A Scholar's Introduction to Stocks, Bonds and Derivatives

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1 Introduction

This course concerns mathematical models of some basic financial assets: stocks, bonds and derivative securities. The real world of financial markets is very complicated and diverse. The models we will use are simplistic in comparison. However they will include enough features of the real world for us to introduce some of the ideas on which the subject of mathematical finance (particularly the pricing of derivatives) is based. For those who have little or no knowledge of financial markets this "Scholar's Introduction" is intended to describe those aspects of stocks, bonds and derivatives which you will need to appreciate the mathematical characterizations of them that our study will be based on. The text by Hull [4] provides additional details of actual market practices. Students who want to pursue the practical side further should consider courses offered by the Finance Department.

2 Stocks

Companies sell stock to raise money (capital). Initially a company may be privately owned, but at some point needs to raise money to expand its business operations. Instead of borrowing money that would need to be paid back later, it can "go public", that is sell shares of ownership of the company (i.e. stock) to raise money. The first time a company does this is called an *initial public offering*. Once shares of a company have been sold, they can be resold from one party to another. (But these resale transactions do not produce new funds for the company itself, any more than Ford Motor Co. benefits if you sell me your used Ford car.) The *stock market* is the place where these stock transactions take place (both initial offerings and resale). Actually the stock market consists of many different stock exchanges: NYSE, AMEX, NASDAQ, London, Tokyo, Paris, Sydney – there are over a hundred such exchanges world-wide. While some are still the traditional chaotic room full of brokers scrambling over each other and shouting bids like the NYSE, others like NASDAQ are based on an electronic network rather than a physical exchange floor.

Stockholders (i.e. owners of shares of stock in a corporation) are thus part owners of the corporation. The corporation's profits and losses are the profits and losses of the stockholders. Profits may be distributed to stockholders as dividends, or they may be reinvested in the company in various ways. Being a stockholder also gives you certain voting rights for major policy decisions. Your vote counts in proportion to the number of shares that you own. So when you hear of someone who owns a controlling interest in a company that means they own enough shares so that the influence of their vote is so large as to control these elections. But the primary reason most people buy stocks is to make money, either through dividend payments or in the expectation that the value of their share in the company will go up, so that it can be sold at a higher price later. The value of your share of the corporation is determined by the stock market — how much it can be sold/purchased for (not simply by some accounting of the company's assets, although that would certainly influence the market value). In essence the stock market functions as an auction. Prices are established as those wishing to buy a particular stock and those wishing to sell negotiate the price through bids and offers. The market price is simply the price at which the sellers and buyers are currently agreeing to trade. The market price reflects not only the estimated value of the company's assets, but also expectations about the company's future prospects, political events, general confidence in the economy, and countless other factors. Its value changes from day to day (even hour to hour) in ways that seem to have a lot of unpredictable (random) variation.

The number and complexity of factors which influence the price of a stock are so large that it is impossible to predict with certainty its future values given all the information available today. Mathematicians, statisticians, and economists have tried to develop approaches to describe this unpredictable behavior of stock prices, so that a careful study can be made of strategies for buying and selling. This course will adopt one of the standard attempts to do this: namely to consider the price S_t of a stock at time t as a stochastic process (i.e. a randomly evolving function of time), using *geometric Brownian motion* as the description of its stochastic behavior.

There are many complexities to the behavior of stock prices that we won't try to account for. For instance when dividends are paid the price of the stock makes an instantaneous jump down. There are stock splits, when each previously existing share is split into 2 (or more) new shares. Of course the price of a share after the split ought to be 1/2 (or less) of the price before the split. Then there is common stock as opposed to preferred stock (which carry different dividend and voting rights). In addition there are typically fees associated with any transaction. We will not concern ourselves with the additional complexities these features would involve.

Something that *is* relevant for us is the practice of *short selling.* Mathematically this amounts to buying a negative number of shares of the stock, so that you *owe* some number of shares rather than own them. In practice your stock brokerage can agree to let you sell some of its shares, under your promise to return them (with some fee or interest, but we will neglect that here) at a future date. Say they "loan" you a share worth \$50 which you immediately sell. You now are holding that \$50 but you also have an obligation to return the share you borrowed, as you promised the brokerage. If the price of the share goes down, say to \$30, by the time you need to return it to your brokerage you can use \$30 of the \$50 to buy a share to return to the brokerage, keeping the \$20 difference. In general, if S_0 denotes the stock price when you borrow it and S_1 the price when your return it, then the profit (or loss if negative) of this short-selling transaction is

$$S_0 - S_1 = (-1)S_1 - (-1)S_0.$$

If we bought n shares at price S_0 per share and sold them for S_1 per share then our profit or loss would be given by

$$nS_1 - nS_0.$$

Thus the effect of short-selling one share is the same as using $n = -1$ in the formula for buying and selling. Because such transactions are possible, for our mathematical discussion we will simply say that it is possible to hold a negative number of shares. (In reality there are limits to how much a brokerage would let you do this, but we will ignore them in our introductory discussion.)

In addition to the prices of individual stocks, the financial industry tracks numerous stock indexes such as the Dow Jones Industrial Average (DJIA) or the Standard and Poor 500 (S&P 500). These are prices of hypothetical portfolios of stock for carefully selected groups of companies, designed to measure the value of the stocks of companies of particular types (e.g. major industrial firms, mining, utilities, . . .) or of the market as a whole. Mutual funds are rather different. When you buy shares of a mutual fund you put your money into a pool that is managed by a firm, according to strategies that vary from fund to fund. The fund manager makes decisions about which stocks (or bonds or other financial instruments) to buy and sell to obtain the best performance possible consistent with the fund's strategy.

3 Bonds

A bond is basically an I.O.U., a promise to pay the holder a prescribed amount of money at some future date (or dates). For instance if you hold a \$100 U.S. savings bond, the U.S. Treasury is obligated to pay you \$100 at a specified time in the future: the *date of maturity* (which was set when the bond was issued/sold and is usually marked on the bond certificate). The U.S. Treasury issues many different kinds of bonds with maturity periods ranging from months (e.g. "T-Bills") up to 10 years (e.g. "T-Notes") or longer (e.g."T-Bonds"). Many other federal agencies issue bonds, as do governments at all levels down to towns, as well as private corporations. Cities and Counties often issue municipal bonds to borrow the money needed for one-time expenses such as building roads or schools. If you buy a certificate of deposit at your bank, the bank essentially issues you a "bond", promising to repay your money (plus interest) at a specified future date. In general, when you own a bond issued by some institution, you are *not* part owner of that institution (as you would be if you owned some of its stock), but you *are* a creditor of the institution - it owes you money and will have to pay up in the future. Another significant difference from stocks is that with a bond you know now exactly how much your future payments will be and when they will be delivered. For longer term maturities, there is usually a series of scheduled payments (called "interest" or "coupon" payments) in addition to the final payment when the bond matures. Bonds that make only a single, final payment are called "zero-coupon" or "discount" bonds.

The reason an agency or corporation issues bonds is to borrow money for current needs that it plans to pay back later. But from our point of view as (theoretical) investors, we will view bonds as the mechanism which allows cash held today to be transferred to an amount of cash of "equivalent value" at a future time. Lets say I am going to buy a bond that will pay \$1000 in 10 years. The price I will pay for that bond had better be less than \$1000, otherwise why would I buy it? I might as well just hold on to my cash. For me to be willing to put my money into this bond purchase, so that it is not available to me until the bond matures in 10 years, I expect to realize some benefit. This should be reflected in a purchase price for the bond that is less than its value at maturity. Lets suppose that I am willing to pay \$620 for the bond today. In essence that says that I consider the value of \$620 today to be equivalent to the value of \$1000 in ten years. We sometimes refer to this as the "time value of money".

The usual way to quantify the time value of money is with an *interest rate*. If B_t dollars today are worth B_{t+1} dollars in one year, then the simple rate of return, or *annual interest rate*, is $r_A = (B_{t+1} - B_t)/B_t$. Usually expressed as a percentage, r_A expresses the change from B_t to B_{t+1} as a fraction of B_t. Said otherwise,

$$B_{t+1} = (1 + r_A)B_t.$$

Over the course of several years this relationship has a compounding effect: if my B_t dollars today are worth B_{t+n} dollars in n years (and the interest rate r_A does not change), then

$$B_{t+n} = (1 + r_A)^n B_t.$$

We describe this by saying that the interest rate r_A is *compounded annually*. In our example above, $B_0 = 620$ and $B_{10} = 1000$ corresponds to an annual interest rate of $r_A = 4.896\%$. That would be the designated interest rate for the particular bond we described.

Annually compounded interest rates are fine if the time variable t is limited to whole numbers of years. But if we need to work with fractional numbers of years, it is more convenient to work with an equivalent *continuously compounded* interest rate r_C:

$$1 + r_A = e^{r_C}, \text{ or } r_C = \ln(1 + r_A).$$

Now the formula connecting equivalent numbers of dollars at two times $t < s$ is

$$B_s = e^{r_C(s-t)}B_t.$$

As we begin our discussion, we will consider only a single bond whose value at time t is given by the formula

$$B_t = e^{rt},$$

using a fixed continuously compounded interest rate r. This is very simplistic compared to the real world of bonds. (It is actually closer to the description of an interest-earning bank account, since the formula takes no account of the term of the bond.) Typically, different interest rates are available for bonds of different maturities. (The farther away the date of maturity the higher the interest rate is, usually.) And of course available interest rates fluctuate up and down for all kinds of reasons associated with the state of the economy. Selling a bond roughly corresponds to borrowing money, and buying a bond is the usual way to hold money over some period of time. In fact the interest rates that apply to my efforts to borrow are usually different (higher) than those that apply to my efforts to buy bonds. There is also a resale market for bonds, which takes place at the New York Stock Exchange and other places. I can sell my bond to someone else, if I want, prior to its maturity. (U.S. Savings bonds are special in this regard; they cannot be resold.) The price I can get is determined by the marketplace, and so can fluctuate from day to day. The interest rate for a bond is fixed when it is issued/sold (unless it is a "floating rate" bond). But as its market (resale) value changes, the ratio of its market value to its value at maturity changes, meaning that its original interest rate is not as meaningful an indicator as it was at the time of original sale. Instead the financial industry calculates and reports the "yield" of a bond in various ways, which can be interpreted as effective interest rates based on the market price, the remaining time to maturity, the scheduled interest payments that remain, and the final value at maturity. To interpret these numbers takes some knowledge of how they are calculated.

Even for newly issued bonds, the way prices are reported can be confusing. For instance T-Bill prices are reported using what is called the *discount rate* or *banker's discount yield*, rather than the actual price or interest rate. On top of all this, bond issuers might default, failing to meet their payment obligations. Hence there are bond ratings to indicate reliability of the issuer: from AAA (most reliable) to C (least reliable; so-called "junk" bonds).

4 Options and Derivatives

Stocks and bonds are what we will call *primary* securities. In addition to them we will be particularly interested in *derivative* securities. A derivative security is typically a contract which either obligates or entitles the parties involved to carry out a transaction in the future, under terms specified in the contract but which may also depend on future events.

A simple example is a *forward contract*. For instance if I manage a restaurant chain perhaps I sign a contract with a beef supplier to sell me a specified quantity of beef a year from now at a price we agree on today. If the market price of beef a year from now turns out to be higher that the price specified in our contract, then I would be glad I made the contract; it protected me from the rise in prices. However if the price ends up being lower than what the contract specified, I still have to pay the higher contractual price. That would please the beef supplier; it would protect him from the decline in prices and guarantee a buyer that he might otherwise loose. From both points of view (mine and the supplier's) changes in market prices for beef will change how valuable that contract is. If it begins to appear that beef prices are going to rise a lot, my supplier may want to get out of his contract with me. Or maybe another restaurant manager is desperate to insure a source of beef at a predictable price, so wants to buy my contract with the supplier from me. For me to release my contract either to the original supplier or to a third party, I would demand some payment, since giving up the contract will mean I have to pay the market prices for beef that I expect to be higher in the future. Thus the contract has a price that is determined by the beef market (and expectations about its future), i.e. its value is derived from the market price of a primary asset, beef in this example. The contract itself can become a tradable asset[1], but one whose value is derived from other primary assets. That is why this type of forward contract is called a derivative asset. Derivaties are also called *contingent claims* because they represent a claim on assets that is contingent on the future state of the market.

Derivative securities can be based on commodities (beef, crude oil, gold, ...) but also on primary financial assets such as stocks and bonds (and many other things such as foreign currencies, and various interest rates). Options are a particular type of derivative security in which the holder has a right but not an obligation to carry out the terms of the agreement. For instance suppose you buy a (European) *call* option for the stock of XYZ Corporation, with a specified exercise price K and date T of expiry. When the specified date T arrives, you have the right to buy 100 shares of XYZ stock for K/share (from whoever sold you the call option) if you want. But if you don't want to you don't have to exercise your option; you can just let it expire. If the market price S_T of a share of XYZ stock on date T is more than K, then of course you would exercise your option. (You will pay K per share for something you can turn around and sell for S_T per share.) However if it turns out that $S_T < K$, then the option is of no value to you. (Why would you pay K per share when you can pay less on the open market?) So the value of your call option at the date of expiry is something that is determined by the prices of other securities in the market. A *call* option gives you the right (but not the obligation) to buy at the price specified in the contract. A *put* option gives you the right (but not the obligation) to sell at a price specified in the contract. *European* options allow you to exercise your right only on the specified date of expiry. *American* options allow you to exercise your right at any time up to the specified date of expiry.

In a sense, an option is very like conventional insurance. You pay a premium up front to enter into a contract with another party (the insurance company) which obligates them to pay you certain amounts

[1] Actually, forward contracts are not usually traded, but many other types of derivates are, such as options.

depending on future events which cannot be predicted now. That is one use of options in finance - to insure against certain events that would be very damaging to your financial holdings. Suppose I buy stock in a promising but risky new company, but can't afford to loose all my investment if its stock price drops too low. To insure against that I might purchase put options for that stock. That would guarantee that I could sell my stock in this company at (or by) a certain date at a prescribed minimum price. The price of buying those put options will be determined by the market, and I would have to decide if the cost of the insurance that the put options would provide is worth it to me. Whoever "writes" or sells those put options to me is accepting my money to take a certain risk for me. They must be careful to "hedge" the obligation that their position entails. That means they must manage their own resources so that they are in a position to fulfill their obligation to buy the stock from me at above market prices if I choose to exercise my option. Their ability to do this will largely determine the price they would demand to issue the option contract, and thereby the market price for the put options in question. We will discuss the idea of *replicating portfolios*, which essentially describe the seller's hedging strategy, i.e. how they would take the sale price of the option and use it to create and manage a portfolio of investments so that they are always prepared to meet their contractual obligation, whatever happens in the market.

In addition to puts and calls on the stock of specific companies there are many types of options and derivative contracts that are actively traded, all differing in the terms of contract. There are puts and calls on the major stock indexes (e.g. the Dow Jones Industrial Average, the S&P 500, and others) rather than on individual stocks. There are *swaps*, which are agreements to exchange the streams of income produced by two investments held by the contracting parties. There are *swaptions*, which give one party the right but not the obligation to carry out a swap of income streams. There are *interest rate caps*, which entitle the holder to a prescribed cash payment if a certain interest rate rises above a level specified in the contract. Then there are *lookback* and *Asian* options whose values at the time T of expiry depend not simply on the market price S_T of the primary asset they are associated with, but also on the history of that price (e.g. its maximum, minimum, or average values): S_t, $0 < t < T$. In principle there is no limit on the design of derivative securities; whatever set of circumstances and possible transactions a "financial engineer" might formulate could be written as a legal contract and sold/bought for a price. The options that are traded on exchanges must conform to standard criteria (maturing on fixed days of the month, priced in multiples of \$2.5 or \$5, and so forth) in order for prices to be listed in a concise form. The firms that write and sell such contracts, i.e. derivative securities, of course have a very strong need to know how to hedge their contracts, i.e. manage their assets so that they are always able to meet their obligations, as well as to know how much to sell the contract for in the first place. This creates the need for people with the sophisticated mathematical skills necessary to analyze hedging strategies and prices, and explains the hiring of many mathematicians and technically trained scientists by large financial firms.

References

[1] D. M. Chance, **An Introduction to Derivatives** (4th edition), Dryden Press, Ft. Worth, TX, 1998.

[2] M. Musiela and M. Rutkowski, **Martingale Methods in Financial Modeling**, Springer, Berlin, 1998.

[3] http://www.cboe.com/education/

[4] J. C. Hull, **Options, Futures, and Other Derivative Securities** (third edition), Prentice Hall, Englewood Cliffs, NJ, 1997.