Robust L_2-Gain Control for Nonlinear Systems with Projection Dynamics and Input Constraints: An Example from Traffic Control

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Abstract

We formulate the L_2-gain control problem for a general nonlinear, state-space system with projection dynamics in the state evolution and hard constraints on the set of admissible inputs. We develope specific results for an example motivated by a traffic signal control problem. A state-feedback control with the desired properties is found in terms of the solution of an associated Hamilton-Jacobi-Isaacs equation (the storage function or value function of the associated game) and the critical point of the associated Hamiltonian function. Discontinuities in the resulting control as a function of the state and due to the boundary projection in the system dynamics lead to hybrid features of the closed-loop system, specifically jumps of the system description between two or more continuous-time models. Trajectories for the closed-loop dynamics must be interpreted as a differential set inclusion in the sense of Filippov. Construction of the storage function is via a generalized stable invariant manifold for the flow of a discontinuous Hamiltonian vector-field, which again must be interpreted in the sense of Filippov. For the traffic control model example, the storage function is constructed explicitly. The control resulting from this analysis for the traffic control example is a mathematically idealized averaged control which is not immediately implementable; implementation issues for traffic problems will be discussed elsewhere.

1 Introduction

We consider a general nonlinear system of the form

$$\begin{aligned} \dot{x} &= f(x,q,u) \\ z &= h(x,u) \end{aligned} \tag{1}$$

where $x(t)$ is the state vector taking values in a state manifold $\mathcal{X}$, $u(t)$ is the control vector taking values in an admissible control set $\mathcal{U}$, $q(t)$ is the disturbance taking values in a space of disturbances $\mathcal{Q}$, and $h(x(t),u(t))$ is a performance or error signal taking values in a Euclidean space $\mathbf{R}^{n_z}$. The associated state-feedback H_∞-control problem (or, more properly, L_2-gain problem) is as follows: *for a preassigned tolerance level γ, find a state-feedback control $u = u_*(x)$ so that the trajectories of the closed-loop system*

$$\begin{aligned} \dot{x} &= f(x,q,u_*(x)), \quad x(0) = x_0 \\ z &= h(x,u) \end{aligned}$$

satisfy

(i) $\int_0^T \|z(t)\|^2\,dt \le \gamma^2 \int_0^T \|q(t)\|^2\,dt + \alpha(x_0)$ *for all* $T > 0$ *for some continuous function* $\alpha : \mathcal{X} \to \mathbf{R}^+$ *with* $\alpha(0) = 0$*, and*

(ii) some appropriate notion of stability.

This problem is now fairly well understood for linear systems, at least for the finite-dimensional, time-invariant case (see [6], [17], [4]). In the past decade there has been remarkable progress, including for the output measurement version of the problem where only a partial measurement of the state vector $x(t)$ is available to the controller u_* at any time t. We refer to [3], [19], [22], [21], and [27] and to [28] and [29] for particularly useful overviews of the subject. In all this work it is usually assumed that the state space $\mathcal{X}$ is a smooth manifold, which for local analysis around the equilibrium point can be assumed to be $\mathcal{X} = \mathbf{R}^n$ with equilibrium point at 0, that the admissible control set $\mathcal{U}$ and the space $\mathcal{Q}$ of possible disturbances are full Euclidean spaces $\mathbf{R}^{n_u}$ and $\mathbf{R}^{n_q}$, and that $f : \mathbf{R}^n \times \mathbf{R}^{n_u} \times \mathbf{R}^{n_q} \to \mathbf{R}^n$ and $h : \mathbf{R}^n \times \mathbf{R}^{n_u} \to \mathbf{R}^{n_z}$ are smooth functions in their respective arguments.

Hybrid systems, unlike (1), have interacting components using more than one modeling paradigm, such as ordinary differential equations, difference equations and differential inclusions. In this paper we will consider hybrid systems obtained from (1) by including additional features in the state dynamics and restricting the control set $\mathcal{U}$. In particular we wish to consider a situation in which the system dynamics include special features on the boundary of a convex set $K \subseteq \mathcal{X}$ which prevent $x(t)$ from leaving K. These are not state constraints in the usual sense that the controller must be designed to enforce them. Rather the fact that $x(t) \in K$ is a feature of the underlying system that will be satisfied regardless of the control $u(t)$. We will start with nominal dynamics $\dot{x} = f(x, u, q)$ and augment them in order to insure that state trajectories never leave K. A mathematical mechanism (which fits with the traffic control example to be described in Section 5 below and with queueing flow models in general) has been described by Dupuis and Nagurney [9] as follows. For K a closed, convex set in $\mathbf{R}^n$, x a point in $\mathbf{R}^n$ and v a direction vector in $\mathbf{R}^n$, there is a notion of "projection of v onto the generalized tangent space of K at x" $(x, v) \to \pi_K(x, v)$. Among other properties, if x is in the interior K° of K and v is any vector in $\mathbf{R}^n$, then $\pi_K(x, v) = v$. The *projected dynamical system* (PDS) proposed in [9] associated with any dynamical system $\dot{x} = f(x)$ and closed, convex set K is

$$\dot{x} = \pi_K(x, f(x)), \quad x(0) = x_0 \in K. \tag{2}$$

When x is in the interior of K we see that the vector field $\pi_K(x, f(x))$ is the same as $f(x)$, while when x is on the boundary of K we use instead the projection of the vector on the tangent space of K. This ensures that the trajectories of the PDS always remain in the set K once the initial condition $x(0) = x_0$ starts the trajectory in K. This analysis is easily adapted to the case where the disturbance signal q and control vector u enter into the dynamics. In this case, the unprojected system is of the form $\dot{x} = f(x, q, u)$ and the associated PDS (with disturbances and control inputs) with respect to the set K is

$$\dot{x} = \pi_K(x, f(x, q, u)) \tag{3}$$

This will be described in more detail in Section 2.

An Example. We will be particularly interested in the example $n = n_u = n_q = n_z = 2$ with

$$f(x, q, u) = q + Bu, \quad B = \begin{bmatrix} -s_1 & 0 \\ 0 & -s_2 \end{bmatrix}, \quad h(x, u) = x. \tag{4}$$

Here $s_1 > 0$ and $s_2 > 0$ are (known) model parameters. We take the set $\mathcal{U}$ of admissible controls to be the triangle in $\mathbf{R}^2$

$$\mathcal{U} = \left\{ \begin{bmatrix} u_1 \\ u_2 \end{bmatrix};\ 0 \le u_i \text{ for } i = 1, 2,\ u_1 + u_2 \le 1 \right\},$$

and let K be the first quadrant:

$$K = \{(x_1, x_2) :\ \text{both } x_i \ge 0\}.$$

As we shall explain in Section 5, the motivation for this particular example comes from a control problem for traffic signal phase allocation at an isolated highway intersection.

For the case of a smooth system of the form (1), the state-feedback H_∞-control problem can be solved as follows (see e.g. [29]). One introduces the pre-Hamiltonian

$$K_\gamma(x,p,q,u) = p^T f(x,q,u) + \frac{1}{2}\|h(x,u)\|^2 - \frac{\gamma^2}{2}\|q\|^2 \tag{5}$$

and seeks a saddle point $(q^*(x,p), u^*(x,p))$ such that

$$K_\gamma(x,p,q,u^*(x,p)) \le K_\gamma(x,p,q^*(x,p),u^*(x,p)) \le K_\gamma(x,p,q^*(x,p),u)$$

for all (x,p). One then defines the Hamiltonian

$$\begin{aligned} H_\gamma(x,p) &= \inf_u \sup_q K_\gamma(x,p,q,u) \\ &= K_\gamma(x,p,q^*(x,p),u^*(x,p)) \end{aligned} \tag{6}$$

and solves the Hamilton-Jacobi inequality

$$H_\gamma(x, \nabla S(x)) \le 0$$

for a C^1-function $S: \mathbf{R}^n \to \mathbf{R}^+$ with $S(0) = 0$. In practice we often replace this inequality with the Hamilton-Jacobi equation

$$H_\gamma(x, \nabla S(x)) = 0. \tag{7}$$

Then, if we use as state feedback the function $u_*(x) = u^*(x, \nabla S(x))$, S will be a storage function for the closed-loop system with supply rate $s(q,z) = \frac{\gamma^2}{2}\|q\|^2 - \frac{1}{2}\|z\|^2$ in the sense that

$$S(x(t_2)) - S(x(t_1)) \le \int_{t_1}^{t_2} \{\frac{\gamma^2}{2}\|q(t)\|^2 - \frac{1}{2}\|z(t)\|^2\}\ dt$$

for all trajectories $(q(t), x(t), z(t))$ of the closed-loop system

$$\begin{aligned} \dot{x} &= f(x,q,u^*(x,\nabla S(x))) \\ z &= h(x,u). \end{aligned}$$

The desired L_2-gain property follows from the storage function inequality:

$$\int_0^T \frac{1}{2}\|z(t)\|^2\ dt \le \int_0^T \frac{1}{2}\|z(t)\|^2\ dt + S(x(T)) \le \int_0^T \frac{\gamma^2}{2}\|q(t)\|^2\ dt + S(x(0)).$$

Moreover, if S enjoys good positivity properties, one can use S as a Lyapunov function to verify the desired stability for the closed-loop system. This completes what we shall call Stage 1 of the solution procedure.

Stage 1 merely reduces the problem to solving a Hamilton-Jacobi equation. What we shall call Stage 2 consists of some procedure for solving the Hamilton-Jacobi equation. One such procedure, very useful for purposes of theoretical analysis if not necessarily for practical computation and also described in [29], is the method of bicharacteristics. For the case of a Hamilton-Jacobi equation (7), this amounts to using the connection between a solution $S(x)$ and an invariant manifold for the Hamiltonian system of ordinary differential equations

$$\begin{aligned} \dot{x} &= \nabla_p H_\gamma(x,p) \\ \dot{p} &= -\nabla_x H_\gamma(x,p), \end{aligned}$$

One can show that in the case where $(0,0)$ is a hyperbolic equilibrium point, at least in a neighborhood Ω_γ of $(0,0)$ the stable invariant manifold $\mathcal{M}$ has the form $\{(x, \nabla S(x)) :\ x \in \Omega_\gamma\}$ where $S:\ \Omega_\gamma \to \mathbf{R}^+$ is the desired solution of the Hamilton-Jacobi equation.

In the present paper we consider modifications needed to this procedure for a system of the form (2) with projection dynamics on the state evolution and hard constraints on the admissible control set. We obtain the analogue of Stage 1 for the general case and an explicit analogue of Stage 2 (the method of bicharacteristics) for the particular example (4). The following is a precise formulation of our result for Stage 1 in the case of a general system of the form (3). The notions of inward normal vector, Filippov solution and the precise definition of π_K will be explained in Section 2.

Theorem 1 *Consider the state-feedback L_2-gain problem for the system (3) as posed above and let $H_\gamma(x,p)$ be the Hamiltonian function as defined in (6). Suppose that $S(x)$ is a C^1 real-valued function on a subset Ω_γ of K satisfying*

$$H_\gamma(x, \nabla S(x)) \le 0, \quad S(x) \ge 0, \tag{8}$$

on $\Omega_\gamma \subset K$ with boundary condition

$$n \cdot \nabla S(x) \le 0 \tag{9}$$

for all $n \in n(x)$, the set of inward normals to $x \in \partial K$. For $x \in \Omega_\gamma$ and $p \in \mathbf{R}^n$ define $u^*(x,p)$ to be the set

$$u^*(x,p) = \{u \in \mathcal{U} :\ H_{\mathrm{pre},\gamma}(x,p,u) = H_\gamma(x,p)\}.$$

where we have set $H_{\mathrm{pre},\gamma}(x,p,u) = \sup_q K_\gamma(x,p,q,u)$. Next define the set-valued state-feedback $x \to u_*(x)$ according to the rule

$$u_*(x) = u^*(x, \nabla S(x)) \tag{10}$$

and define the dynamics of the closed-loop system as solutions of the differential inclusion

$$\dot{x}(t) \in \pi_K(x, f(x,q,u_*(x)) \tag{11}$$

in the sense of Filippov. Then the dissipation inequality

$$S(x(t_2)) - S(x(t_1)) \le \int_{t_1}^{t_2} \frac{1}{2}(\gamma^2\|q(t)\|^2 - \|h(x(t), u_*(x(t)))\|^2)\ dt \tag{12}$$

is satisfied along all trajectories $(x(t), q(t))$ of the closed loop system (11) such that the state vector $x(t)$ remains in the region Ω_γ over the time interval $[t_1, t_2]$. In particular, over all trajectories $(x(t), q(t))$ for which $x(t) \in \Omega_\gamma$ for $0 \le t \le T$, the L_2-gain property

$$\int_0^T \|x(t)\|^2\ dt \le \gamma^2 \int_0^T \|q(t)\|^2\ dt + S(x(0)) \tag{13}$$

holds.

For the Stage 2 step, we obtain a precise, explicit result for the specific example (4). To what extent the result continues to hold for the case of a general PDS system (3) is an interesting topic for further investigation.

Theorem 2 *For the particular $f, h, \mathcal{U}, K$ specified in (4) above, the Hamilton-Jacobi equation (7) can be solved as follows. There exists a region $\Omega_\gamma \subset K$ and a continuous function $P:\ \Omega_\gamma \to \mathbf{R}^n$ so that the Hamiltonian system of ordinary differential equations*

$$\begin{array}{rclrcl} \dot{x} & = & \nabla_p H_\gamma(x,p), & x(0) & = & x_0 \in \Omega_\gamma \\ \dot{p} & = & -\nabla_x H_\gamma(x,p), & p(0) & = & P(x_0) \end{array} \tag{14}$$

has a solution (in the sense of Filippov), defined on an interval $0 \le t \le T_{x_0}$, with the properties that $x(t) \in \Omega_\gamma \setminus (0,0)$ and $p(t) = P(x(t))$ for $0 \le t < T_{x_0}$, and $x(T_{x_0}) = 0$. Although the initial value *problem (14) has nonunique solutions, the particular solution with $p(t) = P(x(t))$ is the only one that remains in the interior of K until reaching 0 for the first time at $t = T_{x_0}$. The function $S:\ \Omega_\gamma \to \mathbf{R}^+$ defined by*

$$S(x_0) = -\int_0^{T_{x_0}} P(x(t))^T \dot{x}(t)\ dt$$

meets all the required conditions in Theorem 1.

Theorem 1 implies that the S of Theorem 2 is a storage function for the controlled system (11) associated with our example (4). We shall show that it is in fact the minimal such storage function.

Theorem 3 *Let the function $S(x)$, $x \in \Omega_\gamma$ be the function constructed as in Theorem 2. $S(x)$ is the minimal or "available" storage function in Ω_γ for the closed-loop system (11) (which becomes (57) below for the system (4)). In other words for any solution of (2) with $x(t) \in \Omega_\gamma$ on $[t_1, t_2]$ we have*

$$S(x(t_2)) - S(x(t_1)) \leq \int_{t_1}^{t_2} \frac{1}{2}(\gamma^2 \|q(t)\|^2 - \|x(t)\|^2)\, dt.$$

If $\widetilde{S}$ is any other nonnegative function with this property, then $S(x) \leq \widetilde{S}(x)$ for every $x \in \Omega$.

We will see that the Hamiltonian vector field in (14) has discontinuities, due to discontinuities in the critical point function $u^*(x,p)$. This necessitates interpreting solutions of (14) in the Filippov sense, as stated in Theorem 2.

The controlled system (11) associated with our example (4) also has discontinuities, due to discontinuities in the state feedback $u_*(x) = u^*(x, \nabla S(x))$. In the optimal control literature (see e.g. [23]), constraints on the admissible input set often lead to bang-bang type controls with a certain number of switchings among the extreme points of the admissible control set. The problem discussed here, however, has an H_∞ or game-theoretic rather than simply optimal control formulation; the closed-loop dynamics involves an unknown disturbance term as well as the state-feedback term. The precise nature of the state-feedback depends on the choice of disturbance driving the dynamics. In particular there can be no a priori upper bound on the number of switchings between extreme control values. For some disturbances the uniquely determined control takes averaged values, which would correspond to infinitely fast switchings. We will discuss the closed-loop dynamics in more detail in Section 4.

The remainder of the paper is organized as follows. In Section 2 we introduce systematically the needed background material on projection dynamics and differential inclusions, and prove Theorem 1. In Section 3 we specialize to the system (4) and verify Theorem 2 by piecing together explicit solutions of the system (14). Section 4 establishes existence and one-sided uniqueness for the closed-loop, controlled system associated with the specific system (4), and analyzes the Hamiltonian flow for the Hamiltonian associated with the bounded-real lemma for this system. Theorem 3 will be proven there. Finally the concluding Section 5 describes the traffic-signal-control origins of the problem, mentions some implementation issues not addressed by the mathematical solution derived in the earlier sections, and formulates some open problems suggested by the results here.

2 Proof of General Results

2.1 Discontinuous Vector Fields and Filippov Solutions

We begin by describing the notion of *projected dynamical systems* as presented in [9]. Let K be a closed convex subset of $\mathbf{R}^2$. Denote by ∂K and K° the boundary and the interior of K. Given $x \in \partial K$, we define the inward unit normals to K at x by

$$n(x) = \{\gamma : \|\gamma\| = 1, \langle \gamma, x - y \rangle \leq 0, \forall y \in K\} \tag{15}$$

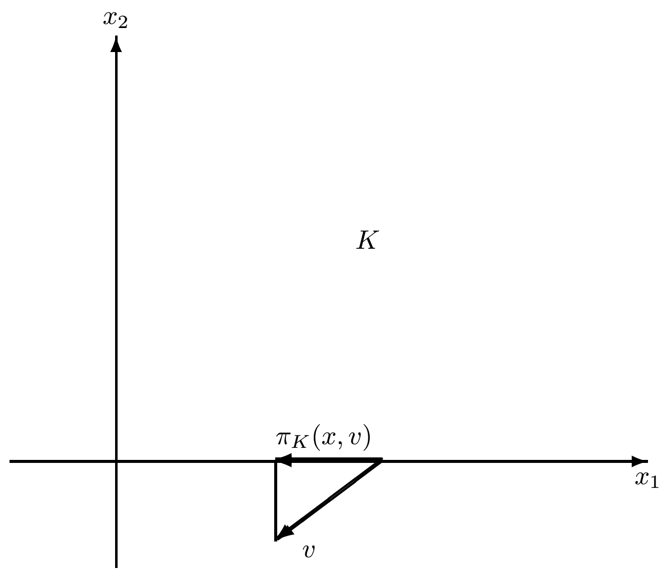

Figure 1: Projected Dynamics of the State Space

Notice that $n(x) = \{0\}$ if x is interior to K. Define the projection map $P_K : \mathbf{R}^2 \to K$ as

$$P_K(x) = \arg\min_{z \in K} \|x - z\|.$$

Given $x \in K$ and $v \in \mathbf{R}^2$, the projection of vector v at x is defined as

$$\pi_K(x,v) = \lim_{\delta \downarrow 0} \frac{P_K(x + \delta v) - x}{\delta}$$

The following properties of π_K were established in [8].

Lemma 1 *1. If $x \in K^\circ$, then $\pi_K(x,v) = v$.*

2. If $x \in \partial K$, then $\pi_K(x,v) = v + \beta(x) n^*(x)$, where

$$n^*(x) = \arg\max_{n \in n(x)} \langle v, -n \rangle \quad \text{and} \quad \beta(x) = \max\{0, \langle v, -n^*(x) \rangle\}.$$

The projected dynamical system (PDS) proposed in Dupuis and Nagurney [9] associated with a (Lipschitz continuous) dynamical system $\dot{x} = f(x)$ and closed, convex set K is

$$\dot{x} = \pi_K(x, f(x)), \quad x(0) = x_0 \in K. \tag{16}$$

From Lemma 1, we see that the vector field $\pi_K(x, f(x))$ is the same as $f(x)$ when x is in the interior of K, while when x is on the boundary of K we use instead the projection of the vector on K. This ensures that the trajectories of the PDS will always remain in the set K once the initial condition $x(0) = x_0$ starts in K.

This analysis generalizes to the case where the disturbance signal q and control vector u enter into the dynamics. In this case the unprojected system is of the form $\dot{x} = f(x,q,u)$ and the associated PDS (with disturbances and control inputs) with respect to the set K is

$$\dot{x} = \pi_K(x, f(x,q,u)), \quad x(0) = x_0 \in K. \tag{17}$$

For our prototype example in (4), (17) becomes

$$\dot{x} = \pi_K(Bu + q). \tag{18}$$

The existence-uniqueness theory for solutions of a PDS is problematical in classical ODE theory since the right hand side of the differential equation in (17) is not even continuous in x. This difficulty is addressed by relating (16) to a more detailed mechanism for the projected dynamics generally referred to as the Skorokhod problem (see [8] for details). In this way Dupuis and Nagurney were able to establish that the PDS (16) has a unique solution for every $x_0 \in K$, provided K is a polyhedron such as $K = [0, \infty) \times [0, \infty)$ for our example.

The Hamiltonian system (52) and the controlled system (57) both have discontinuous dynamics, both in the interior of K and on the boundary. At interior points we interpret these using the notion of solution developed by Filippov [13]. The generalization to the case where boundary dynamics arises as well is in [9]. To explain this in general, consider a differential equation

$$\dot{x} = f(x, t) \tag{19}$$

where $f : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ is essentially locally bounded and measurable. The solution of this differential equation is defined by Filippov as follows.

Definition 1 *A vector function $x(\cdot)$ is the solution of (19) on the interval $[t_0, t_1]$ in the sense of Filippov if $x(\cdot)$ is absolutely continuous on $[t_0, t_1]$ and for almost all t*

$$\dot{x}(t) \in K[f](x, t), \tag{20}$$

where $K[f](x, t)$ is the set-valued function defined in (21) below.

There are two equivalent definitions for $K[f](x, t)$, described in [13], [12], [11], [25] and [30]. The definition most convenient for us is

$$K[f](x, t) = \cap_{\delta>0} \cap_{\mu N=0} \overline{\text{co}} f(B(x, \delta) \backslash N, t), \tag{21}$$

where N ranges over all sets of Lebesgue measure zero. (The notation $\overline{\text{co}}$ denotes closed convex hull.) As described in [25], "the content of Filippov's solution is that the tangent vector to a solution where it exists, must lie in a convex closure of the limiting values of the vector field in progressively smaller neighborhoods around the solution point." We also note that any ambiguities in the definition of f which are confined to a set of measure 0 do not affect $K[f]$.

2.2 Proof of Theorem 1

We now turn to the proof of Theorem 1. From the definition (6) and the hypothesis that S satisfies (8) we obtain the inequality (for any q)

$$\begin{aligned} 0 &\geq H_\gamma(x, \nabla S(x)) \\ &\geq \inf_{u \in \mathcal{U}} K_\gamma(x, \nabla S(x), q, u) \\ &= K_\gamma(x, \nabla S(x), q, u_*(x)) \\ &= \nabla S(x) \cdot f(x, q, u_*(x)) - \frac{1}{2}(\gamma^2 \|q\|^2 - \|h(x, u_*(x))\|^2) \end{aligned}$$

Now since $\pi_K(f) = f + \beta n$ for some $\beta \geq 0$ and inward normal n, (9) implies that $\nabla S(x) \cdot f \geq \nabla S(x) \cdot \pi_K(f)$. Therefore for any q and all $x \in \Omega_\gamma$ we have

$$\nabla S(x) \cdot \pi_K(f(x, q, u_*(x))) \leq \frac{1}{2}(\gamma^2 \|q\|^2 - \|h(x, u_*(x))\|^2).$$

Now consider any $q(t)$ and solution $x(t)$ of (11) which remains in Ω_γ. (In the context of Theorem 1 we are *not* asserting the existence of such solutions. Rather we are deriving a necessary condition for any that do exist.) Since $x(t)$ is absolutely continuous, and S is C^1, $S(x(t))$ is (locally) absolutely continuous, with

$$\frac{d}{dt}S(x(t)) = \nabla S(x(t)) \cdot \dot{x}(t) \leq \frac{1}{2}(\gamma^2\|q(t)\|^2 - \|h(x(t), u_*(x(t)))\|^2).$$

Integrating this yields (12). Since $S(0) \geq 0$, (13) is an immediate consequence of (12). This completes the proof of Theorem 1.

3 Existence of H_∞ Solution for the Traffic Problem

We turn next to the problem of producing a solution of (8) and (9) for the specific system (4), and proving the various assertions of Theorem 2. We begin by writing out the pre-Hamiltonian for our example:

$$K_\gamma(x, p, q, u) = p^T(Bu + q) - \frac{1}{2}(\gamma^2\|q\|^2 - \|x\|^2) \tag{22}$$

This is quadratic in q with quadratic term having negative definite coefficient. Hence the maximum in q is easily computed: the maximal q is given by $q^*(x,p) = \frac{1}{\gamma^2}p$ and the semi-pre-Hamiltonian $H_{\mathrm{pre},\gamma}(x,p,u) = \max_{q\in\mathbf{R}^2} K_\gamma(x,p,q,u)$ is given by

$$H_{\mathrm{pre},\gamma}(x, p, u) = \frac{1}{2\gamma^2}\|p\|^2 + p^T Bu + \frac{1}{2}\|x\|^2 \tag{23}$$

To compute the desired saddle point, it remains to compute arg $\min_{u\in\mathbf{T}} K_\gamma(x,p,q,u)$. Since K_γ is linear in u, this is a simple linear programming problem. In particular, the minimizing $u = u^*(x)$ can always be taken to be one of the vertices of $\mathcal{U}$:

$$u^0 = (0,0), \quad u^1 = (1,0), \quad u^2 = (0,1)$$

Associated with each vertex is an individual Hamiltonian: $H_{i,\gamma}(x,p) = H_{\mathrm{pre},\gamma}(x,p,u^i)$. Explicitly, we have

$$H_{0,\gamma}(x,p) \quad = \quad \frac{1}{2\gamma^2}\|p\|^2 + \frac{1}{2}\|x\|^2 \tag{24}$$

$$H_{1,\gamma}(x,p) \quad = \quad \frac{1}{2\gamma^2}\|p\|^2 - s_1p_1 + \frac{1}{2}\|x\|^2 \tag{25}$$

$$H_{2,\gamma}(x,p) \quad = \quad \frac{1}{2\gamma^2}\|p\|^2 - s_2p_2 + \frac{1}{2}\|x\|^2. \tag{26}$$

The Hamiltonian $H_\gamma(x,p) = \min\limits_{u\in\mathbf{T}} H_{\mathrm{pre},\gamma}(x,p,u)$ is given simply by

$$H_\gamma(x,p) = \min\{H_{0,\gamma}(x,p), H_{1,\gamma}(x,p), H_{2,\gamma}(x,p)\}. \tag{27}$$

We may complete the squares in $H_{1,\gamma}(x,p)$ and $H_{2,\gamma}(x,p)$ to obtain the alternate expressions

$$H_{1,\gamma}(x,p) \quad = \quad \frac{1}{2\gamma^2}(p_1 - \gamma^2 s_1)^2 + \frac{1}{2\gamma^2}p_2^2 - \frac{\gamma^2}{2}s_1^2 + \frac{1}{2}\|x\|^2, \tag{28}$$

$$H_{2,\gamma}(x,p) \quad = \quad \frac{1}{2\gamma^2}p_1^2 + \frac{1}{2\gamma^2}(p_2 - \gamma^2 s_2)^2 - \frac{\gamma^2}{2}s_2^2 + \frac{1}{2}\|x\|^2 \tag{29}$$

An immediate observation is that the Hamilton-Jacobi inequality $H_\gamma(x, \nabla S(x)) \leq 0$ can have no global solutions. Indeed observe that $H_{0,\gamma}(x,p) \leq 0$ only for $x = 0, p = 0$. From (28) we see that $H_{1,\gamma}(x,p) \le 0$ has no solutions p if $\|x\|^2 > \gamma^2 s_1^2$. Similarly, $H_{2,\gamma}(x,p) \le 0$ has no solution if $\|x\|^2 > \gamma^2 s_2^2$.

We next compute $H_\gamma(x,p)$ explicitly. For this purpose define the following regions in the space of costates:

$\Pi_0 = \{(p_1,p_2) : p_1 < 0, p_2 < 0\}$,

$\Pi_1 = \{(p_1,p_2) : p_1 > 0 \text{ and } s_1p_1 > s_2p_2\}$,

$\Pi_2 = \{(p_1,p_2) : p_2 > 0 \text{ and } s_1p_1 < s_2p_2\}$.

We denote the boundary between Π_1 and Π_2 by

$$\Pi_{12} := \{(p_1,p_2)^T : s_1p_1 = s_2p_2,\ p_i > 0\} = \{\rho\cdot(s_2,s_1) : \rho > 0\}.$$

Then it is easily checked that $u^*(p) = \arg\min_{u\in\mathbf{T}} H_{\mathrm{pre},\gamma}(x,p,u)$ is given by

$$u^*(p) = \begin{cases} u^i & \text{if} \quad p\in\Pi_i \\ \lambda u^1 + (1-\lambda)u^2, \ \text{any } 0\le\lambda\le 1 & \text{if} \quad p\in\Pi_{12}. \end{cases} \tag{30}$$

(For p in boundaries between other pairs of the Π_i $u^*(p)$ could be given similarly, but will not be needed below.) The Hamiltonian $H_\gamma(x,p)$ is thus given by

$$H_\gamma(x,p) = \begin{cases} H_{i,\gamma}(x,p) & \text{if} \quad p\in\Pi_i \\ H_{1,\gamma}(x,p) = H_{2,\gamma}(x,p) & \text{if} \quad p\in\Pi_{12}. \end{cases} \tag{31}$$

The associated Hamilton-Jacobi equation (7) can be given piecewise by

$$\begin{aligned} H_{0,\gamma}(x,\nabla S(x)) &= 0 \quad \text{if } \nabla S(x)\in\Pi_0, \\ H_{1,\gamma}(x,\nabla S(x)) &= 0 \quad \text{if } \nabla S(x)\in\Pi_1\cup\Pi_{12}, \\ H_{2,\gamma}(x,\nabla S(x)) &= 0 \quad \text{if } \nabla S(x)\in\Pi_2\cup\Pi_{12}. \end{aligned} \tag{32}$$

Notice that for $\nabla S(x)\in\Pi_{12}$ we can write

$$H_\gamma = \lambda H_{1,\gamma} + (1-\lambda)H_{2,\gamma} \tag{33}$$

for any $0\le\lambda\le 1$. The choice of

$$\bar\lambda = (1/s_1)^2/[(1/s_1)^2 + (1/s_2)^2] \tag{34}$$

will be particularly appropriate below.

3.1 The Strategy

The first step toward constructing a solution $S(x)$ of (32) is to postulate that the state-feedback control $u(x)$ arising from (10) switches between the two values u^1 and u^2 across a half-line Γ:

$$\Gamma = \{(x_1, mx_1) : x_1 > 0\}.$$

We shall see later that there is precisely one value, $m = s_1/s_2$, for which the construction below succeeds. For now we consider $m > 0$ an unspecified parameter. Denote by X_- the region in the first quadrant strictly below Γ and by X_+ the region in the first quadrant strictly above Γ. Thus we consider the following class of controls:

$$u(x) = \begin{cases} u^1 = (1,0)^T & \text{for } x\in X_-, \\ u^2 = (0,1)^T & \text{for } x\in X_+, \\ (\lambda, 1-\lambda)^T, \ \text{some } 0\le\lambda\le 1 & \text{for } x\in\Gamma. \end{cases} \tag{35}$$

For $x \in \Gamma$ $u(x)$ is a convex set of possible control values. The selection of which one to actually use (i.e. the choice of λ) is related to the Filippov notion of solution of a differential equation with discontinuous right hand side; we will discuss this more later.

The general theory stipulates that the control $u(x)$ be given by that value of u which minimizes $H_{\mathrm{pre},\gamma}(x,p,u)$, i.e.

$$u_*(x) = u^*(\nabla S(x)), \tag{36}$$

with u^* as in (30). Under the presumption that rules (35) and (36) are the same, it follows that along the line Γ, $\nabla S(x)$ should have a value in Π_{12}, the boundary between regions Π_1 and Π_2. For this reason we speculate that for x on Γ our solution S of (32) should be such that $\nabla S(x)$ is a scalar multiple of $(1/s_1, 1/s_2)^T$. We develop this idea explicitly by defining unit vectors in to Γ and Π_{12}:

$$\begin{array}{rcll} \kappa &=& C_\kappa(1,m), & C_\kappa = (1+m^2)^{-1/2} \\ \eta &=& C_\eta(1/s_1, 1/s_2), & C_\eta = [(1/s_1)^2 + (1/s_2)^2]^{-1/2} \end{array} \tag{37}$$

(When we determine that $m = s_1/s_2$ below, it will follow that $\kappa = \eta$. But for now we allow them to be distinct.) If we describe Γ in terms of a parameter a,

$$x = a\,\gamma C_\eta \kappa,$$

then our speculation is that for such an x, $\nabla S(x)$ is given, for some $\alpha > 0$, by

$$p = \nabla S(x) = \alpha\,\gamma^2 C_\eta \eta.$$

(The additional scaling factors γ and C_η are included to achieve the simplified equation (39) below.) Consider the Hamilton-Jacobi equation (32) restricted to Γ with $\nabla S(x)$ of this special form. Using (33) with $\bar\lambda$ as in (34), we arrive at the equation

$$\begin{aligned} 0 &= \bar\lambda H_{1,\gamma}(x, \nabla S(x)) + (1-\bar\lambda) H_{2,\gamma}(x, \nabla S(x)) \\ &= \frac{1}{2\gamma^2}\|p\|^2 - C_\eta \eta \cdot p + \frac{1}{2}\|x\|^2 \\ &= \frac{1}{2}\gamma^2 C_\eta^2[\alpha^2 - 2\alpha + a^2]. \end{aligned} \tag{38}$$

Solving this equation for α and taking the smaller of the two solutions (since we are expecting to find the minimal solution of (32)) yields

$$\alpha = \alpha(a) := 1 - \sqrt{1-a^2}. \tag{39}$$

The condition $|a| \leq 1$ is necessary for a solution to exist. We take Γ_0 to be the corresponding segment of Γ:

$$\Gamma_0 = \{x = a\,\gamma C_\eta \kappa : \; 0 \leq a < 1\}. \tag{40}$$

It will be convenient to observe that the solution (39) can be described by

$$a = \sin(\theta), \quad \alpha = 1 - \cos(\theta) \quad \text{for some } 0 \leq \theta \leq \pi/2. \tag{41}$$

We now recover S along Γ_0 by integrating its gradient: for $x = a\,\gamma C_\eta \kappa$

$$\begin{aligned} S(x) &= \int_0^a \frac{d}{dy} S(y\,\gamma C_\eta \kappa)\,dy \\ &= \int_0^a \nabla S(y\,\gamma C_\eta \eta)^T \cdot \gamma C_\eta \kappa\,dy \\ &= \gamma^3 C_\eta^2 \kappa \cdot \eta \int_0^a \alpha(y)\,dy, \end{aligned} \tag{42}$$

which has a simple closed form expression when needed.

Next, since (35) postulates that u^1 is the optimal control in X_- we expect $H_{1,\gamma}$ to be the effective form of the Hamiltonian there. We consider the associated Hamiltonian system

$$\begin{array}{rclcl} \dot{x} & = & \frac{\partial^T H_{1,\gamma}}{\partial p}(x,p) & = & \frac{1}{\gamma^2}p - (s_1,0)^T \\ \dot{p} & = & -\frac{\partial^T H_{1,\gamma}}{\partial x}(x,p) & = & -x \\ \dot{z} & = & p^T\dot{x} & = & \frac{1}{\gamma^2}p^Tp - s_1p_1 \end{array} \tag{43}$$

and use initial conditions

$$\begin{array}{rcl} x(t_1) & = & a\,\gamma C_\eta \kappa \\ p(t_1) & = & \alpha(a)\gamma^2 C_\eta \eta, \\ z(t_1) & = & S(x(t_1)) \end{array} \tag{44}$$

to generate trajectories

$$(x(t,x_1),\, p(t,x_1),\, z(t,x_1)).$$

We hope to cover at least some region $\Omega_- \subseteq X_-$ with such trajectories. This procedure for computing a solution of $H_{1,\gamma}(x,\nabla S(x)) = 0$ in Ω_- amounts to the classical method of characteristics for first order partial differential equations. (For recent work on this method in the context of Hamilton-Jacobi equations as we have here, see [29] and [5].) Indeed, suppose we can define a smooth function $S(x_1,x_2)$ for $x=(x_1,x_2)$ in a neighborhood Ω_- of Γ_0 implicitly (or in parameterized form) by

$$S(x) = z(t,x_1) \text{ if } x = x(t,x_1) \tag{45}$$

where $(x(t,x_1), p(t,x_1), z(t,x_1))$ are as in (43) and (44). Then (according to the method of characteristics) S will solve the Hamilton-Jacobi equation $H_{1,\gamma}(x,\nabla S(x)) = 0$ with boundary condition $S(x)$ given by (42). Moreover the gradient of S will be given implicitly by

$$\nabla S(x) = p(t,x_1) \text{ if } x = x(t,x_1). \tag{46}$$

Moreover, if the costate vector p stays in region Π_1 we will have a solution S of the Hamilton-Jacobi equation (32) in Ω_-. We will see shortly that these hoped for features in the solutions of (43) and (44) are present when (43) is solved *backwards* in time: $t < t_1$. Similarly, starting with (44) on Γ_0 and solving the Hamiltonian flow associated with $H_{2,\gamma}$ backwards in time will allow us to construct $S(x)$ for x in a region $\Omega_+ \subseteq X_+$, with the associated gradient $\nabla S(x)$ remaining in region Π_2.

We still need to determine the slope m of the "switching line" Γ. The bicharacteristic equations associated with $H_{1,\gamma}$ are

$$\dot{x} = \frac{1}{\gamma^2}p - (s_1,0)^T, \quad \dot{p} = -x.$$

For an initial $x = x_1(1,m)^T \in \Gamma$ and $p = \rho(s_2,s_1)^T \in \Pi_{12}$, when integrated in reverse time these should enter the region X_- in state space, and the region Π_1 in costate space. At the initial values this means

$$(-m,1)^T \cdot \dot{x} \geq 0 \quad \text{and} \quad (-s_1,s_2)^T \cdot \dot{p} \geq 0.$$

The second condition in particular says

$$-(-s_1,s_2)^T \cdot x_1(1,m)^T \geq 0, \quad \text{for } x_1 \geq 0,$$

from which we get that $m \leq s_1/s_2$. Repeating the argument for the bicharacteristics associated with $H_{2,\gamma}$ yields the reverse inequality: $m \geq s_1/s_2$. Thus $m = s_1/s_2$ is necessary for the construction proposed above to work. This implies that $\kappa = \eta$ in (37).

Sample Calculations. Now we need to verify that, for $m = s_1/s_2$, the construction described above does indeed succeed in producing a nonnegative C^1 solution of (32) in a region $\Omega = \Omega_- \cup \Gamma \cup \Omega_+$ of the first quadrant, as claimed. We first offer experimental verification using sample calculations. Using the software package Mathematica, we have produced graphical plots of various pieces of the solution of the system of ordinary differential equations (43) with (44). In this way we can view a plot of the graph of S and also check that the costate vector p remains in the appropriate region Π_1 or Π_2.

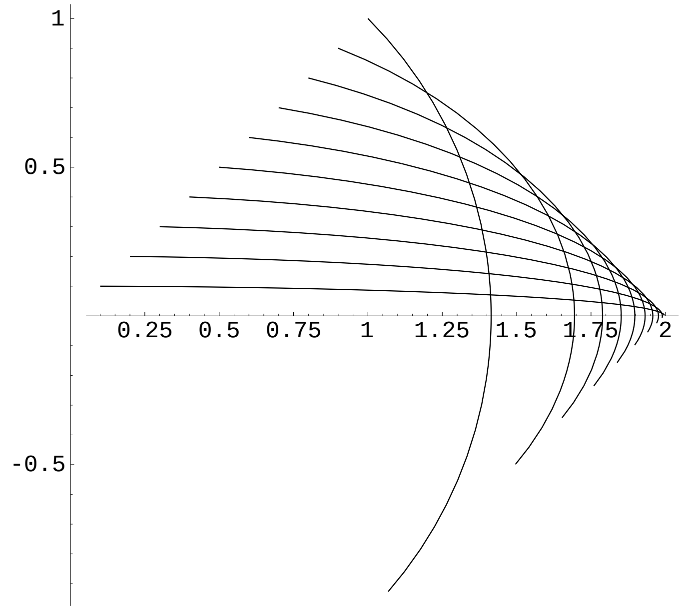

Figure 2: x Plot

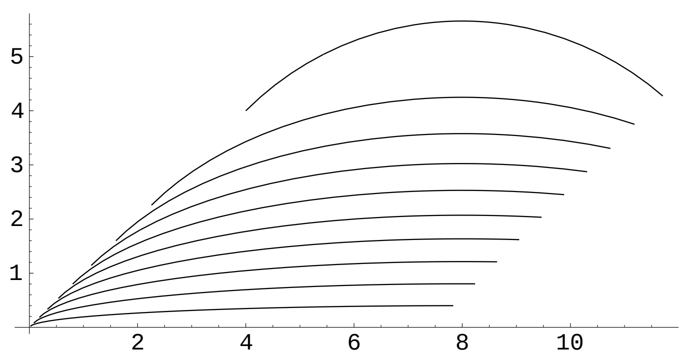

Figure 3: p Plot

We take as an example the case of $s_1 = s_2 = 1$ with $\gamma = 2$. (In this case the slope of the switching line Γ is $m = 1$, as we would expect since the problem is symmetric in x_1 and x_2.) Figure 2 shows the state trajectories of (43) for $t < t_1$, with initial conditions (44). We see that these trajectories cover a region $\Omega_- \subseteq X_-$, consisting of the triangle with vertices at $(0, 0), (1, 1), (2, 0)$. Figure 3 shows

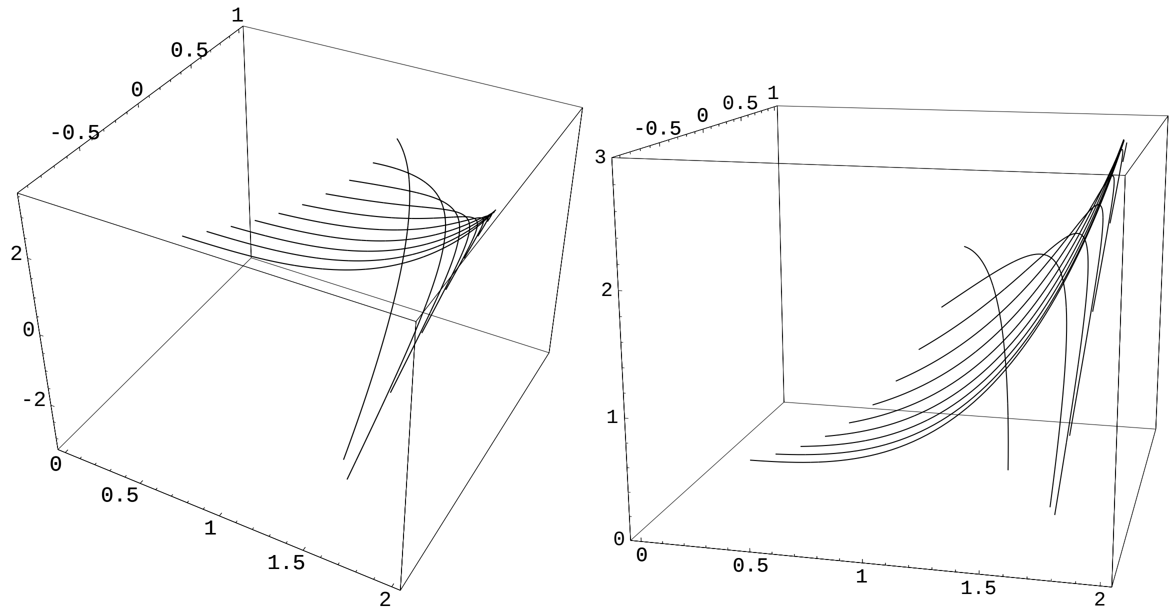

Figure 4: Graph of S with two different views

the associated p values, which do indeed remain in Π_1 as desired. Figure 4 plots the values of z (vertically) with respect to (x_1, x_2) (horizontal axes), providing curves on the graph of S.

We note in Figure 2 that some $x \in \Omega_-$ lie on more than one state trajectory, so that the definition of $S(x)$ is ambiguous. Multivaluedness of S in the context of general Hamilton-Jacobi equations is an interesting phenomenon related to shocks and demanding notions of generalized solutions (such as "viscosity solutions") of the Hamilton-Jacobi equation (see e.g. [20], [2] and [5]). Such notions are not needed here however. Those x for which $S(x)$ is multivalued involve state-trajectories which have already touched the boundary of the region Ω_-. By truncating trajectories at the time they contact $\partial\Omega_-$ and only using the truncated trajectories in the construction of Theorem 1, S becomes single-valued. This truncation is appropriate for several reasons. Notice in Figure 4 that the values of z beyond the truncation point are less than the values for the same x on a trajectory prior to truncation. Only by limiting our construction to truncated trajectories do we obtain a *continuous* function $S(x)$. Secondly, we will see that any nonnegative storage function $S(x)$ that agrees with (42) on Γ_0 must satisfy $z(t) \leq S(x(t))$. This implies that for our construction to be successful we needed to take $S(x)$ to be the largest of the possible values $z(t)$ for $x(t) = x$, which again means discarding the portions of trajectories beyond the truncation point. To see why $z(t) \leq S(x(t))$ is necessary, first notice that we have $H_{1,\gamma} = 0$ along the solutions of (43), (44). (That $H_{1,\gamma}$ is constant along these trajectories is a general porperty of Hamiltonian systems. That $H_{1,\gamma} = 0$ at $t = t_1$ follows from the construction of $\alpha(a)$ in (39) and (38).) With $q(t) = \frac{1}{\gamma^2}p(t)$, the fact that $H_{1,\gamma} = 0$ can be restated as

$$\dot{z}(t) = p(t) \cdot (q(t) + Bu^1) = \frac{1}{2}\gamma^2\|q(t)\|^2 - \frac{1}{2}\|x(t)\|^2.$$

Now our presumption that (35) and (36) agree means that the x equation of (43) can be expressed

$$\dot{x}(t) = q(t) + Bu_*(x(t)).$$

Therefore, applying (12) with $t < 0$ we must have

$$\begin{array}{rcl} S(x(t_1)) - S(x(t)) & \leq & \int_t^{t_1} \{\frac{1}{2}\gamma^2 \|q(s)\|^2 - \frac{1}{2}\|x(s)\|^2\} \, ds \\ & = & \int_t^{t_1} \dot{z}(s) \, ds \\ & = & z(t_1) - z(t) \end{array}$$

Since $z(t_1) = S(x(t_1))$ it follows that $z(t) \leq S(x(t))$, as claimed.

3.2 Explicit Solution and Proof of Theorem 2

We can verify these observed features by calculating solutions of (43), (44) explicitly. To this end we parametrize $x = (x_1, x_2)$ in terms of new parameters (a, b) and $p = (p_1, p_2)$ in terms of new parameters (α, β) according to

$$\begin{array}{rcl} x & = & a\,\gamma C_\eta \eta + b\,\gamma(s_1, 0) \\ p & = & \alpha\,\gamma^2 C_\eta \eta + \beta\,\gamma^2(s_1, 0). \end{array}$$

Note that the region X_- (where now $m = s_1/s_2$) in state space corresponds to $a > 0, b > 0$ while Γ corresponds to $b = 0$. Similarly, the region Π_1 in costate space corresponds to $\alpha > 0, \beta > 0$ while the boundary line Π_{12} corresponds to $\beta = 0$. Then the system (43) reduces to the following system of equations for the coefficients a, α, b, β:

$$\begin{array}{rcl} \dot{a} & = & \frac{1}{\gamma}\alpha \\ \dot{\alpha} & = & -\frac{1}{\gamma}a \\ \dot{b} & = & \frac{1}{\gamma}(\beta - 1) \\ \dot{\beta} & = & -\frac{1}{\gamma}b. \end{array}$$

Consider the following family of solutions, for parameters $t_1, \theta \geq 0$:

$$\begin{array}{rcl} b(t) & = & \sin\left(\frac{t_1 - t}{\gamma}\right) \\ \beta(t) & = & 1 - \cos\left(\frac{t_1 - t}{\gamma}\right) \\ a(t) & = & -\sin\left(\frac{t_1 - t}{\gamma}\right) + \sin\left(\theta + \frac{t_1 - t}{\gamma}\right) \\ \alpha(t) & = & \cos\left(\frac{t_1 - t}{\gamma}\right) - \cos\left(\theta + \frac{t_1 - t}{\gamma}\right). \end{array} \tag{47}$$

Observe that $b(t_1) = \beta(t_1) = 0$, so that $x(t)$ reaches Γ at $t = t_1$. Moreover

$$a(t_1) = \sin(\theta), \quad \alpha(t_1) = 1 - \cos(\theta),$$

so that the values of $x(t_1)$ and $p(t_1)$ correspond with our previous calculations (41), (44) along Γ_0. Now provided $0 \leq \theta + \frac{t_1}{\gamma} \leq \frac{\pi}{2}$ all coefficients will be positive for $0 \leq t \leq t_1$. (For $\alpha(t)$ this follows from $\dot{\alpha}(t) \leq 0$ and $\alpha(t_1) \geq 0$.) This implies $x(t) \in X_-$ and $p(t) \in \Pi_1$ as claimed. So the remaining question is for what $x(0) \in X_-$ such parameters t_1, θ exist. We certainly need $b(0) = \sin(t_1/\gamma) \leq 1$. Beyond this we just need

$$a(0) + b(0) = \sin\left(\theta + \frac{t_1}{\gamma}\right) \leq 1.$$

Note however that $x(0) \cdot \eta = \gamma C_\eta(a(0) + b(0))$, so that for $x(0) \in X_-$ the necessary and sufficient condition for solvability is that $x(0)$ belong to the triangle Ω_γ given by

$$\Omega_\gamma = \{x(0) : \; x(0) \cdot \eta \leq \gamma C_\eta\}. \tag{48}$$

Note also that Γ_0 in (40) is just the set of $x \in \Gamma$ satisfying the same inequality: $x \cdot \eta \leq \gamma C_\eta$. The function $P(x)$ referred to in the statement of Theorem 2 is the result of solving for $\alpha(0), \beta(0)$ subject to $0 \leq \theta \leq \theta + \frac{t_1}{\gamma} \leq \frac{\pi}{2}$, given $x(0) \in \Omega_\gamma$. More explicitly, $P(x)$ is defined as follows: *given* $x(0) = a(0)\gamma C_\eta \eta + b(0)\gamma(s_1, 0) \in \Omega_\gamma \cap X_-$, *solve for parameters* t_1 *and* θ *so that*

$$a(0) = -\sin\left(\frac{t_1}{\gamma}\right) + \sin\left(\theta + \frac{t_1}{\gamma}\right), \quad b(0) = \sin(\frac{t_1}{\gamma}),$$

then define

$$P(x(0)) = \alpha(0)\gamma^2 C_\eta \eta + \beta(0)\gamma^2(s_1, 0) \tag{49}$$

with

$$\alpha(0) = \cos\left(\frac{t_1}{\gamma}\right) - \cos\left(\theta + \frac{t_1}{\gamma}\right) \quad \beta(0) = 1 - \cos(\frac{t_1}{\gamma}).$$

The fact that a, α, b, β are all nonnegative implies $\nabla S(x) \cdot x \geq 0$, which, since $S(0) = 0$, implies that

$$S(x) \geq 0. \tag{50}$$

Finally suppose x is on the portion of the boundary of K where $x_2 = 0$. Then the corresponding a and α values are both 0, so that $\nabla S(x) = P(x) = \beta\gamma^2(s_1, 0)$. Since the only $n \in n(x)$ is $n = (0, 1)$, we find

$$\nabla S(x) \cdot n = 0,$$

confirming (9).

A similar analysis applies for solutions of the Hamiltonian system associated with H_2. In this case we introduce new parameters (a', b') and (α', β') according to

$$\begin{aligned} x &= a' \gamma C_\eta \eta + b' \gamma(0, s_2) \\ p &= \alpha' \gamma^2 C_\eta \eta + \beta' \gamma^2(0, s_2). \end{aligned}$$

Then the region X_+ in state space corresponds to $a' > 0, b' > 0$ while Γ corresponds to $b' = 0$. Similarly, the region Π_2 in costate space corresponds to $\alpha' > 0, \beta' > 0$ while the boundary line Π_{12} corresponds to $\beta' = 0$. One can solve the Hamiltonian system associated with $H_{2,\gamma}$ explicitly

$$\begin{array}{lllll} \dot{x} & = & \frac{\partial^T H_{2,\gamma}}{\partial p}(x, p) & = & \frac{1}{\gamma^2} p - (0, s_2)^T \\ \dot{p} & = & -\frac{\partial^T H_{2,\gamma}}{\partial x}(x, p) & = & -x \\ \dot{z} & = & p^T \dot{x} & = & \frac{1}{\gamma^2} p^T p - s_2 p_2 \end{array} \tag{51}$$

Following the same analysis as above with $H_{2,\gamma}$ in place of $H_{1,\gamma}$ we see that we can cover the region $\Omega_+ \subseteq X_+$ with the state space component $x(t)$ of trajectories $(x(t), p(t))$ of (51) with the associated $p(t)$ in Π_2. In this way we arrive at the required function $x \to S(x)$ for $x \in \Omega_+$ as wanted.

In summary, we have defined a region Ω_γ (see (48)), a function $x \to P(x)$ (see (49) and its counterpart for X_+) and a nonnegative-valued function $x \to S(x)$ (see (42), (45) and (50)) for $x \in \Omega_\gamma$ which meets all the requirements of Theorem 2. The proof of Theorem 2 is now complete apart from the assertion concerning Filippov-sense solutions of (14); this last point is the subject of the next subsection.

3.3 Filippov Bicharacteristics and Completion of the Proof of Theorem 2

To complete the proof of Theorem 2, it remains to relate the analysis above to solutions of (14) in the sense of Filippov. We will consider only (43) in X_-, since (51) in X_+ is analogous. By reference to (31) we see that (14) can be rewritten as

$$\begin{aligned} \dot{x} &= \frac{1}{\gamma^2} p + Bu^i, \quad \text{if } p \in \Pi_i \\ \dot{p} &= -x. \end{aligned} \tag{52}$$

Notice that a discontinuity occurs as a function of the p coordinates, rather the spatial coordinates x as in (11). Since our $P(x) \in \Pi_1$ for $x \in X_- \cap \Omega_\gamma$, the solution $x(t)$, $p(t) = P(x(t))$ that we exhibited for $0 \le t \le t_1$ above solves (52) as well as (43). We can extend it to $t_1 \le t \le T_{x_0}$ using $b(t) = \beta(t) = 0$ (which means $x(t), p(t) \in \Gamma$) and

$$\begin{array}{rcl} a(t) & = & \sin(\theta + \frac{t_1 - t}{\gamma}) \\ \alpha(t) & = & 1 - \cos(\theta + \frac{t_1 - t}{\gamma}). \end{array} \tag{53}$$

This reaches 0 for the first time at $T_{x_0} = \gamma\theta + t_1$. Moreover, as one may check, the resulting $x(t), p(t)$ satisfy

$$\begin{array}{rclcl} \dot{x} & = & \bar{\lambda}\frac{\partial}{\partial p} H_{1,\gamma} + (1-\bar{\lambda})\frac{\partial}{\partial p} H_{2,\gamma} & = & \frac{1}{\gamma^2} p + C_\eta \eta \\ \dot{p} & = & -\bar{\lambda}\frac{\partial}{\partial x} H_{1,\gamma} - (1-\bar{\lambda})\frac{\partial}{\partial x} H_{2,\gamma} & = & -x \end{array} \tag{54}$$

This extension exhibits $x(t)$, $p(t) = P(x(t))$ as a Filippov solution of (14) on the full $[0, T_{x_0}]$ as claimed in Theorem 2.

The general behavior of (52) can be seen in detail by making an orthonormal change of coordinates in both the x and p planes. Define

$$\begin{array}{ll} \mu = (-s_1, s_2) \cdot x/\bar{s} & \mu_D = (s_2, s_1) \cdot x/\bar{s} \\ \nu = (-s_1, s_2) \cdot p/\bar{s} & \nu_D = (s_2, s_1) \cdot p/\bar{s} \end{array}$$

where $\bar{s} = \sqrt{s_1^2 + s_2^2}$. The coordinates μ_D and ν_D measure the components of x and p in the "diagonal" direction (Γ and Π_{12} respectively), while μ and ν give their components orthogonal to this. In particular $\nu > 0$ indicates $p \in \Pi_2$ so that we should be using $i = 2$ in (52). (We are ignoring x in the third quadrant in saying this.) When (52) is converted to these coordinates the diagonal components are independent of i:

$$\dot{\mu}_D = \frac{1}{\gamma^2}\nu_D - \frac{s_1 s_2}{\bar{s}}; \quad \dot{\nu}_D = -\mu_D. \tag{55}$$

Only the orthogonal components involve the discontinuities:

$$\dot{\mu} = \frac{1}{\gamma^2}\nu \pm \frac{s_i^2}{\bar{s}}; \quad \dot{\nu} = -\mu. \tag{56}$$

Here we use $+$ when $i = 1$ (i.e. for $\nu < 0$) and $-$ when $i = 2$ (i.e. $\nu > 0$). All three of these systems are periodic with period $2\pi\gamma$ having constant angular velocity about their respective centers. The solutions of (55) are ellipses centered at $(0, \gamma^2 s_1 s_2/\bar{s})$. The solutions of (56) are ellipses centered at $(0, v_i)$ where

$$v_1 = -\gamma^2 s_1^2/\bar{s}, \quad v_2 = +\gamma^2 s_2^2/\bar{s}.$$

The resulting phase portrait for the orthogonal components of (52), i.e. (56) using $i = 1$ if $\nu < 0$ and $i = 2$ if $\nu > 0$, is illustrated in Figure 5. Inspection should convince the reader that Filippov solutions are unique, except those passing through the point $\mu = \nu = 0$. We see several solutions through this point: an ellipse in the upper half-plane, an ellipse in the lower half-plane, as well as $\nu(t) \equiv 0, \mu(t) \equiv 0$. This latter is a Filippov solution since the μ component of $K[f]$ at the origin is the interval $[-s_1^2/\bar{s}, s_2^2/\bar{s}]$, which contains 0. In addition $\mu(t), \nu(t)$ can switch from one of these basic solutions to another whenever it passes through the origin.

The solution using $P(x)$ as in Theorem 2 on $[t_1, T]$ corresponds to $\mu(t) \equiv \nu(t) \equiv 0$ since $x(t) \in \Gamma$ and $p(t) \in \Pi_{12}$. It is then a Filippov solution of (52). We now see that there are other solutions of (52) starting from the same $x(t_1), p(t_1)$. However, all these alternate solutions will leave the first quadrant (K in x-space) before reaching the origin. To see this, notice that any such solution must have the same diagonal components, $\mu_D(t), \nu_D(t)$ which approach $(0, 0)$ monotonically on $[t_1, T]$. This means that the length of $[t_1, T]$ is at most a quarter period of (55). But once a solution of (56) leaves $\mu = 0$ it takes exactly a half period until it returns. Thus any solution of (52) other than the one we constructed but having the same values $x(t_1), p(t_1)$ will have $\mu(T) \neq 0$ but $\mu_D(T) = 0$. This means the solution has left the first quadrant K. So although (52) has many solutions for initial conditions $x(0) \in \Gamma_0$ with $p(0) = \nabla S(x(0))$, there is only one which reaches the origin without first leaving the first quadrant. This completes the proof of all aspects of Theorem 2.

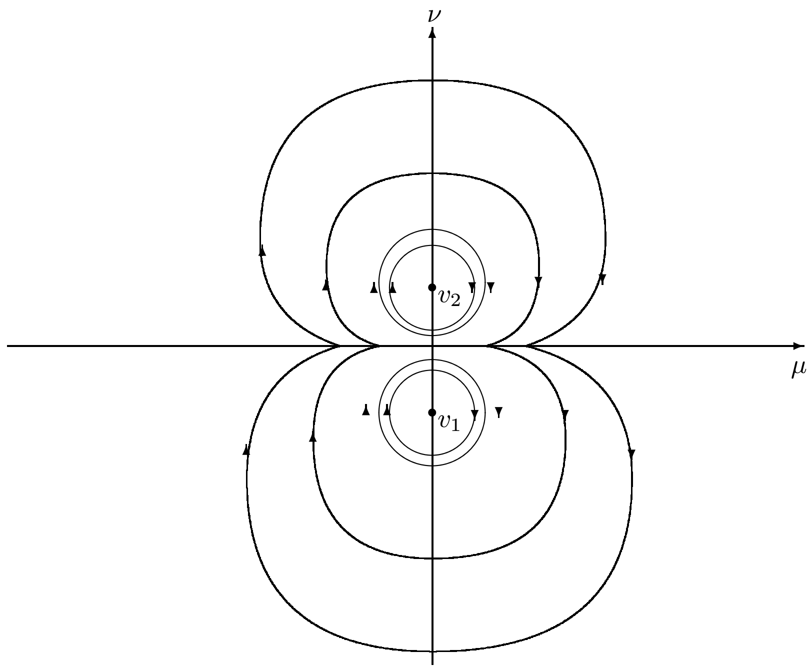

Figure 5: Off-diagonal components of Hamiltonian dynamics H_γ

4 Minimal Storage Function and Properties of the Closed Loop system

We turn now to issues concerning the closed loop control system (11), or (57) just below. In particular we will show that $S(x)$ is the minimal storage function (in Ω_γ) for our PDS (18) using control (35), proving Theorem 3. Secondly we will see that $(x, \nabla S(x))$, $x \in \Omega_\gamma$ is an invariant manifold (in a Filippov sense) for the bounded-real-lemma Hamiltonian system associated with the closed-loop system.

The control associated with our solution S is that of (35). Using it in our system (18) results in the closed-loop differential inclusion

$$\dot{x} \in \pi_K(Bu_*(x) + q), \quad x(0) = x^0. \tag{57}$$

The assertion that $S(x)$ is the minimal storage function in Ω_γ is with reference to this system.

4.1 Proof of Theorem 3

Our argument that S is minimal is an adaptation of that used for Proposition 7.1.8 in [29]. Consider any $x(0) \in \Omega_\gamma$. We will exhibit a pair $x(t)$, $q(t)$ solving (57) on an interval $[0, T]$ with $x(T) = 0$ and for which the storage function inequality becomes an equality:

$$S(x(T)) - S(x(0)) = -S(x(0)) = \int_0^T \frac{1}{2}(\gamma^2\|q(t)\|^2 - \|x(t)\|^2)\, dt, \tag{58}$$

since $S(x(T)) = S(0) = 0$. Given that such $x(t), q(t)$ exist, consider any other nonnegative storage function $\widetilde{S}$. It follows that

$$-\widetilde{S}(x(0)) \leq \widetilde{S}(x(T)) - \widetilde{S}(x(0)) \leq \int_0^T \frac{1}{2}(\gamma^2\|q(t)\|^2 - \|x(t)\|^2)\,dt = -S(x(0)).$$

If $x(0) = x \in \Omega_\gamma$ is arbitrary, then $S \leq \widetilde{S}$, establishing our assertion that S is the smallest of all possible storage functions in Ω_γ.

Consider $x \in \Omega_\gamma \cap X_-$ and the solution $x(\cdot), p(\cdot)$ of (43) used in our construction of S at x. Define an associated disturbance by

$$q(t) = \frac{1}{\gamma^2} p(t).$$

On $[0, t_1]$, $x(t) \in X_-$ so $u_*(x(t)) = u^1$ and therefore (43) and (57) coincide. (Since $x(t) \in \Omega_\gamma$, the projected dynamics are not involved.) Noting that $q(t)$ achieves the supremum defining $H_{1,\gamma}$ (see the discussion preceding (23)) we find that

$$\begin{aligned}
\nabla S(x(t)) \cdot \dot{x}(t) - \tfrac{1}{2}(\gamma^2\|q(t)\|^2 - \|x(t)\|^2) &= p(t)(Bu(x(t)) + q(t)) - \tfrac{1}{2}(\gamma^2\|q(t)\|^2 - \|x(t)\|^2) \\
&= H_{1,\gamma}(x(t), \nabla S(x(t))) \\
&= 0.
\end{aligned} \tag{59}$$

In other words, for $x(t), q(t)$ so constructed on $[0, t_1]$, equality is achieved in the dissipation inequality:

$$S(x(t_1)) - S(x(0)) = \int_{t_1}^0 \frac{1}{2}(\gamma^2\|q(t)\|^2 - \|x(t)\|^2)\,dt. \tag{60}$$

Continuing with the solution (53), (54) on $[t_1, T]$, we continue to consider $q(t) = \frac{1}{\gamma^2}p(t)$ as a disturbance. Then $x(t)$ continues to be a solution of (57). We are using the value $\bar{\lambda}$ to choose the value of $u(x)$ in (35), since $x(t) \in \Gamma$. This gives a Filippov sense solution; see (61). Again we have (from (33))

$$\nabla S(x(t)) \cdot \dot{x}(t) - \frac{1}{2}(\gamma^2\|q(t)\|^2 - \|x(t)\|^2) = H_{\mathrm{pre},\gamma}(x(t), p(t), u(x(t))) = H_\gamma(x(t), p(t)) = 0.$$

Thus

$$S(x(T)) - S(x(t_1)) = \int_T^{t_1} \frac{1}{2}(\gamma^2\|q(t)\|^2 - \|x(t)\|^2)\,dt.$$

Combining this with (60) establishes (58) used above. and therefore completes the proof of Theorem 3.

4.2 Filippov Solution of the Control System

We now consider (57) in general, which has discontinuities for $x \in \Gamma$. We see that $K[f](x,t)$ is essentially $Bu(x) + q(t)$ using what we have already written down in (35):

$$K[f] = \{q(t) - (\lambda s_1, (1-\lambda)s_2) : \ 0 \leq \lambda \leq 1\}, \quad \text{for } x \in \Gamma \tag{61}$$

If $q(t)$ is parallel to Γ, as it was in our construction above, the opposing directions of the vector fields on opposite sides of Γ imply that the solution of the differential inclusion in this case is unique and remains on Γ. The Filippov solution then corresponds to choosing $u = (\bar{\lambda}, 1-\bar{\lambda})$ with $\bar{\lambda}$ selected so that Bu is parallel to Γ. See Figure 6.

Other choices of $q(t)$ can translate the directions of the vectors by any fixed amount (depending on t). The resulting control $u(t)$ for a Filippov solution of the closed loop system may then have any number of switchings between u^1 and u^2 or involve any averaging $u = (\lambda, 1-\lambda)$ of u^1 and u^2, all depending on what disturbance $q(t)$ actually appears. Nevertheless we have the following general result.

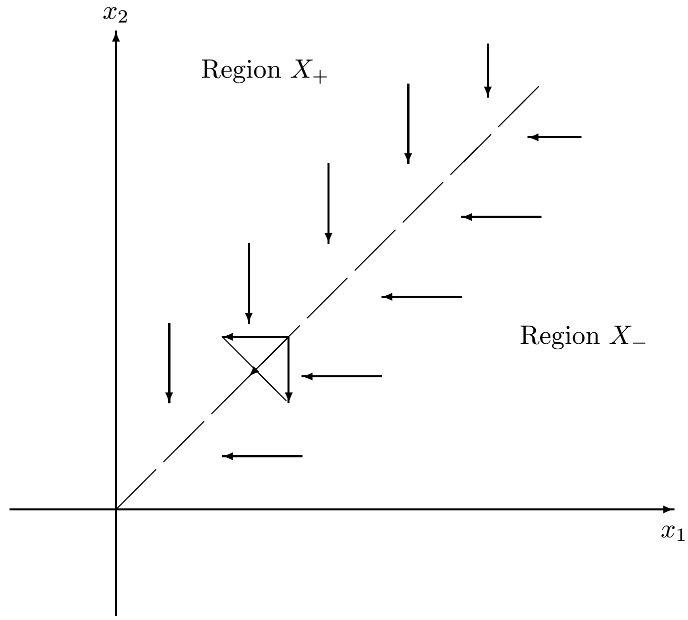

Figure 6: Dynamics of the discontinuous flow

Theorem 4 *Let $q(t)$ be an $\mathbf{R}^2$-valued integrable function on the interval $[t_0, t_1]$ for some $t_0 < t_1$ and let $x^0 \in K = \{(x_1, x_2) : \ x_1 \geq 0 \text{ for } i = 1, 2\}$. Then existence and right uniqueness holds for the system (57) with initial condition $x(t_0) = x_0$, i.e., there exists a unique absolutely continuous function $x(t)$ satisfying (57) a.e. with a given initial condition $x(t_0) = x_0 \in Q$ on the interval $[t_0, t_1]$.*

Proof. Define a time-dependent set-valued function $b(t, x)$ on all of $[t_0, t_1] \times \mathbf{R}^2$ by

$$b(x,t) = \begin{cases} Gu^1 + q(t), & s_1 x_1 > s_2 x_2 \\ Gu^2 + q(t), & s_1 x_1 < s_2 x_2 \\ \{G(\lambda u^1 + (1 - \lambda u^2) + q(t) : \ 0 \leq \lambda \leq 1\}, & s_1 x_1 = s_2 x_2 \end{cases}$$

and consider the differential inclusion with no boundary dynamics for an $\mathbf{R}^2$-valued absolutely continuous function $x(t)$:

$$\dot{x}(t) \in b(t, x(t)), \quad x(t_0) = x^0. \tag{62}$$

One can easily check that the set-valued function $b(t, x)$ satisfies the generalized Lipschitz condition

$$\langle f(t,x) - f(t,y), x - y \rangle \leq L \|x - y\|^2 \tag{63}$$

for all vectors $f(t, x) \in b(t, x)$ and $f(t, y) \in b(t, y)$ and all $x, y \in \mathbf{R}^2$ for a fixed constant L (where in fact one can take $L = 0$). Existence of solutions of the unconstrained differential inclusion (62) for $t \in [t_0, t_1]$ follows from Theorem 8 of Section 7 (page 85) of [13] while uniqueness follows from the generlized Lipschitz condition (63) by Theorem 1 of Section 10 (page 106) of [13]. Now existence and uniqueness for the system (57) with the projection dynamics on the boundary follows from Theorem 2 of [9], and the Theorem follows.

We close this section with a couple of additional remarks on the closd-loop system.

Remark 1. The controlled system (57) using (35) satisfies the usual criteria of suboptimal H_∞ control: it stabilizes the system with $q(t) \equiv 0$ (Figure 6 is the phase portrait) and has the required gain property (this follows from our successful construction of storage function $S(x)$). However the storage function is only defined in the bounded region Ω_γ. Increasing γ will produce a larger Ω_γ, but it will be bounded for each finite γ.

Remark 2. We have worked with the Hamiltonian H_γ of (27) to construct $S(x)$ from the H_∞ point of view. It is interesting to note that a different Hamiltonian function results if we simply postulate $u(x) = u_*(x)$ and then consider the L_2-gain property of the the closed-loop system. Again leaving out the projection dynamics, this leads to considering

$$\widetilde{H}_\gamma(x,p) = \sup_q \{p \cdot (Bu(x) + q) - \frac{1}{2}(\gamma^2 \|q\|^2 - \|x\|^2)\},$$

which is $\sup_q K_\gamma(x,p,q,u(x))$ in terms of the pre-Hamiltonian (22). This a distinct Hamiltonian from H_γ. In general it is true that $H_\gamma \leq \widetilde{H}_\gamma$, because

$$\begin{array}{rcl} \widetilde{H}_\gamma(x,p) & = & \sup_q K(x,p,q,u(x)) \\ & \geq & \sup_q \inf_{u \in \mathbf{T}} K(x,p,q,u) \enspace . \\ & = & H_\gamma(x,p) \end{array}$$

However when $p = \nabla S(x)$ both Hamiltonians agree since the minimizing $u \in \mathcal{U}$ is then given by $u^*(\nabla S(x)) = u(x)$:

$$H_\gamma(x, \nabla S(x)) = \widetilde{H}_\gamma(x, \nabla S(x)) = 0.$$

It is actually $\widetilde{H}_\gamma(x, \nabla S(x)) = 0$ that we used in (59) above. One could construct the available storage for (57) directly by considering the stable invariant manifold for the Hamiltonian system associated with $\widetilde{H}_\gamma$:

$$\begin{array}{rcl} \dot{x} & = & \frac{1}{\gamma^2}p \;+ Bu^i, \quad i = 1(2) \text{ if } x \in X_{-(+)} \\ \dot{p} & = & -x. \end{array} \tag{64}$$

We can again study the Filippov solutions of this by looking at the orthogonal components μ, ν as before. We now use (56) with $i = 1$ for $\mu < 0$ and $i = 2$ for $\mu > 0$ (instead of using the sign of ν as previously). The resulting phase portrait is illustrated in Figure 7. Now we see that *all* solutions are unique. Our $(x, \nabla S(x))$, $x \in \Omega_\gamma$ is again an invariant manifold of solutions, all reaching the origin in finite time. Now however (64) has a whole line segment of equilibrium points, connecting v_1 and v_2 along the ν-axis in Figure 7. Thus we would not say that the origin is a hyperbolic critical point for (64)!

5 Applications Issues and Conclusions

Our example (4) originates with a simple model for traffic flow at an isolated intersection of two one-way streets; see Figure 8. The variables and parameters appearing in the model are as follows:

State Variables:
x_1: the queue length of the traffic stream in approach A
x_2: the queue length of the traffic stream in approach B

Exogenous inputs:
q_1: the arrival rate of the vehicles at approach A
q_2: the arrival rate of the vehicles at approach B

Parameters:
s_1: the saturation flow rate of approach A (vehicles/lane/second)
s_2: the saturation flow rate of approach B (vehicles/lane/second)

We allow the q_i to be negative, corresponding to departure of vehicles from the two approaches. We assume both s_i to be strictly positive.

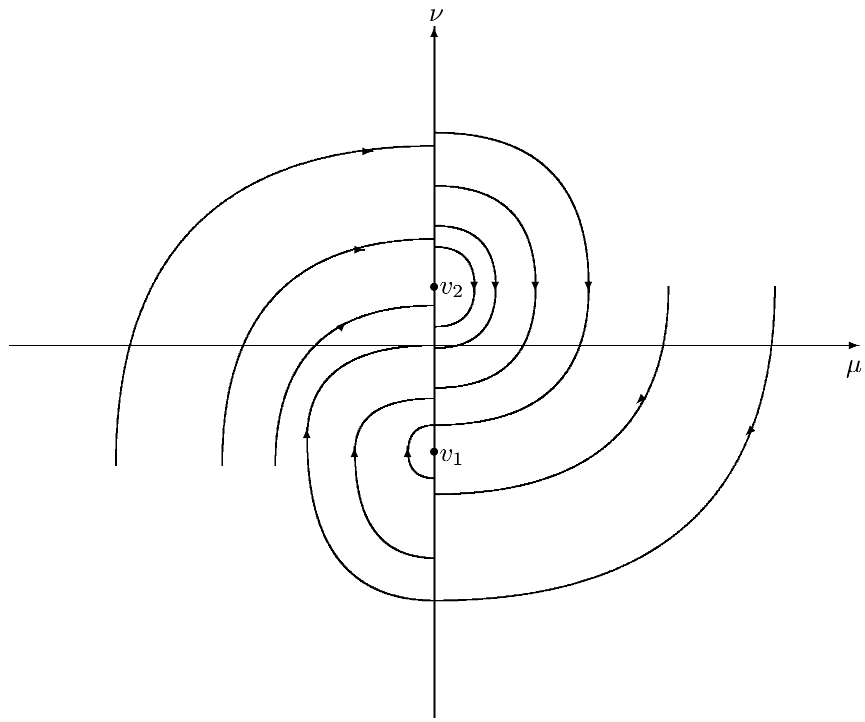

Figure 7: Off-diagonal components of alternate Hamiltonian dynamics $\tilde{H}_\gamma$

The set $\mathcal{U}$ of control actions (u_1, u_2) is

$$0 \le u_i, \quad u_1 + u_2 \le 1. \tag{65}$$

The interpretation is that $u_1 = 1$ corresponds to a green light for approach A (and red light for B). Likewise $u_2 = 1$ corresponds to a red light for A and a green light for B. While letting both $u_i > 0$ makes sense mathematically in our system equations (67), it would correspond to an infinitely fast switching between green for A and green for B, which is unrealistic physically. In practice, the control is a finite-state machine with two possible states, namely green or red for a given direction of traffic. There are other practical issues ignored by our simple mathematical model, for example, the need for a minimum green time to avoid unnecessary lost time in transition, and a maximum green time so as not to shut out an individual driver waiting in the less busy direction. We address such practical implementation issues in a more systematic manner in another publication [1].

Our state equations are based on the following assumption: *the difference between the arrival density and the flow served represents exactly the dynamic rate of the queue length at that approach, as long as the queue length to be served is positive.* That is,

$$\dot{x}_i = \delta(x_i, q_i - s_i u_i), \qquad i = 1, 2 \tag{66}$$

where the function $\delta(\cdot,\cdot)$ is defined by

$$\delta(y, v) = \begin{cases} v & \text{if } y > 0, \text{ or if } y = 0 \text{ and } v > 0 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, in the situation where both lines of traffic are oversaturated (both $x_i > 0$), equation (66) assumes the simpler form

$$\begin{cases} \dot{x}_1 &= q_1 - s_1 u_1 \\ \dot{x}_2 &= q_2 - s_2 u_2, \end{cases} \tag{67}$$

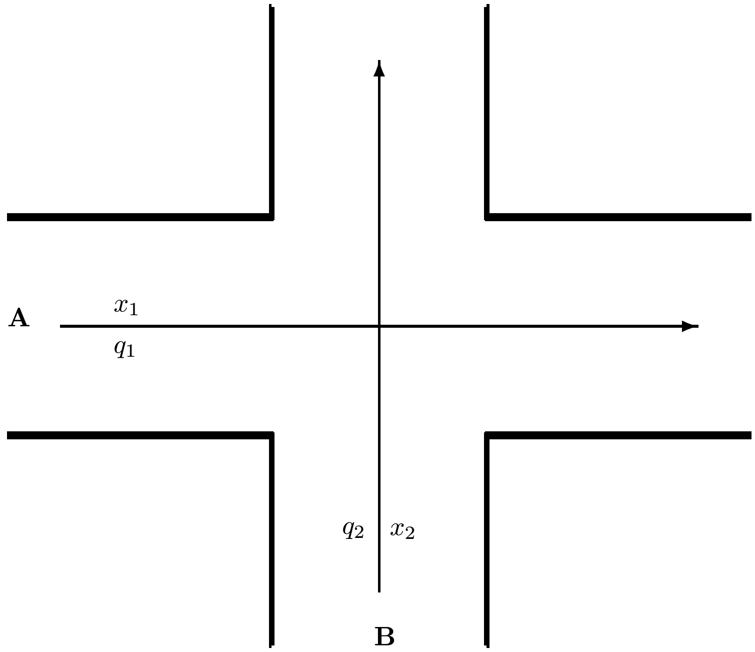

Figure 8: A Simple Two-way Intersection

which can be rewritten in matrix format as

$$\dot{x} = Bu + q. \tag{68}$$

The effect of δ is to produce the projected version, as in (18) above.

This simple model for traffic flow using saturation flow rate parameters s_1 and s_2 (for the over-saturated case where projection dynamics can be ignored) is usually attributed to Webster (see [32]). The above model can also be used to handle physically more complicated intersections or networks as long as the detailed traffic signal control configuration can be condensed to two phases.

The problem of traffic signal control has been studied extensively over the years; a sample of references is [7], [15], [24], [26], [14] and [18]. A state of the art summary of what is done in practice can be found in the Traffic Control Systems Handbook of the Federal Highway Administration [16]. Mainly open loop, rolling horizon, and heuristic techniques have been used by researchers and engineers in the control design, with most of the work coming out of the operations research rather than the control community. One notable exception is the paper of Gazis [15], where a simple two-phase and a two-intersection network problem were considered (with the disturbance $q(t)$ assumed to be known and of a simple analytical form) by using Pontryagin's maximum principle.

In conclusion, the contribution of this paper is to analyze the general nonlinear H_∞-control problem in the situation where discontinuities arise due to projection dynamics on the boundary of the state manifold and discontinuities in the state feedback due to constraints on the admissible control set. Just as in the smooth case, we have shown that the solution of the problem can be reduced to constructing a positive definite solution of a certain Hamilton-Jacobi equation, but now subject to an inequality constraint on the boundary of the state manifold, and with the resulting state-feedback in general set-valued. For the specific example of a system of this sort arising in a simple traffic signal control problem (4), we have shown that one can construct a solution of the Hamilton-Jacobi equation from a version of the stable invariant manifold for the associated nonsmooth Hamiltonian system of ordinary differential equations. We then went on to show, again just for the example (4), that the storage function $S(x)$ resulting from this construction is a *minimal* storage function for the closed-loop system in the sense of van der Schaft [29]. We were also able to analyze the Hamiltonian flow associated with the closed-loop system having gain equal to at most the preassigned tolerance level γ. Solutions of both Hamiltonian systems and of the closed-loop system must be interpreted in the sense of Filippov.

An open question is to understand to what extent the results obtained for the special case (4) extend to the general system (2), or to develope other special cases of interest for applications where these results extend. One such special case, where B is replaced by a general $n \times n$ diagonal matirx $B = \text{diag}(-s_i)$, can be used to model an intersection with a multiphase traffic signal, and is a topic of current research. Of interest for applications is to extend our results to more complicated models, e.g. traffic models including allowance for lag times and models for a traffic network rather than an isolated intersection. Finally a natural problem suggested by our work is the extension of the analysis to the measurement feedback case, where only a measurement $k(x)$ of the state vector x (rather than the whole state vector x itself) is available to the controller.

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