Matrix Powers

Yoichiro Mori

March 10, 2015

One of the reasons why eigenvalues/eigenvectors of diagonalization is so useful is that it allows us to compute matrix powers (or more generally matrix functions) very easily. The idea is the following. Suppose we have successfully diagonalized a matrix A as follows:

$$P^{-1}AP = D, D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \tag{1}$$

We are interested in computing the matrix power A^k. Using the above,

$$A = PDP^{-1}. \tag{2}$$

Therefore,

$$A^k = PDP^{-1}PDP^{-1} \cdots PDP^{-1} = PD^kP^{-1} \tag{3}$$

since all the intervening $P^{-1}P = I$, where I is the identity matrix. The key here is that D^k is very easy to compute. It is just:

$$D^k = \begin{pmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{pmatrix}. \tag{4}$$

Let us look at an example. Consider the matrix:

$$A = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix} \tag{5}$$

Its eigenvalues and eigenvectors are:

$$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; A\begin{pmatrix} 3 \\ -2 \end{pmatrix} = -3\begin{pmatrix} 3 \\ -2 \end{pmatrix}. \tag{6}$$

We have two distinct eigenvalues and the matrix is diagonalizable. Let

$$P = \begin{pmatrix} 1 & 3 \\ 1 & -2 \end{pmatrix}. \tag{7}$$

We may use P to diagonalize A as follows:

$$P^{-1}AP = D = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}. \tag{8}$$

Now, for matrix powers, we have:

$$\begin{aligned} A^k = PD^kP^{-1} &= \begin{pmatrix} 1 & 3 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 2^k & 0 \\ 0 & (-3)^k \end{pmatrix} \frac{1}{5} \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 2^{k+1} - (-3)^{k+1} & 3 \cdot 2^k + (-3)^{k+1} \\ 2^{k+1} - 2 \cdot (-3)^k & 3 \cdot 2^k + 2 \cdot (-3)^{k+1} \end{pmatrix}. \end{aligned} \tag{9}$$