x \cdot Gu > 0$. Observe that for any $i \in I(x)$, since $x_i = 0$ and all off-diagonal entries of G are nonpositive, $x \cdot Ge_i \leq 0$. Thus the i-coordinate of u_* must be 0, from which we can conclude that $n_i \cdot Gu_* \leq 0$. Since $q_i \geq 0$ by hypothesis, we see that

$$n_i \cdot (q - Gu_*) \geq 0, \text{ for all } i \in I(x).$$

This means that $\pi(x, q - Gu_*) = q - Gu_*$. Also, for any $i \in I(x)$ we have $x \cdot d_i \leq 0$ because $x_i = 0$ and only the i coordinate of d_i is positive. Therefore, for any $u \in \mathcal{U}$ we can say that

$$x \cdot \pi(x, q - Gu) = x \cdot (q - Gu + \sum_{I(x)} \beta_i d_i) \leq x \cdot (q - Gu) \leq x \cdot (q - Gu_*) = x \cdot \pi(x, q - Gu_*).$$

Thus the policy

$$\alpha_*(x) = \{u_* \in \mathcal{U} :\; x \cdot Gu_* = \max_{\mathcal{U}} x \cdot Gu\}. \tag{15}$$

is an obvious candidate for the optimal service policy. We will see below that it is indeed optimal in the sense to be made precise in Section 2.4.

Several comments should be made at this point. First observe that $\alpha_*(x)$ is set-valued. There are inherent discontinuities in $\alpha_*(x)$, as the optimum u jumps among the extreme points of $\mathcal{U}$. When we replace $u(t)$ by $\alpha_*(x(t))$ in (5), we will want the resulting feedback system to have good existence properties. This is addressed using the Filippov theory of differential inclusions. For that it is important that $\alpha_*(x)$ have closed graph and be convex set-valued. (The notion of closed graph is often called "lower semi-continuity" for set-valued functions.) It is easy to see that (15) has these properties.

Another important point to make is that the naive reasoning which suggests $\alpha_*(x(t))$ to us does *not* actually imply that it achieves the smallest possible value of $\frac{1}{2}\|x(T)\|^2$ for a given $q(\cdot)$ and target time T. It is conceivable that it might be better to forgo pointwise minimization of $\frac{d}{dt}\frac{1}{2}\|x(t)\|^2$ in order to drive $x(t)$ into a different region (or section of the boundary) where larger reductions of $\|x(t)\|$ could be achieved. Some sort of dynamic programming argument, such as the Hamilton-Jacobi equation developed in Section 3, is needed to adequately address such global optimality issues.

Although it may not have much practical import, one might ask whether allowing $q_i < 0$ when $x_i = 0$ might affect the choice of $u_* \in \mathcal{U}$ which minimizes $x \cdot \pi(x, q - Gu)$. Indeed it can. If $q_i < 0$ is large enough its effect through the Skorokhod dynamics can produce cases in which a $u \notin \alpha_*(x)$ minimizes $x \cdot \pi(x, q - Gu)$. This consideration would lead to an enhanced optimal policy which agrees with $\alpha_*(x)$ on the interior of K, but depends on *both* q and x when $x \in K$. Although no longer state-feedback, this enhanced control would (we expect) produce lower values of (14), but only for those negative loads $q(t)$ which, as we described above, have the effect of drawing customers backwards through the system. Even so, this enhanced control would not improve the performance of the system in the worst case sense of the differential game formulated in Section 2.4, as Theorem 1 below will assert.

2.3. **Minimum Performance Criteria.** Our strategy α_* is expressed in state feedback form. Given a load $q(\cdot)$, the associated control function $u_*(t)$ would be what results from solving the system

$$\begin{aligned} \dot{x} &= \pi(x(t), q(t) - Gu_*(t)), \\ u_*(t) &\in \alpha_*(x(t)). \end{aligned} \tag{16}$$

This system is a combination of a differential inclusion, in the sense of Filippov, and a Skorokhod problem as described above. The discussion in [2, Section 1.4] outlined how the arguments of [6] can be adapted to establish the existence of a solution. A proof of uniqueness is more elusive. The usual Filippov uniqueness condition would be that for some L

$$(x_a - x_b) \cdot [(q - Gu_a) - (q - Gu_b)] = (x_a - x_b) \cdot (Gu_b - Gu_a) \leq L\|x_a - x_b\|^2.$$

This is immediate (using $L = 0$), since by definition of $\alpha_*(\cdot)$,

$$x_a \cdot Gu_b \leq x_b \cdot Gu_a \quad \text{and} \quad x_b \cdot Gu_a \leq x_b \cdot Gu_b$$

for all $u_a \in \alpha_*(x_a)$, $u_b \in \alpha_*(x_b)$. However, as noted in [2], when coupled with Skorokhod dynamics (16) we are unable to conclude uniqueness based on existing results in the literature. Until that issue can be addressed, we must allow the possibility of multiple solutions to (16). The uniqueness question is not essential to our main result Theorem 1, however. We simply need to formulate its statement in such a way that strategies are allowed to produce more than one control function $u(t)$ for a given load $q(t)$.

In general a service *strategy* $\alpha(\cdot)$ maps a pair $x(0), q(\cdot)$ to one or more control functions $u(\cdot)$. We will write $u(t) = \alpha[x(0), q(\cdot)](t)$, although this notation is not quite proper if there are actually more than one $u(\cdot)$ associated with $x(0), q(\cdot)$ by α. Rather than formulating a cumbersome notation to accommodate this, we will simply use phrases like "for any $u(t) = \alpha[x(0), q(\cdot)](t)$" to refer to all possible $u(t)$. A strategy should produce one or more control functions for any $x(0) \in K$ and load function $q(\cdot)$ which is locally square-integrable. We insist that a strategy be nonanticipating, in the sense that if $q(s) = \tilde{q}(s)$ for all $s \leq t$, then for any $u(t) = \alpha[x(0), q(\cdot)](s)$ there is a $\tilde{u}(t) = \alpha[x(0), \tilde{q}(\cdot)](s)$ with $u(s) = \tilde{u}(s)$ for all $s \leq t$. Given any such $x(0)$, $q(\cdot)$ and a resulting $u(t)$, the general existence and uniqueness properties of the Skorokhod problem (e.g. [5]) provide a unique state trajectory $x(t) \in K$.

We will call a strategy $\alpha(\cdot)$ *non-idling* if for any nonnegative load $q_i(t) \geq 0$ for all i and all $t \geq 0$, any $x(0) \in K$, and any $u(t) = \alpha[x(0), q(\cdot)](t)$, the resulting state trajectory $x(t)$ has the property that $u_i(t) > 0$ and $x_i(t) = 0$ occur simultaneously for some i only if $x(t) = 0$. In other words, all service effort is allocated to nonempty queues, unless all queues are empty. In particular, our strategy (15) is non-idling, because if $x \in K$ and $x_i = 0$, then $x \cdot Ge_i \leq 0$, while $x \cdot Ge_j > 0$ if j is the index of the largest nonzero coordinate of x.

One of the features of single servers as in Figure 1 is that for nonnegative loads, a non-idling strategy will never invoke the Skorokhod dynamics on ∂K, until it reaches $x(T) = 0$. Indeed if $x(t) \in K \setminus \{0\}$ but $x_i(t) = 0$, the non-idling property means that $u_i(t) = 0$, from which the structure of G implies that

$$n_i \cdot Gu(t) \leq 0.$$

Since $q_i(t) \geq 0$, we conclude that

$$n_i \cdot (q(t) - Gu(t)) \geq 0.$$

Thus unless $x(t) = 0$, $\pi(x(t), q(t) - Gu(t)) = q(t) - Gu(t)$. Multiple servers do not have this property. In the case of Figure 2 for instance, if both $x_2 = x_4 = 0$, then the service effort at B is wasted and the Skorokohd dynamics will definitely come into play, regardless of x_1 and x_3. The Skorokhod dynamics will thus have a stronger influence on the design of optimal strategies for multiple server models.

When considering those fluid models that arise as limits of discrete/stochastic queueing systems, the stability criterion of Dai [4] is important for purposes of positive recurrence of the stochastic model. In that setting the load $q(t)$ is typically constant, with $1/q_i$ equal to the mean time between new arrivals in queue x_i. The stability property of [4] is simply that for any $x(0) \in K$, the state $x(t)$ reaches $x(T) = 0$ at a finite

$T \geq 0$. For our single server model, all non-idling strategies are equivalent in this respect. To see why, consider the vector

$$\nu = (\sum_{j=1}^{n} \frac{1}{s_j}, \sum_{j=2}^{n} \frac{1}{s_j}, \dots, \frac{1}{s_n})^T. \tag{17}$$

Observe that $\nu^T G = (1, 1, \dots, 1)$, so that for all $u \in \mathcal{U}$ we have $\nu \cdot Gu = 1$. For any nonnegative load $q(t)$ and the $u(t)$ resulting form *any* non-idling strategy, we have (on any interval prior to the first time T when $x(T) = 0$)

$$\begin{aligned} \frac{d}{dt}\nu \cdot x(t) &= \nu \cdot \pi(x(t), q(t) - Gu(t)) \\ &= \nu \cdot (q(t) - Gu(t)) \\ &= \nu \cdot q(t) - 1. \end{aligned}$$

Said another way, $W(x) = \nu \cdot x$ is a sort of universal Lyapunov function for all non-idling controls. Thus, the first time T for which $x(T) = 0$ does not depend on the choice of non-idling control; it only depends on the load $q(t)$. For constant nonnegative loads $q(t) \equiv q$, the Dai stability property simply boils down to

$$\nu \cdot q < 1. \tag{18}$$

Moreover if $q = (q_1, 0, \dots, 0)^T$ then this reduces to the familiar load condition of [4, (1.9)]:

$$q_1 \sum_{1}^{n} \frac{1}{s_j} < 1.$$

Figure 3 illustrates this stability property. We have taken the optimal strategy $\alpha_*(x)$ for our model with $n = 2$ and $s_1 = s_2 = 1$ and subjected the system to the constant disturbance $q(t) \equiv (.4, 0)$. For these parameters we find $\nu = (2, 1)$ and $q \cdot \nu = .8$, so that the load condition (18) is indeed satisfied. The figure illustrates the resulting trajectories of (16). When $x(t)$ reaches the ray from the origin in the direction of ν, the solution of (16) in the Fillipov sense uses the averaged control value

$$u(t) = (\frac{8}{25}, \frac{17}{25}),$$

which takes $x(t)$ to the origin in finite time directly along the ν ray. One may check that this ray consists of those x for which $\alpha_*(x) = \mathcal{U}$ is multiple-valued.

Theorem 1 below considers the optimality of α_* with respect to all strategies α that satisfy the following *minimum performance criterion*: given $x(0) \in K$ with

$$\nu \cdot x(0) < 1,$$

there exists $\delta < 1$ so that whenever $q(\cdot)$ is a nonnegative load satisfying

$$\nu \cdot q(t) \leq 1 \text{ for all } t,$$

and any $u(t) = \alpha[x(0), q(\cdot)](t)$ the resulting state trajectory satisfies

$$\nu \cdot x(t) < \delta \text{ for all } t \geq 0.$$

It is clear from our discussion above that every non-idling control satisfies the minimum performance criterion; simply take $\delta = \nu \cdot x(0)$.

2.4. **The Robust Control Problem.** We now want to define more carefully the sense in which our service strategy $\alpha_*(x)$ is optimal. We follow the general approach of Soravia [14] to formulate a differential game based on (14). The focus is on a *value function* of the form

$$V_\gamma(x) = \inf_{\alpha(\cdot)} \sup_{q(\cdot), T} \int_0^T \frac{1}{2}\|x(t)\|^2 - \frac{\gamma^2}{2}\|q(t)\|^2 \, dt. \tag{19}$$

Here $x \in K$, the outer infimum is over strategies $\alpha(\cdot)$, the inner supremum is over locally square integrable loads $q(\cdot)$, all $u(t) = \alpha[x, q(\cdot)](t)$ and bounded time intervals $[0, T]$, with $x(t)$ the resulting solution of (1) for $x(0) = x$.

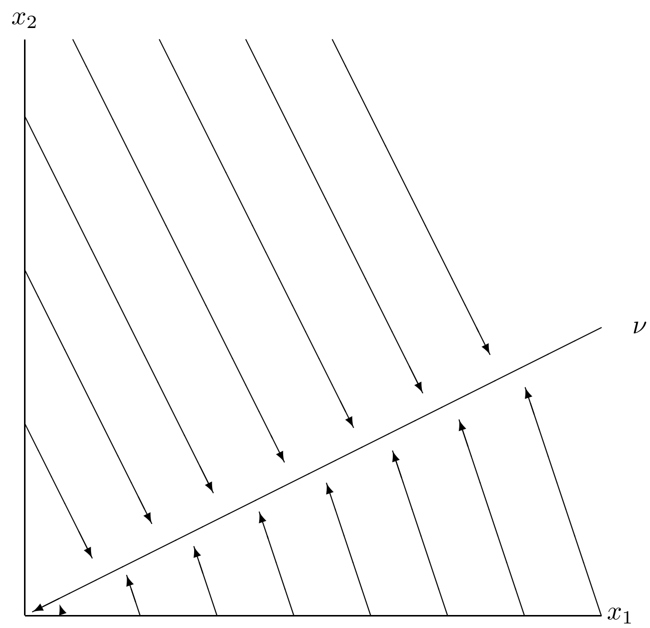

FIGURE 3. Controlled Trajectories for $q(t) \equiv (.4, 0)$.

The gain parameter $\gamma > 0$ is customary in robust control formulations. However for the structure of our problems γ scales out of the game (19) in a natural way. To see this, consider a particular load $q(t)$, control $u(t)$, and solution $x(t)$ of (5). Make the change of variables

$$s = \gamma^{-1} t, \quad \bar{x}(s) = \gamma^{-1} x(t), \quad \bar{q}(s) = q(t), \quad \bar{u}(s) = u(t).$$

Then $\dot{x}(t) = \frac{d}{ds}\bar{x}(s)$, and because K is a cone, $\pi(x, q - Gu) = \pi(\bar{x}, \bar{q} - G\bar{u})$. Thus $\bar{x}(s)$ solves (5) on the new time scale. With $S = T/\gamma$ we have

$$\int_0^T \frac{1}{2}\|x(t)\|^2 - \frac{\gamma^2}{2}\|q(t)\|^2 \, dt = \gamma^3 \int_0^S \frac{1}{2}\|\bar{x}(t)\|^2 - \frac{1}{2}\|q(s)\|^2 \, ds.$$

If $V(\cdot) = V_1(\cdot)$ is the value (19) for $\gamma = 1$, then the above implies that

$$V_\gamma(x) = \gamma^3 V(\gamma^{-1} x).$$

From this point forward we simply take $\gamma = 1$ and write V instead of V_γ.

We can only expect $V(x) < \infty$ to hold in a bounded region. To see why, imagine a load $q(t)$ which is large on some initial interval $0 \le t \le s$ so as to drive the state out to a large value X, and then $q(t)$ is chosen for $t > s$ so as to maintain $x(t) = X$ for $t > s$: $q(t) = Gu(t)$. If $\|X\| > \sup_{u \in \mathcal{U}} \|Gu\|$, the integral in (19) grows without bound as $T \to \infty$, producing infinite value. We must exclude such scenarios from the definition of the game. It turns out that the region Ω in which $V(x)$ will be finite is described using the vector ν of (17) above:

$$\Omega = \{x \in K : \; x \cdot \nu < 1\}. \tag{20}$$

We restrict the T in (19) to those for which $x(t)$ remains in Ω for all $0 \le t \le T$.

This qualification on the state in turn requires us to place some limitations on the strategies $\alpha(\cdot)$ considered as well. We need to exclude controls that "cheat" by encouraging $x(t)$ to run quickly to the outer boundary of Ω to force an early truncation of the integral in (19). Such controls could achieve an artificially low value by having actually destabilized the system. To exclude such policies we insist that all control strategies $\alpha(\cdot)$ satisfy the minimum performance criterion stated at the end of Section 2.2. With these qualifications, we can now state precisely the optimality properties of the feedback strategy $\alpha_*(x)$.

Theorem 1. *Let Ω and $\alpha_*(x)$ be as defined above and suppose the boundary verifications of Section 4 have been successfully completed. Using the control $\alpha_*(x)$, for $x \in \Omega$, define*

$$V(x) = \sup_{q(\cdot),T} \int_0^T \frac{1}{2}\|x(t)\|^2 - \frac{1}{2}\|q(t)\|^2 \, dt, \tag{21}$$

where the supremum is over all loads $q(t)$, all resulting control functions $u(t) = \alpha_*[x, q(\cdot)](t)$, and those $0 < T < \infty$ such that the controlled state from $x(0) = x$ satisfies $x(t) \in \Omega$ for all $0 < t < T$. Then, for any other control strategy $\alpha(\cdot)$ satisfying the minimum performance criterion,

$$V(x) \leq \sup_{q(\cdot),T} \int_0^T \frac{1}{2}\|x(t)\|^2 - \frac{1}{2}\|q(t)\|^2 \, dt \tag{22}$$

with the same qualifications on the supremum.

The proof of these assertions will be discussed in Section 5 below. The qualification regarding the boundary verifications of Section 4 will be explained in the last paragraph before Section 3.1.

3. Construction of the Value Function by Staged Characteristics

The proof in Section 5 of Theorem 1 is based on showing that the function $V(x)$ of (21) solves the *Hamilton-Jocobi-Issacs equation* associated with the game (19):

$$0 = H^\pi(x, DV(x)). \tag{23}$$

The Hamiltonian function is complicated by the special reflection effects on ∂K:

$$H^\pi(x,p) = \sup_q \inf_{u \in \mathcal{U}} \left\{ p \cdot \pi_K(x, q - Gu) - \frac{1}{2}\|q\|^2 + \frac{1}{2}\|x\|^2 \right\}. \tag{24}$$

The essential property of our strategy $\alpha_*(x)$ for the proof is that, given $x \in \Omega$ and $p = DV(x)$, a *saddle point* for the $\sup_q \inf_{\mathcal{U}}$ defining $H^\pi(x, DV(x))$ in (24) is given by $q^* = DV(x)$ and any $u_* \in \alpha_*(x)$. To be specific, the minimum value of

$$DV(x) \cdot \pi(x, q^* - Gu) - \frac{1}{2}|q^*|^2 + \frac{1}{2}|x|^2 \tag{25}$$

over $u \in \mathcal{U}$ is 0, achieved at $u = u_*$; and the maximum value of

$$DV(x) \cdot \pi(x, q - Gu_*) - \frac{1}{2}|q|^2 + \frac{1}{2}|x|^2 \tag{26}$$

over $q \in \mathbb{R}^n$ is 0, achieved at q^*. Together these imply (23). Our primary task is to produce $V(x)$ and establish this property of α_*.

In general (23) must be considered in the viscosity sense. Lions [13] considers the viscosity-sense formulation of a general class of problems involving Skorokhod dynamics on ∂K. Instead of working with H^π as in (24), the viscosity sense solutions are described using only the interior form of the Hamiltonian (27), together with special viscosity sense boundary conditions on ∂K. In our case it will turn out that the solution V is actually a classical one. We find the direct formulation in terms of H^π more natural for our development.

We will construct the desired solution $V(x)$ by working in the interior K°, where the complicating effects of $\pi(x,v)$ are not present: $\pi(x,v) = v$ so $H^\pi = H$ where

$$\begin{aligned} H(x,p) &= \sup_q \inf_{u \in \mathcal{U}} \left\{ p \cdot (q - Gu) - \frac{1}{2}\|q\|^2 + \frac{1}{2}\|x\|^2 \right\} \\ &= \inf_{u \in \mathcal{U}} H_u(x,p). \end{aligned} \tag{27}$$

Here H_u refers to the the *individual Hamiltonian* for $u \in \mathcal{U}$:

$$\begin{aligned} H_u(x,p) &= \sup_q \{ p \cdot (q - Gu) - \frac{1}{2}|q|^2 + \frac{1}{2}|x|^2 \} \\ &= \frac{1}{2}|p|^2 - p \cdot Gu + \frac{1}{2}|x|^2. \end{aligned} \tag{28}$$

The supremum is achieved for $q^* = p$. Also observe that for $u_* \in \mathcal{U}$ to achieve the infimum in (27) means simply that u_* maximizes $p \cdot Gu$. So for $x \in K^\circ$, (23) and the saddle point conditions (25) and (26) simply reduce to the statement that for any $u_* \in \alpha_*(x)$,

$$DV(x) \cdot Gu_* = \max_{u \in \mathcal{U}} DV(x) \cdot Gu, \quad \text{and } H_{u_*}(x, DV(x)) = 0. \tag{29}$$

We turn now to the construction of $V(x)$ of by a generalized method of characteristics. We cover Ω with a family of paths $x(t)$ as described below. The idea is that at a point $x = x(t)$ the gradient of $DV(x(t))$ should be given by the costate trajectory $p(t)$ that accompanies $x(t)$. Thus a simple covering of Ω by a family of such paths will determine the values of $DV(x)$ in Ω. Knowing $V(0) = 0$ will then determine $V(x)$ in the region. We itemize the essential features of this family of $x(t), p(t)$ in (30)–(33), and then explain their relation to the Hamilton-Jacobi-Isaacs equation and saddle point property above. To begin, the paths $x(t), p(t)$ must solve the system of ODEs

$$\begin{aligned} \dot{x} &= p - Gu_* \\ \dot{p} &= -x \end{aligned} \tag{30}$$

for some piecewise constant $u_*(t) \in \mathcal{U}$. The value of $u_*(t)$ may change from one time interval to another, but at each time t we require the optimality condition

$$p(t) \cdot Gu_*(t) = \max_{u \in \mathcal{U}} p(t) \cdot Gu. \tag{31}$$

Given an initial condition $x = x(0) \in \Omega$, we require $x(t) \in \Omega$ for $0 < t \leq T$ for some time T (depending on $x(0)$) at which both x and p reach the origin:

$$x(T) = 0 = p(T). \tag{32}$$

Lastly $0 \leq t < T$ we require

$$\|p(t)\| < \|x(t)\|. \tag{33}$$

Observe that (30) is the Hamiltonian system $\dot{x} = \frac{\partial}{\partial p} H_{u_*}(x, p)$, $\dot{p} = -\frac{\partial}{\partial x} H_{u_*}(x, p)$. This is intimately connected with the propertyt that $p(t) = DV(x(t))$ for a solution of $H_{u_*}(x, DV(x)) = 0$. We will return to this issue, near the end of Section 3.2 explaining why the manifold of (x, p) formed by our solution family is truly the graph of a gradient $p = DV(x)$. Notice also that for $0 \leq t \leq T$ we have

$$H_{u_*(t)}(x(t), p(t)) = 0 \tag{34}$$

The formula (28) for H_u shows that (34) is indeed satisfied at $t = T$ since $x(T) = 0 = p(T)$ according to (32). It is a general property of Hamiltonian systems as in (30) that the value of $H_{u_*}(x(t), p(t))$ is constant with respect to t. Property (31) implies that the jumps in $u_*(t)$ do not produce discontinuities with respect to t in (34). Therefore (34) follows as a consequence of (30) – (32). Thus (34) and (31) give us (29) for $u_*(t)$ in particular. The construction of $x(t), p(t)$ in Section 3.2 will show that $u_*(t) \in \alpha_*(x(t))$ and that (29) extends to all $u_* \in \alpha_*(x(t))$. This will provide the saddle point conditions (29) on the interior.

The equation $H(x, DV(x)) = 0$ has many solutions, if it has any at all. One property of the particular solution we want is that V be associated with the stable manifold of (30), in accord with the general approach of van der Schaft [15, 16, 17] to robust nonlinear control. We see this in the convergence to the origin of (32) above. Another important property is that

$$V(0) = 0 < V(x), \text{ for } x \neq 0. \tag{35}$$

To this end, notice that the formula (28) for an individual Hamiltonian, together with (34), impliess that

$$p(t) \cdot \dot{x}(t) = p(t) \cdot (p(t) - Gu_*) = \frac{1}{2}\|p(t)\|^2 - \frac{1}{2}\|x(t)\|^2. \tag{36}$$

So for $p(t) = DV(x(t))$, (33) is the same as saying

$$\frac{d}{dt} V(x(t)) = p(t) \cdot \dot{x}(t) < 0. \tag{37}$$

If we stipulate that $V(0) = 0$, then it will follow that $0 = V(0) = V(x(T)) < V(x(0))$, which is (35). One may wonder why we have insisted on $p(0) = 0$ in (32). Observe that (33) (in the limit as $t \to 0$) implies that $\|p(0)\|^2 \leq \|x(0)\|^2$, so $p(0) = 0$ is necessary if $x(0) = 0$.

A family of $x(t), p(t)$ as described above will give us a function $V(x)$ which has the desired saddle point properties at interior points. However for $x \in \partial K$ both (25) and (26) are complicated by the nontrivial structure of $\pi(x, v)$. We claim the $V(x)$ so constructed does in fact satisfy the saddle point conditions (25) and (26) at $x \in \partial K$ as well. We do not give a mathematical proof of this. Instead we have developed a scheme of numerical confirmation that can be applied to test this claim for any specification of s_i. This is described in Section 4. We also note the requirement in (32) above that $x(t) \in \Omega$ for all $0 \le t < T$, given $x(0) \in \Omega$. This follows if we can verify that whenever $x(t) \in \partial_i K$ then

$$n_i \cdot (p - Gu_*) \ge 0.$$

We rely on numerical tests for this fact as well. (See the discussion of (52) in Section 4.) Based on the success of these tests for numerous examples, we conjecture that (25) and (26) are true in general. The reference to the "boundary verifications of Section 4" in Theorem 1 indicates that the validity of that result depends on the success of those tests.

3.1. **Identification and Properties of the Invariant Control Vectors.** We will construct the family $x(t), p(t)$ as above by generalizing the development of [2]. The key is to look for solutions that approach the origin as in (32) using a constant control u_*. The solution of (30) with constant $\eta = Gu_*$ and terminal conditions $x(T) = 0 = p(T)$ is

$$\begin{aligned} x(t) &= -\sin(t-T)\eta \\ p(t) &= (1-\cos(t-T))\eta. \end{aligned} \tag{38}$$

Observe that for $0 \le t \le T \le \pi/2$ the values of both $-\sin(t-T)$ and $1-\cos(t-T)$ will be positive. Now consider what (31) requires of (38):

$$\eta \cdot \eta = \max_{u \in \mathcal{U}} \eta \cdot Gu. \tag{39}$$

There are only a finite number of such $\eta = Gu_*$, and they provide the key to the explicit representation of the family of solutions $x(t), p(t)$ that we desire.

We will call any $\eta \in G\mathcal{U}$ satisfying (39) an *invariant control vector*. To simplify our discussion here, let $g_i = Ge_i$ denote the columns of G. (In more general models, g_i would be the extreme points of $G\mathcal{U}$.) To say $\eta \in G\mathcal{U}$ means that η is a convex combination of the g_i: $\eta = \sum \lambda_i g_i$, some $0 \le \lambda_i$, $\sum \lambda_i = 1$. For an η as in (39) consider the set of indicies

$$J = \{i : \lambda_i > 0\}.$$

It follows from (39) that every $j \in J$ achieves the maximum value of $\eta \cdot g_i$ over $i \in N$. Therefore

$$\begin{aligned} \eta = \sum_{j \in J} \lambda_j g_j \text{ for some } \lambda_j \ge 0 \text{ with } \sum_{j \in J} \lambda_j = 1, \text{ and} \\ \eta \cdot g_j = \max_i \eta \cdot g_i, \text{ for all } j \in J. \end{aligned} \tag{40}$$

Our construction of $V(x)$ depends on the fact that there is a unique such $\eta = \eta_J$ associated with every nonempty subset $J \subseteq N$. The existence of η_J depends on properties of our particular set of g_i, but the uniqueness does not. So we present the uniqueness argument separately as the following lemma.

Lemma 2. *Suppose g_i, $i = 1, \dots, m$ are nonzero vectors in $\mathbb{R}^n$ and $J \subseteq \{1, \dots, m\}$ is nonempty. If there exists a vector η_J as described in* (40) *then it is unique. Suppose η_J and $\eta_{\tilde{J}}$ exist for both $J \subseteq \tilde{J}$. Then*

$$\eta_J \cdot \eta_{\tilde{J}} = \eta_{\tilde{J}} \cdot \eta_{\tilde{J}}. \tag{41}$$

Proof. We establish (41) first. Without assuming uniqueness, suppose $\eta_J = \sum_J \lambda_j g_j$ exists as in (40). It follows that

$$\eta_J \cdot \eta_J = \sum_J \lambda_j \, \eta_J \cdot g_j = \max_i \eta_J \cdot g_i. \tag{42}$$

Now suppose both η_J and $\eta_{\tilde{J}}$ exist for $J \subseteq \tilde{J}$. Then the same reasoning implies that

$$\eta_{\tilde{J}} \cdot \eta_J = \sum_J \lambda_j \eta_{\tilde{J}} \cdot g_j = \max_i \eta_{\tilde{J}} \cdot g_i = \eta_{\tilde{J}} \cdot \eta_{\tilde{J}},$$

which is (41). Regarding uniqueness, suppose $\eta = \sum_{j\in J} \lambda_j g_j$ and $\tilde{\eta} = \sum_{j\in J} \tilde{\lambda}_j g_j$ both satisfy (40) for the same $J = \tilde{J}$. In that case (41) implies

$$\tilde{\eta}\cdot\tilde{\eta} = \tilde{\eta}\cdot\eta = \eta\cdot\eta.$$

But then (42) implies $g_j\cdot(\eta-\tilde{\eta}) = 0$ for all $j \in J$. This means $\eta-\tilde{\eta}$ is orthogonal to the span of $\{g_j, :\ j \in J\}$. But since it is also in the span, we are forced to conclude that $\eta = \tilde{\eta}$. □

It is not difficult to determine whether or not η_J exists for a given J. If it does, the values $\mu_j = \frac{1}{\eta\cdot\eta}\lambda_j$ must be a nonnegative solution of the linear system

$$g_{j'}\cdot\sum_{j\in J}\mu_j g_j = 1,\ \text{each } j' \in J. \tag{43}$$

From such a nonnegative solution we can recover λ_j from $\lambda_j = \mu_j/\sum_{j'\in J}\mu_{j'}$, then form $\eta = \sum_J \lambda_j g_j$ and check (40). With this observation we can prove that η_J exists for all $J \subseteq N$ in our single-server model.

Theorem 2. *Assume the specific G and U_0 of our model (see Section 2). For every nonempty $J \subseteq N$ there exists a unique invariant control vector η_J. Moreover the λ_j, $j \in J$ in (40) are strictly positive.*

Proof. Let $G_J = [g_j]_{j\in J}$ be the matrix whose columns are the g_j for just those $j \in J$. Observe that (43) simply says $\mu_J = [\mu_j]_{j\in J}$ must solve

$$G_J^T G_J \mu_J = 1_J.$$

For the existence of nonnegative μ_j in (43) it is enough to show that $G_J^T G_J$ is invertible and that all entries of its inverse are nonnegative. Consider the diagonal matrix

$$S_J = \operatorname{diag}(1/s_j,\ j \in J)$$

and let

$$M_J = G_J S_J.$$

Note that M_J is nothing but G_J for the particular case of all $s_j = 1$. Since

$$(G_J^T G_J)^{-1} = S_J(M_J^T M_J)^{-1}S_J,$$

it is enough to show that $(M_J^T M_J)^{-1}$ exists and has nonnegative entries. Now observe that $M_J^T M_J$ is block diagonal

$$M_J^T M_J = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & \ddots & \vdots & \\ \vdots & & A_{k-1} & 0 \\ 0 & \dots & 0 & B \end{bmatrix}$$

where the A_ℓ and B are tridiagonal of the form

$$A_\ell = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & -1 \\ 0 & \dots & 0 & -1 & 1 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{bmatrix}.$$

One may check by explicit calculation that $(B^{-1})_{i,j} = \min(i,j)$, and with c_ℓ denoting the size of A_ℓ,

$$(A_\ell^{-1})_{i,j} = \min(i,j) - \frac{ij}{c_\ell+1}.$$

Since all entries are positive in both cases, it follows that all entries of $(M_J^T M_J)^{-1}$ and hence $(G_J^T G_J)^{-1}$ are nonnegative, as desired. Since no rows are identically 0, all μ_i are positive in (43) and therefore the respective $\lambda_j > 0$ can always be found.

Next, we need to show that $g_j\cdot\eta_J > g_i\cdot\eta_J$ for $i \notin J$, $j \in J$. First observe that $g_j\cdot\eta_J = \eta_J\cdot\eta_J > 0$ is constant over $j \in J$. Also note that for $i \neq j$

$$g_i\cdot g_j = \begin{cases} -s_i s_j & \text{if } |j-i| = 1 \\ 0 & \text{if } |j-i| \neq 1. \end{cases} \leq 0.$$

We know that $\lambda_j > 0$. So if $i \notin J$,

$$g_i \cdot \eta_J = \sum_{j \in J} \lambda_j g_i \cdot g_j \leq 0.$$

Therefore, $g_i \cdot \eta_J \leq 0$ for every $i \notin J$, and $g_j \cdot \eta_J > g_i \cdot \eta_J$. Lemma 2 gives the uniqueness. □

Observe that ν of (17) is a scalar multiple of η_N. Indeed, in the notation of the above proof, $\nu = G_N \mu_N$. It follows that $\nu = \eta_N / \eta_N \cdot \eta_N$. In particular the Ω of (20) is alternately described as

$$\Omega = \{x \in K : \; x \cdot \eta_N < \eta_N \cdot \eta_N\}. \tag{44}$$

Fundamental to our construction is the existence and uniqueness of the following representation of $x \in \Omega$ using a "nested" sequence of invariant control vectors.

Theorem 3. *Assume the specific G and U_0 of our model. Every nonzero $x \in \Omega$ has a unique representation of the form*

$$x = \sum_{j=1}^{k} a_j \eta_{J_j} \tag{45}$$

for some $0 < a_j$, $\sum a_j < 1$ and $J_1 \subsetneq \ldots \subsetneq J_k \subseteq N$. Moreover,

$$J_1 = \{j : \; g_j \cdot x = \max_i g_i \cdot x\}. \tag{46}$$

Proof. Consider any nonzero $x \in \Omega$. We first solve

$$x = G\beta.$$

The reader can check that G^{-1} has all nonnegative entries, which implies that all $\beta_i \geq 0$. Therefore every $x \in K$ can be written as

$$x = \sum_{1}^{n} \beta_j g_j \quad \text{with } \beta_j \geq 0. \tag{47}$$

Next, let $J_* = \{j : \beta_j > 0\}$, and consider the invariant control vector $\eta_{J_*} = \sum_{J_*} \lambda_j g_j$. Let $a_* = \min_{j \in J_*} \beta_j / \lambda_j$. Now consider

$$x - a_* \eta_{J_*} = \sum_{J_*} (\beta_j - a_* \lambda_j) g_j$$

Our choice of a_* implies

$$\beta_j - a_* \lambda_j \geq 0$$

for all $j \in J_*$. However for one or more $j \in J_*$, $\beta_j - a_* \lambda_j = 0$. By induction on the number of positive coefficients in (47) it is possible to write

$$x - a_* \eta_{J_*} = \sum_{1}^{m-1} a_i \eta_{J_i} \quad \text{with } a_i > 0$$

and $J_1 \subset J_2 \subset \cdots \subset J_{m-1} \subset J_*$. Simply taking $a_m = a_*$ and $J_m = J_*$ completes the induction argument.

Next notice that since $\eta_{J_i} \cdot \eta_N = \eta_N \cdot \eta_N$ (Lemma 2), (45) implies

$$\eta_N \cdot x = (\sum a_j) \|\eta_N\|^2.$$

From the hypothesis that $x \in \Omega$ we conclude that $\sum a_j < 1$.

Now consider J_1 in (45) and any $j, j' \in J_1$. Then $j, j' \in J_i$ for all i, which from (40) tells us that $g_j \cdot \eta_{J_i} = g_{j'} \cdot \eta_{J_i}$ for all i and so

$$g_j \cdot x = g_{j'} \cdot x$$

However if $j \in J_1$ but $j' \notin J_1$ then

$$g_j \cdot \eta_{J_1} > g_{j'} \cdot \eta_{J_1}$$

while for $i > 1$, $g_j \cdot \eta_{J_i} \geq g_{j'} \cdot \eta_{J_i}$ (depending on whether $j' \in J_i$ or not). We conclude that

$$g_j \cdot x > g_{j'} \cdot x.$$

This proves that J_1 is the set of j for which $g_j \cdot x$ takes its largest possible value, as claimed.

Regarding uniqueness, since G is nonsingular the β in $x = G\beta$ are uniquely determined, and then J_k from the last term of (45) is necessarily the J_* above. Since $J_{k-1} \subsetneq J_k$,

$$\sum_{j=1}^{k-1} a_j \eta_{J_j} = \sum_{j \in J_*} (\beta_j - a_k \lambda_j) g_j$$

still must have nonnegative coefficients $\beta_j - a_k\lambda_j$. But only those for $j \in J_{k-1}$ can be positive. Hence $\beta_j - a_k\lambda_j = 0$ for some $j \in J_*$. This implies that $a_k = a_*$ as well. Thus J_k and a_k are uniquely determined. Repeating the argument on

$$x - a_k \eta_{J_k} = \sum_{1}^{k-1} a_j \eta_{J_j}$$

gives the uniqueness of the other a_j, η_{J_j}. □

The following lemma records two other facts that will be used below.

Lemma 3. *Assuming the G and U_0 of our model,*

a) *All coordinates of η_N are positive.*
b) *If $x \in K$ is $x = \sum_i a_i \eta_{J_i}$ as in Theorem 3, and if $x_i = 0$, then $i \notin J_1$.*

Proof. We have already observed that $\eta_N = \nu \|\eta_N\|^2$, where ν is as in (17). This proves a). Next suppose $x_i = 0$. It follows that $x \cdot g_i \le 0$. On the other hand, there does exist j with $x \cdot g_j > 0$ (take j to be largest with $x_j > 0$ for instance). Thus the set J_1 of those j with

$$x \cdot g_j = \max_k x \cdot g_k$$

does *not* include i. □

It is important to realize that the single server re-entrant queue being studied here is special in that a unique η_J is defined for every $J \subseteq N$. This is not the case for more multiserver systems. For example, consider the re-entrant line with two servers in Figure 2. The g_i are Gu_i, $u_i \in U_0$ as in (3). Simple test calculations reveal many J for which no η_J exists. Moreover it turns out that the g_i are linearly dependent ($g_1 + g_4 = g_2 + g_3$), so the points representable as in (45) can account for at most a 3-dimensional subset of K.

3.2. **Construction of the Characteristic Family.** We can now exhibit the desired family of solutions to (30). Consider a nested sequence $J_1 \subset J_2 \subset \cdots \subset J_k$, and parameters

$$0 = \theta_0 \le \theta_1 \le \cdots \le \theta_k \le \pi/2.$$

Define coefficient functions $a_i(t)$ and $\alpha_i(t)$ according to the formulas

$$\begin{aligned} a_1(t) &= \sin((\theta_1 - t)^+) & \alpha_1(t) &= 1 - \cos((\theta_1 - t)^+) \\ a_1(t) + a_2(t) &= \sin((\theta_2 - t)^+) & \alpha_1(t) + \alpha_2(t) &= 1 - \cos((\theta_2 - t)^+) \\ &\vdots & &\vdots \\ a_1(t) + \cdots + a_k(t) &= \sin((\theta_k - t)^+) & \alpha_1(t) + \cdots + \alpha_k(t) &= 1 - \cos((\theta_k - t)^+). \end{aligned} \tag{48}$$

(Here again we use y^+ to denote the positive part: $y^+ = \max(0, y)$.) Take $T = \theta_k$. For all $0 \le t \le T$ we have

$$0 \le (\theta_1 - t)^+ \le \ldots \le (\theta_k - t)^+ \le \pi/2,$$

so that all $a_i(t)$ and $\alpha_i(t)$ are nonnegative. We claim that

$$x(t) = \sum_{i=1}^{k} a_i(t) \eta_{J_i}, \quad p(t) = \sum_{i=1}^{k} \alpha_i(t) \eta_{J_i}, \tag{49}$$

provide a solution of of (30)–(33). The derivation of (48) is based on the calculations in [2]. Here we will simply present as direct a calculation as possible. To that end, consider the partial sums appearing on the left in (48):

$$\bar{a}_i(t) = \sum_{j=1}^{i} a_j(t), \quad \bar{\alpha}_i(t) = \sum_{j=1}^{i} \alpha_j(t).$$

Consider t in one of the intervals $\theta_{\ell-1} < t < \theta_\ell$. Then for $i < \ell$ we have $\bar{a}_i(t) = \bar{\alpha}_i(t) = 0$, so that

$$\dot{\bar{a}}_i(t) = \bar{\alpha}_i(t); \quad \dot{\bar{\alpha}}_i(t) = -\bar{a}_i(t).$$

For $i \geq \ell$,

$$\dot{\bar{a}}_i(t) = \bar{\alpha}_i(t) - 1; \quad \dot{\bar{\alpha}}_i(t) = -\bar{a}_i(t).$$

Taking pairwise differences we see that

$$\begin{aligned} \dot{a}_i &= \alpha_i, & \dot{\alpha}_i &= -a_i \quad \text{for } i \neq \ell; \text{ and} \\ \dot{a}_\ell &= \alpha_\ell - 1, & \dot{\alpha}_\ell &= -a_\ell. \end{aligned}$$

Using this in (49), we find that for $\theta_{\ell-1} < t < \theta_\ell$

$$\begin{aligned} \dot{x} &= p - \eta_{J_\ell} \\ \dot{p} &= -x. \end{aligned}$$

Thus (30) is satisfied for $\theta_{\ell-1} < t < \theta_\ell$ using $Gu_* = \eta_{J_\ell}$. To confirm property (31) on that interval, observe that since $\alpha_j(t) = 0$ for $j < \ell$,

$$p(t) = \sum_{j=\ell}^{k} \alpha_j(t)\, \eta_{J_j}. \tag{50}$$

Since $\alpha_\ell(t) > 0$ and $\alpha_j(t) \geq 0$ for $j \geq \ell$ we know from (46) that $p(t) \cdot Gu$ is maximized over $u \in \mathcal{U}$ at any u for which only the $j \in J_\ell$ coordinates are positive, in particular for $Gu_* = \eta_{J_\ell}$.

Implicit in this construction is a function $p : \Omega \to \mathbb{R}^n$ defined by means of (49). Starting with $x \in \Omega$, express x as in (45). Then determine $p(x)$ using

$$p(x) = \sum_{1}^{k} \alpha_i \eta_{J_i} \tag{51}$$

where the partial sums of the α_i are determined from those of the a_i according to

$$\sum_{j=1}^{i} a_j = \bar{a}_i = \sin(\theta_i), \quad \sum_{j=1}^{i} \alpha_j = \bar{\alpha}_i = 1 - \cos(\theta_i),$$

for $0 < \theta_i \leq \pi/2$. This is the gradient map $p(x) = DV(x)$ of our solution to (23). There are several facts to record about $p(x)$ before proceeding.

Theorem 4. *The map $p(x)$ described above is locally Lipschitz continuous in Ω and satisfies the strict inequality*

$$\|p(x)\| < \|x\|$$

for all $x \in \Omega$, $x \neq 0$.

To see this, first consider what we will call a *maximal sequence* $J_1 \subset J_2 \subset \ldots \subset J_n$, i.e. $k = n$ and each J_i has precisely i elements, with $J_n = N$. Consider the x representable as in (45) for this particular maximal sequence. The maps $x \mapsto a_i \mapsto \bar{a}_i$ and $\bar{\alpha}_i \mapsto \alpha_i \mapsto p$ are linear. The maps $\bar{a}_i \mapsto \bar{\alpha}_i$ are simply

$$\bar{\alpha}_i = 1 - \sqrt{1 - \bar{a}_i^2}.$$

These are Lipschitz so long as $\bar{a}_i$ remains bounded below 1. Since $\bar{a}_i \leq \sum_1^n a_i = x \cdot \nu < 1$ in Ω, we see that $p(x)$ is indeed Lipschitz in any compact subset of Ω, constrained to those x associated with a fixed maximal sequence of J_i. If in (45) we relax the positivity assumption to $a_i \geq 0$, then we can include additional J_i so that every $x \in \Omega$ is associated with one or more maximal sequence of J_i. The Lipschitz continuity argument extends to the x associated with any given maximal sequence in this way. To finish the continuity assertions of the theorem we need to consider x on the boundary between the regions associated with distinct maximal sequences:

$$\sum a_i \eta_{J_i} = x = \sum \tilde{a}_i \eta_{\tilde{J}_i}.$$

The uniqueness assertion of Theorem 3 means that the nonzero terms of both representations agree, which implies that the corresponding terms of the expressions for p also agree:

$$\sum \alpha_i \eta_{J_i} = p(x) = \sum \tilde{\alpha}_i \eta_{\tilde{J}_i}.$$

Thus $p(x)$ is continuous across the boundaries of the regions associated with different maximal sequences. From this it follows that the (local) Lipschitz continuity assertion of the theorem is valid in all of Ω.

The argument that $\|p(x)\|^2 < \|x\|^2$ is the same as [2, pg. 334, 335]. The strict inequality comes from the fact that

$$1 - \cos(\theta_n) < \sin(\theta_n).$$

Equality occurs only for $\theta_n = 0$ ($x = 0$) or $\theta_n = \pi/2$ ($x \cdot \nu = 1$). Notice that for $x \in \Omega$, $x \cdot \nu = \sum_1^n a_i = \sin(\theta_n) < 1$ implies that $p(x) \cdot \nu = \sum_1^n \alpha_i = 1 - \cos(\theta_n) < 1$ as well.

The argument given in [2] that $p(x)$ is indeed the gradient of a function $V(x)$, $x \in \Omega$ also generalizes to the present context. In brief, the standard reasoning from the method of characteristics can be applied to each of the individual Hamiltonians H_{u_*} where $Gu_* = \eta_{J_i}$, $i = 1, \ldots n$ is succession to see that $DV(x) = p(x)$ in the region associated with a given maximal sequence J_i. Continuity across the boundaries between such regions allows us to conclude that there is indeed a C^1 function V in Ω with $DV(x) = p(x)$. Taking $V(0) = 0$ implies that $V(x) > 0$ for $x \neq 0$, by virtue of the discussion of (35) above.

Finally, we return to the connection of (31) with $\alpha_*(x)$ and (29). Since $\mathcal{U}$ is convex,

$$\max_{\mathcal{U}} x \cdot Gu = \max_i x \cdot g_i$$

and so $\alpha_*(x)$ consists precisely of those u in the convex hull of e_j, $j \in J_1$, J_1 being as in (45) for x. Because in $p(x) = \sum \alpha_i \eta_{J_i}$ the α_i are positive when the corresponding a_i are positive, we see that $\alpha_*(x)$ has the alternate description

$$\alpha_*(x) = \{u_* \in \mathcal{U} : \; p(x) \cdot Gu_* = \max_{\mathcal{U}} p(x) \cdot Gu\}.$$

In particular, the $u_*(t)$ of (31) belongs to $\alpha_*(x(t))$. Moreover $p(x) \cdot Gu_*$ has the same value for all $u_* \in \alpha_*(x)$, so

$$H_{u_*}(x, p(x)) = 0 \text{ for all } u_* \in \alpha_*(x),$$

as desired in (29).

4. Verification of Conditions on the Boundary

We have completed the construction of $V(x)$ satisfying (29) on the interior of Ω. We now consider the assertion of that for $x \in \partial K$ the resulting V remains a solution when the H of (27) is replaced by H^π as in (27), and that $q^* = DV(x)$ and any $u_* \in \alpha_*(x)$ is a saddle point, as in (25) and (26). Specifically, we want to confirm that for a given $x \in \Omega \cap \partial K$, its associated $p = DV(x)$, and any $u_* \in \alpha_*(x)$, the following hold:

$$\pi(x, p - Gu_*) = p - Gu_* \tag{52}$$

$$p \cdot (p - Gu_*) \leq p \cdot \pi(x, p - Gu) \text{ for all } u \in \mathcal{U} \tag{53}$$

$$p \cdot \pi(x, q - Gu_*) - \frac{1}{2}\|q\|^2 \leq p \cdot (p - Gu_*) - \frac{1}{2}\|p\|^2 \text{ for all } q \tag{54}$$

Since we know $H_{u_*}(x, p) = 0$, (52) and (53) imply (25), and (52) and (54) imply (26). Together these imply (23).

Our validation of (52) – (54) consists of extensive numerical testing, as opposed to a deductive proof. We will describe computational procedures below. Test calculations have been performed on numerous examples (see Section 4.4), confirming (52) – (54) to within machine precision in each case. This gives us confidence in the theoretical validity of (52) – (54), but until deductive arguments can be presented, their theoretical validity must be considered conjectural.

4.1. **Inactive Projection.** Given $x \in \partial\Omega$, the corresponding $p = p(x)$, and any $u_* \in \alpha_*(x)$, (52) is equivalent to the statement that

$$n_i \cdot (p - Gu_*) \geq 0 \text{ for all } i \in I(x).$$

This would be easy to check by direct calculation at a given x. However the second part of the following lemma provides an equivalent condition which is even easier to check.

Lemma 4. *The following are equivalent*

1. $n_i \cdot (p(x) - Gu_*) \geq 0$ *for all* $x \in \Omega \cap \partial K$ *with* $x \neq 0$*, all* $u_* \in \alpha_*(x)$*, and* $i \in I(x)$*;*
2. $p(x)_i \geq 0$ *for all* $x \in \Omega \cap \partial K$ *and* $i \in I(x)$*;*
3. $p(x)_i > 0$ *for all* i *and all* $x \neq 0$ *in the interior of* Ω*.*

Proof. Clearly (2) follows from (3) by continuity of $p(x)$. To see that (2) implies (1), recall from our discussion in Section 2.2 of the fact that α_* is a nonidling policy that that $n_i \cdot Gu_* \leq 0$ for any $u_* \in \alpha_*(x)$. Therefore (2) implies

$$n_i \cdot (p - Gu_*) \geq p_i \geq 0.$$

Finally, observe that (1) implies that the characteristic curves (49) do not exit K in forward time. From any $x(0)$ in the interior of Ω, $x(t)$ remains in K up to the time T at which $x(T) = 0$. Since $\dot{p} = -x$ and $p(T) = 0$, it follows that $p_i(x) > 0$ for all i in x is in the interior of Ω. ☐

4.2. **Control Optimality.** Now we consider an approach to checking (53) at a given $x \in \partial K$ with its associated $p = p(x)$ and any $u_* \in \alpha_*(x)$. We want to check that u_* is the minimizer of

$$p \cdot \pi(x, p - Gu)$$

over $u \in \mathcal{U}$. Observe that by virtue of (52)

$$p \cdot \pi(x, p - Gu_*) = p \cdot (p - Gu_*).$$

Since $p \cdot Gu_*$ has the same value for all $u_* \in \alpha_*(x)$ it suffices to consider any single $u_* \in \alpha_*(x)$ and to show that it gives the minimum of $p \cdot \pi(x, p - Gu)$ over $u \in \mathcal{U}$. Since this is a continuous function of u and $\mathcal{U}$ is compact, we know that there does exist a minimizing $\bar{u}$. Moreover for some $F \subseteq I(x)$, $p \cdot \pi(s, p - G\bar{u}) = p \cdot R_F(p - G\bar{u})$, according to (13). So given F we can identify $\bar{u}$ as a maximizer of $p \cdot R_F Gu$ subject to the constraints of Lemma 1 part 2). If (53) were false then an exception $\bar{u}$ would occur as a solution of such a constrained minimization problem, for some $F \subseteq I(x)$.

There is no exception to (53) for $F = \emptyset$, because in that case

$$p \cdot \pi(x, p - G\bar{u}) = p \cdot (p - G\bar{u}) \geq p \cdot (p - Gu_*).$$

If $F \neq \emptyset$ then $\bar{u}$ solves a standard linear programming problem:

$$\begin{aligned} &\text{maximize } p \cdot R_F G u \\ &\text{subject to } u \in \mathcal{U}, \\ &\qquad B_F(p - Gu) \geq 0, \text{ and} \\ &\qquad N^T_{I(x)\setminus F} R_F(p - Gu) \geq 0. \end{aligned} \tag{55}$$

If $\bar{u}$ is an exception to (53) then so is any feasible maximizer u_F to (55):

$$p \cdot \pi(x, p - Gu_F) = p \cdot \pi(x, p - G\bar{u}).$$

To verify (53) computationally we invoke a standard linear programming algorithm for (55) for each nonempty subset F of $I(x)$, and for each feasible maximizer so found, check that

$$p \cdot \pi(x, p - Gu_F) \geq p \cdot (p - Gu_*).$$

We note that when $I(x) = \{i\}$ is a singleton we only need to check $F = \{i\}$ itself. In this case it is sufficient to check that

$$p \cdot \pi(x, p - Ge_j) \leq p \cdot (p - Gu_*)$$

directly for each of $j = 1, \ldots, n$. To see why, first observe that the last constraint in (55) is satisfied vacuously. Since $B_F(p - G\bar{u}) > 0$, the same must hold for all $u \in \mathcal{U}$ sufficiently close to $\bar{u}$. It follows that $\bar{u}$ gives a local maximum of $p \cdot R_F Gu$ over $\mathcal{U}$. Since $\mathcal{U}$ is convex, it must be a global maximum. Therefore $\bar{u}$

must be a convex combination of those e_j for which $p \cdot R_F G e_j = p \cdot R_F G\bar{u}$. But observe that since $F = \{i\}$ the constraint

$$B_F(p - Gu) = -n_i \cdot (p - Gu) > 0$$

is a scalar constraint. It must therefore be satisfied by one of the e_j for which $p \cdot R_F G e_j = p \cdot R_F G\bar{u}$. This means that this e_j also solves (55). Hence when $I(x)$ is a singleton it suffices to check just the e_j as candidates for $\bar{u}$, rather than invoking the linear programming algorithm.

4.3. **Load Optimality.** Once (52) is confirmed we know that for any $u_* \in \alpha_*(x)$, $\pi(x, q^* - Gu_*) = q^* - Gu_*$ and that $q^* = p(x)$ maximizes

$$p \cdot \pi(x, q - Gu_*) - \frac{1}{2}\|q\|^2 + \frac{1}{2}\|x\|^2$$

with respect to those q for which $\pi(x, q - Gu_*) = q - Gu_*$, and that the maximal value is 0. To verify (54) we need to be sure that there are not some other $\bar{u} \in \alpha_*(x)$ and $\bar{q}$ with $\pi(x, \bar{q} - G\bar{u}) \neq \bar{q} - G\bar{u}$ and for which

$$p \cdot \pi(x, \bar{q} - G\bar{u}) - \frac{1}{2}\|\bar{q}\|^2 + \frac{1}{2}\|x\|^2 > 0. \tag{56}$$

Since $\pi(x, v)$ is continuous and piecewise linear, and since $\alpha_*(x)$ is a compact set, it follows that there does exist a $\bar{u} \in \alpha_*(x)$ and $\bar{q}$ which maximizes (56) over $q \in \mathbb{R}^n$ and $u \in \alpha_*(x)$. We derive necessary conditions contingent on the specification of the subset $F \subseteq I(x)$ for which $\pi(x, \bar{q} - G\bar{u}) = R_F(\bar{q} - G\bar{u})$. Using part 3 of Lemma 1 we know that $u = \bar{u}$ and $q = \bar{q}$ satisfy

$$B_F(q - Gu) > 0, \quad N^T_{I(x)\setminus L} R_F v > 0, \quad \text{and} \tag{57}$$

$$N^T_{L\setminus F} R_F (q - Gu) = 0. \tag{58}$$

Consider the affine set of all q satisfying (58). Since the inequalities are strict in (57), all q near $\bar{q}$ and satisfying (58) must also have $\pi(x, q - G\bar{u}) = R_F(q - G\bar{u})$. Thus $q = \bar{q}$ is a local maximum of

$$p \cdot R_F(q - G\bar{u}) - \frac{1}{2}\|q\|^2, \text{ subject to the constraint } N^T_{L\setminus F} R_F(q - G\bar{u}) = 0. \tag{59}$$

A simple calculation shows that this implies

$$\bar{q} - G\bar{u} = P_{L\setminus F}(R_F^T p - G\bar{u}),$$

where $P_{L\setminus F}$ is the orthogonal projection onto the kernel of $N^T_{L\setminus F}$. (If $F = L$ the constraint (58) is vacuous and we take $P_{L\setminus F} = I$.) Substituting this back into (59) and considering the result as a function of $\bar{u}$, it follows that $u = \bar{u}$ is a local (and hence global by convexity) solution of the quadratic programming problem:

$$\begin{aligned} &\text{maximize } p \cdot R_F P_{L\setminus F}(R_F^T p - Gu) - \frac{1}{2}\|(I - P_{L\setminus F})Gu\|^2 - \frac{1}{2}\|P_{L\setminus F} R_F^T p\|^2 + \frac{1}{2}\|x\|^2 \\ &\text{subject to } u \in \alpha_*(x), \\ &\qquad\qquad B_F P_{L\setminus F}(R_F^T p - Gu) \geq 0, \text{ and} \\ &\qquad\qquad N^T_{I(x)\setminus L} R_F P_{L\setminus F}(R_F^T p - Gu) \geq 0. \end{aligned} \tag{60}$$

To verify (54) computationally, we consider all pairs of subsets $F \subseteq L \subseteq I(x)$. For each, we invoke a standard quadratic programming algorithm to find a feasible maximizer $\bar{u}$, if any exists. If such a $\bar{u}$ is found, we take

$$\bar{q} = P_{L\setminus F} R_F^T p + (I - P_{L\setminus F})G\bar{u}$$

and then check by direct calculation whether this is an exception to (54), as in (56). If we consider all $F \subseteq L \subseteq I(x)$ but find no such exceptions, then (54) is confirmed for this x, p.

Again we note that the quadratic programming calculation can be skipped in some cases. If $F = \emptyset$, then $\pi(x, \bar{q} - G\bar{u}) = \bar{q} - G\bar{u}$ and we know there are no such exceptions to (54). Thus only $F \neq \emptyset$ need be considered. Secondly, suppose $I(x) = \{i\}$ is a singleton. Then the only case to check is $F = L = I(x)$. In that case if there is an exception to (54), $\bar{q}$ must maximize

$$p \cdot R_F(q - G\bar{u}) - \frac{1}{2}\|q\|^2 + \frac{1}{2}\|x\|^2$$

and satisfy

$$B_F(\bar{q} - G\bar{u}) > 0.$$

It follows that

$$\bar{q} = R_F^T p, \text{ and } B_F(R_F^T p - G\bar{u}) > 0.$$

But for $F = \{i\}$, the latter inequality simplifies to

$$n_i \cdot (p - G\bar{u}) < d_i \cdot p.$$

Moreover since $d_i = n_i - n_{i+1}$ (with $n_{n+1} = 0$), this is equivalent to

$$n_i \cdot G\bar{u} > n_{i+1} \cdot p. \tag{61}$$

But $i \in I(x)$ means $i \notin J_1$, so $n_i \cdot G\bar{u} \leq 0$ for all $\bar{u} \in \alpha_*(x)$. So (61) would imply $p_j < 0$ for some j. If we have already checked that $p \geq 0$ in accord with Lemma 4 and our confirmation of (52), then we can be sure no exceptions to (54) occur when $I(x)$ is a singleton. Thus we only need to appeal to the quadratic programming calculations when two or more x_i are zero.

4.4. **Test Cases.** We begin our test of (30)–(33) for a specific choice of parameters $s_1, \ldots, s_n$ by calculating all the invariant control vectors η_J. Then on each face $\partial_i K$ a rectangular grid of points $x \in \partial_i K$ with $x \cdot \eta_N \leq \eta_N \cdot \eta_N$ is constructed. For each grid point x we then compute the representation (45) and then the associated $Gu_* = \eta_{J_1}$ and $p(x)$ according to (51). We then check that all $p_i \geq 0$ in accord with Lemma 4 and carry out the constrained optimization calculations described above for all possible $\emptyset \subsetneq F \subseteq L \subseteq I(x)$. Obviously, the amount of computation involved will be prohibitive if the number of dimensions n is significant. However, for modest n the calculations can be completed in a reasonable amount of time. We have carried out these computations for numerous examples, including the following:

$$\begin{aligned}(s_1, \ldots, s_n) &= (1,1,1)\\ &= (1,3,5)\\ &= (10,5,2)\\ &= (2,16,4)\\ &= (1,1,1,1)\\ &= (1,3,7,10)\\ &= (1,12,4,23)\\ &= (12,1,23,4)\\ &= (1,1,1,1,1).\end{aligned}$$

No exceptions to (52)–(54) were found.

5. Proof of Optimality: Theorem 1

We turn now to the proof of the optimality assertions of Theorem 1. By hypothesis $V(x)$ is as constructed in Section 3.2, the saddle point conditions (25) and (26) have been confirmed, as well as the equivalent conditions of Lemma 4. We know that $\alpha_*(\cdot)$ satisfies the minimum performance criterion of Section 2.4. As explained above, $V(x) > 0$ for all $x \in \Omega$ with $x \neq 0$. Consider any load $q(\cdot)$. The argument of [2, Theorem 2.1] shows that with respect to $\alpha_*(\cdot)$, on any interval $[0,T]$ on which $x(t)$ remains in Ω, we have

$$V(x) \geq \sup_{q(\cdot),T} \int_0^T \frac{1}{2}\|x(t)\|^2 - \frac{1}{2}\|q(t)\|^2\, dt.$$

For a given $x(0) \in \Omega$, let $x(t), p(t)$ be the particular path constructed according to (30), with $x(T) = 0$. We know that $u_*(t) \in \alpha_*(x(t))$ so $x(t)$ is the controlled path produced by α_* in response to the load $q^*(t)$. Along it we have from (36) that

$$-\frac{d}{dt}V(x(t)) = \frac{1}{2}\|x(t)\|^2 - \frac{1}{2}\|q^*(t)\|^2,$$

and therefore

$$V(x(0)) = \int_0^T \frac{1}{2}\|x(t)\|^2 - \frac{1}{2}\|q^*(t)\|^2\, dt.$$

This establishes (21).

Next we consider an arbitrary strategy α satisfying the minimum performance criterion. We would like to produce a load $q(t)$ which is related to the resulting state trajectory $x(t)$ by $q(t) = DV(x(t))$. In [2, Theorem 2.3] this was accomplished by limiting α to state-feedback strategies and appealing to an existence result for Filippov solutions of the differential inclusion [2, (2.23)]. Here we only approximate such a load. By taking advantage of the properties of $\pi(x, \cdot)$, our argument will not be limited to state-feedback strategies, and will not need the Filippov existence result.

Given $x \in \Omega$ we will show that for any $\epsilon > 0$ there exists a load $q(t)$ satisfying $q_i(t) \geq 0$ and $q(t) \cdot \nu \leq 1$ for all $t > 0$, and such that (for some $u(t) = \alpha[x, q(\cdot)](t)$)

$$V(x(0)) - V(x(T)) \leq \epsilon + \frac{1}{2}\int_0^T \|x(t)\|^2 - \|q(t)\|^2 \, dt \tag{62}$$

holds for all T. The difficult question of existence for the closed loop system

$$\begin{aligned} \dot{x} &= \pi(x, q(t) - G\alpha[q(\cdot)](t)) \\ q(t) &= DV(x(t)) \end{aligned}$$

for an arbitrary strategy is easily resolved by introducing a small time lag:

$$q(t) = \begin{cases} DV(x(t - \epsilon e^{-t})) & \text{if } \epsilon e^{-t} < t \\ DV(x(0)) & \text{if } t \leq \epsilon e^{-t}. \end{cases}$$

The system can now be solved incrementally on a sequence of time intervals $[t_{n-1}, t_n]$ where $t_{n-1} = t_n - \epsilon e^{-t_n}$. For $t \in [t_{n-1}, t_n]$ the values of $q(t)$ are determined by $x(t)$ on the previous interval $[t_{n-2}, t_{n-1}]$, so the basic existence properties of the system under α subject to a prescribed $q(t)$ insure the existence of $x(t)$ and $q(t)$ as above. Let $u(t) = \alpha[x, q(\cdot)](t)$ be the associated control function. Since $q(t)$ is always a value of $DV(x)$ at some $x \in \Omega$, we know $q_i(t) \geq 0$ and $q(t) \cdot \nu \leq 1$, and the minimum performance hypothesis insures that $x(t)$ remains in a compact subset of Ω: $x(t) \cdot \nu \leq \delta$, $\delta < 1$. We must explain how the time lag leads to the $+\epsilon$ term in (62).

Observe that because $\pi(x, v) = R_F v$ for one of only a finite number of possible matrices R_F, and because $v = DV(x) - Gu$ is bounded over $x \in \Omega$, $u \in \mathcal{U}$, there is a uniform upper bound on $\dot{x}$:

$$\|\dot{x}\| = \|\pi(x, q(t) - G\alpha[q(\cdot)](t))\| \leq B.$$

Consequently,

$$\|x(t) - x(t - \epsilon e^{-t})\| \leq B \, \epsilon e^{-t}.$$

Next, on the subset of $x \in \Omega$ with $x \cdot \nu \leq \delta$, $DV(x)$ is Lipschitz; see Theorem 4. It follows that for some constant C_1 (independent of ϵ) such that

$$\|q(t) - DV(x(t))\| = \|DV(x(t - \epsilon e^{-t})) - DV(x(t))\| \leq C_1 \, \epsilon e^{-t}. \tag{63}$$

We observed previously that $\pi(x, v)$ in Lipschitz in v. It follows that for some constant C_2 and all $\epsilon, t > 0$

$$DV(x) \cdot \pi(x, q(t) - Gu(t)) - \frac{1}{2}\|q(t)\|^2 + C_2 \epsilon e^{-t} \geq DV(x) \cdot \pi(x, DV(x) - Gu(t)) - \frac{1}{2}\|DV(x)\|^2.$$

We know that

$$0 \leq DV(x) \cdot \pi(x, DV(x) - Gu(t)) - \frac{1}{2}\|DV(x)\|^2 + \frac{1}{2}\|x\|^2.$$

So it follows that

$$-DV(x) \cdot \pi(x, q(t), Gu(t)) \leq C_2 \epsilon e^{-t} + \frac{1}{2}\|x(t)\|^2 - \frac{1}{2}\|q(t)\|^2.$$

Since $\dot{x} = \pi(x, q(t) - Gu(t))$, integrating both sides over $[0, T]$ and replacing ϵ by ϵ / C_2 yields (62).

With this $q(t)$ in hand the remainder of the argument proceeds as in [2]: if there exists a sequence T_n with $x(T_n) \to 0$, then $V(x(T_n)) \to 0$ in (62) which implies

$$V(x(0)) \leq \epsilon + \sup_T \frac{1}{2}\int_0^T \|x(t)\|^2 - \|q(t)\|^2 \, dt. \tag{64}$$

Suppose no such sequence T_n exists. Then in addition to $x(t) \cdot \nu \leq \delta < 1$ we know $x(t)$ does not approach 0; it must remain in a compact subset M of $\Omega \setminus \{0\}$. From (63),

$$\int_0^T \frac{1}{2}\|x(t)\|^2 - \frac{1}{2}\|q(t)\|^2 \, dt \geq -C_1 \epsilon + \int_0^T \frac{1}{2}\|x(t)\|^2 - \frac{1}{2}\|DV(x(t))(t)\|^2 \, dt.$$

We also know from Theorem 4 that $\frac{1}{2}\|x\|^2 - \frac{1}{2}\|DV(x(t))\|^2$ has a positive lower bound. Therefore the right side in (64) is infinite. Thus (64) holds in either case. Since $\epsilon > 0$ was arbitrary, (22) follows.

6. An Example with Restricted Entry

In this section we reconsider our model in $n = 2$ dimensions, but modified so that the exogenous load $q = q_1$ is *only* applied to queue x_1. This is illustrated in Figure 4. The system equations are now

$$\dot{x}(t) = \pi(x, Mq(t) - Gu(t)) \tag{65}$$

where $q(t)$ is a scalar and

$$M = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

while the control matrix

$$G = \begin{bmatrix} s_1 & 0 \\ -s_1 & s_2 \end{bmatrix},$$

the control values $u \in \mathcal{U}$ and the constraint directions d_i all remain as before. We carry out the same general approach to constructing $V(x)$ as outlined at the beginning of Section 3. The details of the analysis are different in several regards. This is significant because it shows that our general approach is not exclusive to all the structural features of Sections 3.1 and 3.2. We will find that the optimal policy is the same $\alpha_*(x)$ as given in (15) above. In higher dimensions ($n > 2$) it is interesting to speculate whether the optimal policy would likewise remain unchanged if we removed the exogenous loads $q_i(t)$, $i > 1$. However, at present this has only been explored in 2 dimensions.

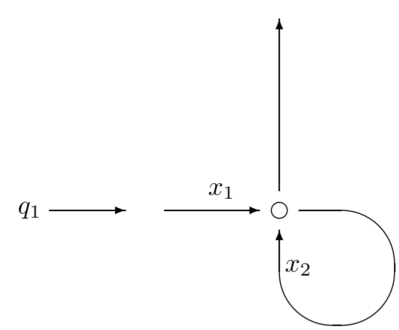

Figure 4. Re-entrant Loop with Single Input Queue

The presence of M in (65) changes the individual Hamiltonian:

$$\begin{aligned} H_u(x, p) &= \sup_q \left\{ p \cdot (Mq - Gu) - \frac{1}{2}q^2 + \frac{1}{2}|x|^2 \right\} \\ &= \frac{1}{2}p_1^2 - p \cdot Gu + \frac{1}{2}|x|^2, \end{aligned}$$

p_1 being the first coordinate of $p = (p_1, p_2)$. The supremum is achieved for $q^* = p_1$. The corresponding Hamiltonian system, for a given $u \in \mathcal{U}$, is

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} p(s) - Gu \\ \dot{p} &= -x. \end{aligned} \tag{66}$$

We calculate the invariant control vector

$$\eta_N = \frac{s_1 s_2}{s_1^2 + (s_1+s_2)^2} \begin{bmatrix} (s_1+s_2) \\ s_1 \end{bmatrix}$$

as described in Section 3.1. Other than $g_1 = Ge_1$, $g_2 = Ge_2$ and η_N, there are no additional η_J to consider. To simplify notation we will drop the subscript N:

$$\eta = \eta_N = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}.$$

The first place we find a significant difference from our previous analysis is in the calculation of a (Filippov) solution to the Hamiltonian system associated with η, analogous to (38). Previously, we did this using $\eta_J = Gu_*$, $u_* = \sum_J \lambda_j$, the λ_j being as determined by the construction of $\eta_J = \sum_J \lambda_j g_j$; see (40). But now, because of the missing p_2 term in the $\dot{x}_2$ equation of (66), we must use a $Gu_*(t)$ which is both different from η and time dependent. We seek a solution $\bar{x}(t) = a(t)\eta$, $\bar{p}(t) = \alpha(t)\eta$ (both $a(t)$ and $\alpha(t)$ nonnegative) to

$$\begin{aligned} \dot{\bar{x}}(t) &= \begin{bmatrix} \bar{p}_1 \\ 0 \end{bmatrix} - G \begin{bmatrix} \mu(t) \\ (1-\mu(t)) \end{bmatrix} \\ \dot{\bar{p}}(t) &= -\bar{x}(t) \end{aligned} \tag{67}$$

for some function $0 \le \mu(s) \le 1$. The overbar on $\bar{x}, \bar{p}$ distinguishes this special solution from the others encountered below. In light of the $\dot{\bar{p}}$ equation and the terminal conditions (32), the solution we seek must be of the form

$$\bar{p}(t) = \alpha(t) \cdot \eta, \quad \bar{x}(t) = -\dot{\alpha}(t) \cdot \eta, \quad \alpha(T) = 0, \quad \dot{\alpha}(T) = 0$$

for some function $\alpha(t) \ge 0$. Since, $\dot{\bar{x}} = -\ddot{\alpha}\eta$ is a scalar multiple of η, the right side of the $\dot{\bar{x}}$ equation in (67) must also be a scalar multiple of η. Since $\bar{p}_1 = \alpha(t)\eta_1$ this implies a relationship between $\mu(t)$ and $\alpha(t)$, which works out to be

$$\mu(t) = \frac{s_2(s_1+s_2)}{(s_1^2 + (s_1+s_2)^2)^2}(1 + s_1^2 \alpha(t)).$$

Using this we can reduce (67) to a single second order differential equation for $\alpha(t)$:

$$\ddot{\alpha}(t) + A\alpha(t) = 1,$$

where A is the constant

$$A = \frac{(s_1+s_2)^2}{s_1^2 + (s_1+s_2)^2}.$$

The solution (for initial conditions $\alpha(T) = \dot{\alpha}(T) = 0$) is $\alpha(t) = A^{-1}[1 - \cos(\sqrt{A}\,(t-T))]$. It will be convenient for the rest of this discussion to fix $T = 0$, so that

$$\alpha(t) = A^{-1}[1 - \cos(\sqrt{A}\,t)].$$

(One consequence of fixing $T = 0$ is that for a given x the $t < 0$ for which $x(t) = x$ depends on x.) For $-\frac{\pi}{2\sqrt{A}} \le t \le 0$ we confirm that $0 \le \mu(t) \le 1$, $\alpha(t) \ge 0$ and $a(t) = -\dot{\alpha}(t) \ge 0$, as we wished.

We now have the desired solution:

$$\begin{aligned} \bar{x}(t) &= -A^{-1/2} \sin(\sqrt{A}\,t) \cdot \eta \\ \bar{p}(t) &= A^{-1}[1 - \cos(\sqrt{A}\,t)] \cdot \eta. \end{aligned}$$

This special solution provides the final stage of our family of paths $x(t), p(t)$ as in (30)–(33) of Section 3, but with some adjustment. In contrast to Section 3, our $u_*(t) = [\mu(t), 1-\mu(t)]^T$ varies continuously, instead of being piecewise constant. This means we have to pay closer attention to (34). Once again it is satisfied at $t = 0$ by virtue of the terminal conditions. When we calculate $\frac{d}{dt}H_{u_*(t)}(\bar{x}(t), \bar{p}(t))$, one term does not automatically drop out:

$$\frac{d}{dt}H_{u_*(t)}(\bar{x}(t), \bar{p}(t)) = -\bar{p}(t) \cdot Gu'_*(t) = -\bar{p}(t) \cdot (g_1 - g_2)\mu'(t).$$

Since $\bar{p}$ is a scalar multiple of η and we know $\eta \cdot g_1 = \eta \cdot g_2$, we do indeed find that $H_{u_*(t)}(\bar{x}(t), \bar{p}(t)) \equiv 0$.

The analogue of (33) for this example is

$$(p_1)^2 < \|x\|^2. \tag{68}$$

This is because $\|q\|^2 - \|x\|^2 = (p_1(t))^2 < \|x(t)\|^2$ when $q^*(t) = p_(t)$. Also note that

$$\begin{aligned} p \cdot \dot{x} &= p \cdot (Mp_1 - g_i) \\ &= H_{u_*}(x, p) + \frac{1}{2}p_1^2 - \frac{1}{2}\|x\|^2 \\ &= \frac{1}{2}p_1^2 - \frac{1}{2}\|x\|^2, \end{aligned}$$

taking advantage of the fact that $H_{u_*}(x, p) = 0$. So to verify (33) along $\bar{x}, \bar{p}$ in particular, simply observe that

$$\bar{p} \cdot \dot{\bar{x}} = -\alpha\ddot{\alpha}|\eta|^2 < 0,$$

since both α and $\ddot{\alpha}$ are positive (excepting $t = 0$). In the following $\bar{x}_i(t), \bar{p}_i(t)$ will refer to the individual coordinates of this particular solution.

Our special solution $\bar{x}(t), \bar{p}(t)$ provides the final stage ($t_1 < t \leq 0$) for each of the solutions in the family described at the beginning of Section 3. The initial stage ($t < t_1 < 0$) will be a solution of (66), with $u_* =$ either e_1 or e_2, which joins $\bar{x}, \bar{p}$ at some $t_1 < 0$: $x(t_1) = \bar{x}(t_1)$, $p(t_1) = \bar{p}(t_1)$. In other words we solve (66) backwards from $\bar{x}(t_1), \bar{p}(t_1)$,for the appropriate choice of u_*. It turns out that using $u_* = e_1$ produces that part of the family which covers a region below the line $\bar{x}(\cdot)$ in the first quadrant, and using $u_* = e_2$ gives the $x(t)$ which cover a region above $\bar{x}(\cdot)$. This is illustrated in Figure 5, for parameter values $s_1 = 4$, $s_2 = 1$. Note that the region covered by this family, and hence the domain Ω of $V(x)$, is no longer the simple polygon of (20).

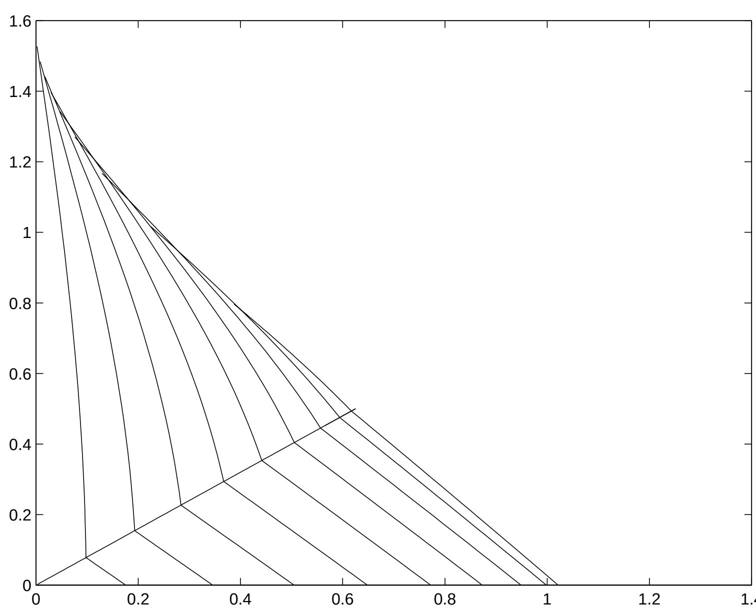

FIGURE 5. Characteristics for Restricted Entry Loop

We will need to verify that the resulting family indeed satisfies all the conditions outlined in Section 3. These verifications are discussed below. Once confirmed, this implies that the optimal control $\alpha_*(x)$ produces

e_1 if x is below the line $\bar{x}(t)$, e_2 if x is above the line, and any $u \in \mathcal{U}$ if x is on the line. So although we will not produce as explicit a construction for $x \mapsto p$ as we did in Section 3, we still find the same optimal control

$$\alpha_*(x) = \{u \in \mathcal{U} : \; x \cdot Gu = \sup_{u \in \mathcal{U}} x \cdot Gu\}.$$

6.1. **Interior Verifications.** We have already discussed properties (30)–(33) of Section 3 for the final stage of our family of solutions: $x(t) = \bar{x}(t)$, $p(t) = \bar{p}(t)$ for $t_1 \le t \le 0$. However we still need to verify (31) and (68) for the initial segment $t < t_1$. In Section 3.2 this followed from properties of the η_J and the rather explicit formulae for $x(t)$ and $p(t)$ in terms of them. Here we have not developed such an elaborate general structure. Instead we resort to direct evaluation of the needed inequalities. By solving (66) for $u_* = e_1$ with $x(t_1) = \bar{x}(t_1)$, $p(t_1) = \bar{p}(t_1)$ we obtain the formulas for the "lower" half of our family: for $t < t_1 < 0$,

$$x^{(1)}(t) = \begin{bmatrix} \bar{x}_1(t_1) \cdot \cos(t - t_1) + [\bar{p}_1(t_1) - s_1] \cdot \sin(t - t_1) \\ s_1 \cdot (t - t_1) + \bar{x}_2(t_1) \end{bmatrix}, \tag{69}$$

$$p^{(1)}(t) = \begin{bmatrix} -\bar{x}_1(t_1) \cdot \sin(t - t_1) + [\bar{p}_1(t_1) - s_1] \cdot \cos(t - t_1) + s_1 \\ -s_1 \cdot \frac{(t-t_1)^2}{2} - \bar{x}_2(t_1) \cdot (t - t_1) + \bar{p}_2(t_1) \end{bmatrix}.$$

For any $-\frac{\pi}{2\sqrt{A}} < t_1 < 0$, the above will be valid for $t < t_1$ down to the first time at which $x^{(1)}(t)$ either reaches the horizontal axis, $t = \tau_1(t_1)$, or reaches the outer boundary of Ω, $t = \tau_b(t_1)$. (For all curves appearing in the figure, $\tau_b(t_1) < \tau_1(t_1)$.) A formula for $\tau_1(\cdot)$ is easily obtained from the expressions in (69). The value $t = \tau_b(t_1)$ can be identified as the point at which the determinant of the Jacobian of $x^{(1)}$ with respect to (t_1, t) vanishes. An explicit formula is possible for $\tau_b(\cdot)$ as well. (For brevity we omit both formulas.) Thus (69) is valid for

$$\max(\tau_1(t_1), \tau_b(t_1)) \le t \le t_1 \le 0.$$

The points on the horizontal boundary $\partial_2 K$ are $x^{(1)}(\tau_1(t_1))$ for those t_1 with $\tau_b(t_1) \le \tau_1(t_1)$.

The analogous formulas for the "upper" half of our family are obtained by solving (66) for $u_* = e_2$ with $x(t_1) = \bar{x}(t_1)$, $p(t_1) = \bar{p}(t_1)$ to obtain the following expression for $t < t_1 < 0$:

$$x^{(2)}(t) = \begin{bmatrix} \bar{x}_1(t_1) \cdot \cos(t - t_1) + \bar{p}_1(t_1) \cdot \sin(t - t_1) \\ -s_2 \cdot (t - t_1) + \bar{x}_2(t_1) \end{bmatrix}, \tag{70}$$

$$p^{(2)}(t) = \begin{bmatrix} -\bar{x}_1(t_1) \cdot \sin(t - t_1) + \bar{p}_1(t_1) \cdot \cos(t - t_1) \\ s_2 \cdot \frac{(t-t_1)^2}{2} - \bar{x}_2(t_1) \cdot (t - t_1) + \bar{p}_2(t_1) \end{bmatrix}.$$

This time, for a given $-\frac{\pi}{2\sqrt{A}} < t_1 < 0$, the valid range of $t < t_1$ is slightly different. It turns out that $x^{(2)}(t)$ always reaches the outer boundary of Ω, at a time $t = \sigma_b(t_1)$, prior to contacting the vertical boundary $\partial_1 K$. (Once again, an explicit formula for $\sigma_b(\cdot)$ is obtained by setting the Jacobian of (70) equal to 0.) Thus given $t_1 < 0$, (70) is valid for

$$\sigma_b(t_1) < t < t_1 \le 0.$$

The vertical boundary itself is traced out by the solution for $t_1 = 0$:

$$x^{(2)}(t) = \begin{bmatrix} 0 \\ -s_2 t \end{bmatrix} \tag{71}$$

$$p^{(2)}(t) = \begin{bmatrix} 0 \\ \frac{1}{2} s_2 t^2 \end{bmatrix},$$

valid for $\sigma_b(0) < t \le 0$.

The availability of these formulas makes it possible to check the inequalities we need for (31) and (33). For (31) we want to verify

$$p^{(1)}(t) \cdot G(e_1 - e_2) \ge 0, \tag{72}$$

$$p^{(2)}(t) \cdot G(e_2 - e_1) \ge 0 \tag{73}$$

For (73), it turns out that

$$
\begin{aligned}
p^{(2)}(t)\cdot G(e_2-e_1) =& s_1\cdot \bar{x}_1(t_1)\cdot(\sin(t-t_1)-(t-t_1))\\
&+ s_1\cdot \bar{p}_1(t_1)\cdot(1-\cos(t-t_1)+\frac{(s_1+s_2)\cdot s_2\cdot(t-t_1)^2}{2}),
\end{aligned}
$$

which certainly is positive for $t<t_1$. We resort to numerical calculation to confirm (72). We have already noted that (33) should be replaced by (68):

$$\|x^{(i)}(t)\|^2-(p_1^{(i)}(t))^2>0,$$

for both $i=1,2$. It is a straightforward task to prepare a short computer program that, given values for s_1,s_2, evaluates (72) and (68) for a large number of $t<t_1$ pairs extending through the full range of possibilities. In this way we have confirmed the above inequalities numerically.

6.2. **The Horizontal Boundary.** Finally we must consider the influence of the projection dynamics at points $x\in\partial K$, confirming as we did in Section 4 that our $q^*=p_1$ and $u_*=\alpha_*(x)$ remains a saddle point when $\pi(x,v)$ is taken into account. This entails checking the same three facts, (52), (54), and (53) as before. We consider the two faces of ∂K separately.

The reflection matrix for $\partial_2 K$ is

$$R_{\{2\}}=I-\frac{1}{n_2\cdot d_2}d_2n_2^T=\begin{bmatrix}1&0\\0&0\end{bmatrix},$$

and

$$\pi(x,v)=\begin{cases}v & \text{if } n_2\cdot v\geq 0\\ R_{\{2\}}v & \text{if } n_2\cdot v<0\end{cases}.$$

Now observe that

$$n_2\cdot(Mq-Gu)=-n_2\cdot Gu,$$

which is independent of q. Since $u_*=e_1$ and $-n_2\cdot Ge_1=s_1>0$, it follows that $\pi(x,Mq^*-Gu_*)=Mq^*-Gu_*$, so that (52) reduces to (72). Moreover this independence of q also implies (54) since we know $q^*=p_1$ is the saddle point in the absence of projection dynamics.

Next consider (53). The $u\in\mathcal{U}$ are just

$$u=\begin{bmatrix}\mu\\1-\mu\end{bmatrix},\quad 0\leq\mu\leq 1. \tag{74}$$

Note that $u_*=e_1$ corresponds to $\mu=1$. So for (53) we want to show that the minimum of

$$p\cdot\pi(x,Mp_1-Gu) \tag{75}$$

over $0\leq\mu\leq 1$ occurs at $\mu=1$. A little algebra shows that for $0\leq\mu\leq\frac{s_2}{s_1+s_2}$ we have $n_2\cdot(Mp_1-Gu)\leq 0$, so that

$$
\begin{aligned}
p\cdot\pi(x,Mp_1-Gu)&=p\cdot R_{\{2\}}(x,Mp_1-Gu)\\
&=p_1^2-s_1p_1\mu.
\end{aligned}
$$

For $\frac{s_2}{s_1+s_2}\leq\mu\leq 1$ we have $n_2\cdot(Mp_1-Gu)\geq 0$, so that

$$
\begin{aligned}
p\cdot\pi(x,Mp_1-Gu)&=p\cdot(Mp_1-Gu)\\
&=p_1^2-s_2p_2+p\cdot G(e_2-e_1)\mu.
\end{aligned}
$$

Thus the function of μ in (75) is piecewise linear, in two segments. The slope of the right segment ($\frac{s_2}{s_1+s_2}\leq\mu\leq 1$) is

$$p\cdot G(e_2-e_1),$$

which we already know to be negative, by virtue of our work in checking (72). Thus to establish (53) we only need to check that the value for $\mu=1$ is no greater than that for $\mu=0$:

$$p_1^2\leq p_1^2-s_2p_2+p\cdot(g_2-g_1),$$

which is equivalent to $p\cdot g_1\geq 0$. This we have confirmed numerically, by evaluating

$$p^{(1)}(\tau_1(t_1)))\cdot g_1$$

for various choices of s_1, s_2 and t_1 throughout its range.

6.3. **The Vertical Boundary.** Recall that along $\partial_1 K$ we have $u_* = e_2$ and that from (71) we know $q^* = p_1 = 0$ and $p_2 > 0$. Therefore

$$n_1 \cdot (Mq^* - Gu_*) = -n_1 \cdot g_2 = 0.$$

Thus $\pi(x, Mq^* - Gu_*) = (Mq^* - Gu_*)$, confirming (52).

The reflection matrix on $\partial_1 K$ is

$$R_{\{1\}} = I - \frac{1}{n_1 \cdot d_1} d_1 n_1^T = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$

We already know that $q^* = p_1 = 0$ maximizes

$$p \cdot \pi(x, Mq - g_2) - \frac{1}{2}q^2 + \frac{1}{2}\|x\|^2$$

over those q for which $\pi(x, Mq - g_2) = Mq - g_2$. We need to consider the possibility of a global maximum among those q with $\pi(x, Mq - g_2) = R_{\{1\}}(Mq - g_2)$, namely q with $n_1 \cdot (Mq - g_2) = q \leq 0$. However,

$$p \cdot R_{\{1\}}(Mq - g_2) - \frac{1}{2}q^2 + \frac{1}{2}\|x\|^2$$

is maximized at $q = p_2 > 0$. So its maximum over $q \leq 0$ must occur at $q^* = 0$. This confirms (54).

Finally, we turn to (53). Since $q^* = p_1 = 0$, for any u as in (74) we have

$$n_1 \cdot (Mq^* - Gu) = -\mu \leq 0.$$

Therefore, after a little algebra,

$$p \cdot \pi(x, Mq^* - Gu) = p \cdot R_{\{1\}}(Mq^* - Gu) = (\mu - 1)p_2,$$

which is minimized at $\mu = 0$, since $p_2 > 0$. Since $\mu = 0$ corresponds to $u = e_2$, this verifies (53).

References

[1] R. Atar and P. Dupuis, *Characterization of the value function for a differential game formulation of a queueing network optimization problem* (Lefschetz Center for Dynamical Systems preprint #99-2), 1999.

[2] J. A. Ball, M. V. Day, and P. Kachroo, *Robust feedback control of a single server queueing system*, Mathematics of Control, Signals, and Systems **12** (1999), pp. 307–345.

[3] A. Budhiraja and P. Dupuis, *Simple necessary and sufficient conditions for the stability of constrained processes*, SIAM J. Appl. Math. **59** (1999), pp. 1686–1700.

[4] J. G. Dai, *On the positive Harris recurrence for multiclass queueing networks: a unified approach via fluid models* Ann. Appl. Prob. **5** (1995), pp. 49–77.

[5] P. Dupuis and H. Ishii, *On Lipschitz continuity of the solution mapping of the Skorokhod problem, with applications* Stochastics and Stochastics Reports **35** (1991), pp. 31–62.

[6] P. Dupuis and A. Nagurney, *Dynamical systems and variations inequalities*, Annals of Op. Res. **44** (1993), pp. 9–42.

[7] P. Dupuis and K. Ramanan, *Convex duality and the Skorokhod problem, I and II*, Prob. Theor. and Rel. Fields **115** (1999), pp. 153–195, 197–236.

[8] A. F. Filippov, *Differential Equations with Discontinuous Right Hand Sides*, Kluwer Academic Publishers, 1988.

[9] **Stochastic Networks**, F. P. Kelly and R. J. Williams ed.s, IMA Volumes in Mathematics and its Applications, volume 71, Springer-Verlag, NY (1995).

[10] D. Kinderlehrer and G. Stampacchia, **An Introduction to Variational Inequalities and their Applications**, Academic Press, 1980, New York.

[11] P. R. Kumar, *Re-entrant lines*, Queueing Systems: Theory and Applications, **13** (1993), pp. 87–110.

[12] J. M. Harrison and M. I. Reiman, *Reflected Brownian motion on an orthant*, Ann. Prob. **9** (1981), pp. 302-308.

[13] P. L. Lions, *Neumann type boundary conditions for Hamilton-Jacobi equations*, Duke Math. J. **52** (1985), pp. 793–820.

[14] P. Soravia, *H_∞ control of nonlinear systems: differential games and viscosity solutions*, SIAM J. Control and Optimization **34** (1996), pp. 1071–1097.

[15] A. J. van der Schaft, *L_2-gain analysis of nonlinear systems and nonlinear state feedback H_∞ control*, IEEE Trans. Auto. Control **AC-37** (1992), pp. 770–784.

[16] A. J. van der Schaft, *Nonlinear state space H 1 control theory*, in Essays on Control: Perspectives in the Theory and its Applications (Eds. H. L. Trentelman and J. C. Willems), Birkhauser, Basel, 1993, pp. 153-190.

[17] A. J. van der Schaft, *L2-Gain and Passivity Techniques in Nonlinear Control*, Lecture Notes in Control and Information Sciences **218**, Springer-Verlag, 1996.

[18] G. Weiss, *On optimal draining of re-entrant fluid lines*, in Stochastic Networks (F. P. Kelly and R. J. Williams, ed.s), Springer-Verlag, NY (1995).

Department of Mathematics, Blacksburg, Virginia 24061
E-mail address: day@math.vt.edu