Partial Notes 2373

Andrew Binder

October 10, 2016

Problem. *Consider a mass-spring system which satisfies the following second-order differential equation:*

$$\ddot{x} + 5\dot{x} + 4x = 0. \tag{1}$$

Here, $x(t)$ represents the position at time t of the mass on a spring, and we use the usual dot notation to denote time derivatives. Solve for the motion of this spring system with the initial values $x(0) = 2$ and $\dot{x}(0) = 0$. Is this system overdamped, critically damped, or underdamped?

Solution. The first step is to find the general solution for the motion of the mass-spring system. In order to do so, it is necessary to find the characteristic equation. The characteristic equation is the quadratic equation that results when we attempt a trial solution of $x(t) = e^{rt}$. Substituting this guess into the differential equation yields

$$r^2 e^{rt} + 5re^{rt} + 4e^{rt} = e^{rt}(r^2 + 5r + 4) = 0. \tag{2}$$

As the exponential can never be zero, the only way the above statement can hold is if r is a root of the characteristic equation $r^2 + 5r + 4$; that is, if r is such that $r^2 + 5r + 4 = 0$. For such an r, our trial solution is indeed an actual solution to the mass-spring system. In order to find the roots to the characteristic equation, we simply use the quadratic formula. The two distinct roots can be shown to be $r_1 = -1$ and $r_2 = -4$. As we have two distinct, real roots, the general solution is then

$$x(t) = c_1 e^{-t} + c_2 e^{-4t}, \tag{3}$$

where c_1 and c_2 are constants. This corresponds with the solution to second-order differential equations with constant coefficients in the case of a positive discriminant ($\Delta = b^2 - 4mk > 0$) discussed in class. The constants c_1 and c_2 are determined from the initial conditions. In order to find these constants, let us first write the derivative of $x(t)$:

$$\dot{x}(t) = -c_1 e^{-t} - 4c_2 e^{-4t}. \tag{4}$$

Now,

$$2 = x(0) = c_1 + c_2, \quad \text{and} \tag{5}$$
$$0 = \dot{x}(0) = -c_1 - 4c_2. \tag{6}$$

This leaves us with a system of two linear equations with two variables. One can write this into a matrix system and then reduce it into reduced row echelon form to find the solution. Add the two equations together to remove c_1 and show that $c_2 = -2/3$, from which we can easily compute that $c_1 = 8/3$. The solution to this initial-value problem is then

$$x(t) = \frac{8}{3}e^{-t} - \frac{2}{3}e^{-4t}. \tag{7}$$

To finish the problem, we note that mass-spring systems with two distinct, real roots (which have $\Delta > 0$) are called overdamped. This is an overdamped system. Mass-spring systems are called critically damped if they have a repeated, real root ($\Delta = 0$), and they are called underdamped if they have a pair of complex conjugate roots ($\Delta < 0$).

Definition. *A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is linearly dependent if there are scalars $c_1, c_2, \ldots, c_k$ with at least one nonzero such that*

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}. \tag{8}$$

Otherwise, they are linearly independent.

Problem. *Are the vectors*

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \tag{9}$$

linearly independent?

Solution. From the definition of linear independence, we can see that we need to find any scalars such that

$$\begin{aligned} 2c_1 &+ 1c_2 &- c_3 &= 0 \\ 3c_1 &+ 0c_2 &+ c_3 &= 0 \\ 4c_1 &+ 3c_2 &+ 3c_3 &= 0 \end{aligned} \tag{10}$$

holds. Note that the above equation is simply (8) with the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ substituted in explicitly. We can write this in matrix format in the following way:

$$\begin{bmatrix} 2 & 1 & -1 \\ 3 & 0 & 1 \\ 4 & 3 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{11}$$

Let

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 0 & 1 \\ 4 & 3 & 3 \end{bmatrix}. \tag{12}$$

An obvious solution to this problem is the case when all the scalars are zero; that is, when $c_1 = c_2 = c_3 = 0$. Since A is a square matrix, recall from previous classes that the above system will have only the trivial solution (all scalars c_k equal to 0) if the matrix A is invertible, which would imply that the vectors are linearly independent by definition. We can easily check the invertibility of A by computing its determinant. If the determinant of A is non-zero, then as we have shown previously, the matrix is invertible. So, we compute the determinant using the expansion method along the first column:

$$\begin{aligned} \det(A) &= 2\begin{vmatrix} 0 & 1 \\ 3 & 3 \end{vmatrix} - 3\begin{vmatrix} 1 & -1 \\ 3 & 3 \end{vmatrix} + 4\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} && (13) \\ &= 2(0-3) - 3(3+3) + 4(1-0) && (14) \\ &= -20 \neq 0. && (15) \end{aligned}$$

The determinant is non-zero, so we can only have that $c_1 = c_2 = c_3 = 0$. By definition, these vectors are linearly independent. Linear dependence requires that at least one of the c_k's be non-zero.

Definition. *A vector* $\mathbf{v}$ *is called a linear combinatin of vectors* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ *if there are scalars* $c_1, c_2, \ldots, c_k$ *such that* $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$.

Problem. (Linear Combination) *Is the vector*

$$\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{16}$$

a linear combination of the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \quad \textit{and} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}? \tag{17}$$

Solution. The vector $\mathbf{w}$ is a linear combination of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ if there exist scalars c_1 and c_2 such that

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2. \tag{18}$$

We can write the above equation as the following:

$$\begin{aligned} c_1 - \ \ c_2 &= 1 && (19) \\ 0c_1 + \ \ c_2 &= 2 && (20) \\ 3c_1 - 3c_2 &= 3. && (21) \end{aligned}$$

By defining,

$$\mathbf{A} := \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 3 & -3 \end{bmatrix}, \quad \mathbf{x} := \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \quad \mathbf{b} := \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \tag{22}$$

we see that finding c_1 and c_2 amount to solving the linear system of equations given by $\mathbf{Ax} = \mathbf{b}$. This can be solved using Gauss-Jordan Reduction. To do so, we write the problem in its augmented matrix form. The augmented matrix form is simply the matrix $\mathbf{A}$ with the column vector $\mathbf{b}$ appended at the end. The augmented matrix for this problem is

$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 3 & -3 & 3 \end{array}\right]. \tag{23}$$

We can now attempt to solve for the scalars c_1 and c_2 by using row operations to convert the matrix into reduced row echelon form. We find that

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 3 & -3 & 3 \end{array}\right] \xrightarrow{R_3^* = R_3 - 3R_1} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \tag{24}$$

$$\xrightarrow{R_1^* = R_1 + R_2} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]. \tag{25}$$

Thus, we have found that $c_1 = 3$ and $c_2 = 2$ satisfies the problem. Hence, $\mathbf{w} = 3\mathbf{v}_1 + 2\mathbf{v}_2$. By definition, $\mathbf{w}$ is a linear combination of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.