Math 2373: Linear Algebra and Differential Equations

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1 First-order differential equations

1.1 *Dynamical systems and models*

Systems that change over time are often better described by the *continuous* variations of some quantity (temperature, height, amount) than by discrete variations, although ultimately minute discrete quantities can be isolated (atoms). The attempt of such a description leads to the process of modeling:

- What are the important processes?
- What variables are necessary to describe the system?
- How do I describe this evolution?

The process of modeling is iterative: refinements to the model are brought in over time as errors are detected and analyzed.

Note that time is not the only possible independent variable which can be considered, e.g. space can be also an interesting parameter. It is however a very intuitive variable, and we will study many models which take time as the independent variable. Differential equations relate rate of changes to other variables. Such equations are natural candidates for mathematical modeling, for example:

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$$x(t), \qquad \frac{dx}{dt} = x^2 + t^2,$$

or

$$y(t), \qquad \frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 7y = 0.$$

Definition: A Differential Equation is an equation which contains *derivatives* of one, or more, dependent variables with respect to one, or more independent variables.

- If there is only one independent variable, the equation contains only *ordinary* derivatives, and it is called an **ordinary differential equation.**
- If there is more than one independent variable, the equation contains only *partial* derivatives, and it is called an **partial differential equation.**

The **order** of a differential equation is the highest order of the derivatives that appear in the equation.

1.2 *Example: the Malthus model.*

The problem is to model world population growth over time.

Step I: Discover a differential equation describing the situation.

- $P(t)$: the population at time t,
- $\frac{dP}{dt}(t)$: the rate of change of the population.
- *Modeling hypothesis:* both are proportional!

$$\frac{dP}{dt} = \kappa P(t),$$

where κ is a **constant of proportionality**.

Step II: Solve the proposed model equation. Here, we know the solution:

$$P(t) = Ce^{\kappa t},$$

where C is an arbitrary constant.

Indeed, we may compute

$$P'(t) = C\kappa e^{\kappa t} = \kappa(Ce^{\kappa t}) = \kappa P(t). \checkmark$$

Furthermore, we can determine the constant C by some additional information. Here, this could be the population at the starting time $t = 0$:

$$P(0) = P_0,$$

since then

$$P(0) = Ce^{\kappa 0} = C = P_0.$$

We find the solution to the problem:

$$P(t) = P_0 e^{\kappa t}.$$

A problem of the form

$$\begin{cases} \text{Differential equation: } y' = f(y), \\ \text{Initial condition: } y(t_0) = P_0 \end{cases}$$

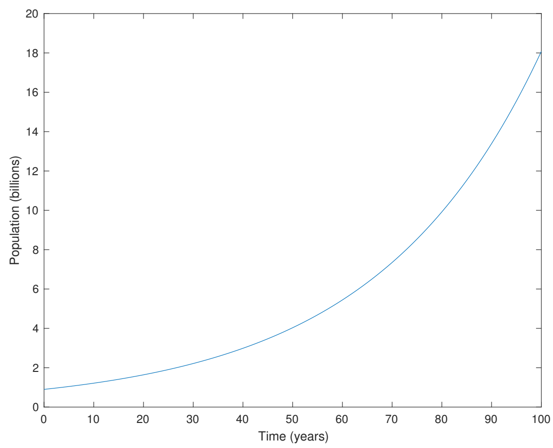
with t_0, P_0 given is called an **initial value problem (IVP).**

We remark that the Malthus problem is an IVP:

$$\begin{cases} P' = \kappa P, \\ P(0) = P_0. \end{cases}$$

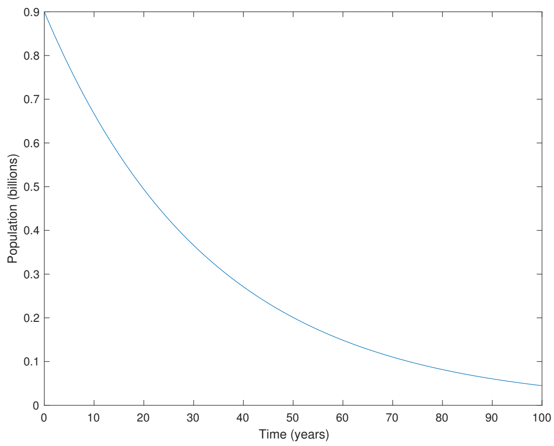
Step III: Interpret the result and compare to reality.

Malthus (in 1798) estimated $\kappa = .03$ and $P_0 = 0.9$ billion humans. This led to a prediction 60 times too high (363.81 billion people in 2000). Clearly this was an oversimplified model!



- In this case, $\kappa = 0.03 > 0$: the population *grows exponentially* to infinity.

- We can also look at the case when $\kappa = -.03 < 0$: the population *decays exponentially* to zero:



1.3 *Simple first-order models*

Typically, the most simple models assume **proportionality** between a rate of change and some function of the quantity.

- We have seen above the case where the rate of change of y is proportional to y:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \kappa y.$$

- The rate of change of y is inversely proportional to y:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\kappa}{y}.$$

- The rate of change of y is proportional to y^2 and inversely proportional to e^t:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\kappa y^2}{e^t} = \kappa y^2 e^{-t}.$$

- *Logistic growth* (disease modeling): the rate of increase of the number N of infected people in a total population N_{tot} is proportional to the product of the number of people infected and the number of still healthy people:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = kN(N_{tot} - N)$$

- *Newton's law of cooling / heating:* the rate of change of the temperature of an object is proportional to the difference between the temperature T_f of the surroundings and the temperature of the object:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \kappa(T_f - T)$$

1.4 *Higher order equations*

A first possibility of higher order system is when higher order derivatives are involved. An example is Newton's law of gravity,

$$\frac{\mathrm{d}^2 h}{\mathrm{d}t^2} = -g,$$

where h is the height of a mass in free fall. Such an equation is **a 2^{nd} order ordinary differential equation**.

Hooke's law

Let us model the displacement x of mass attached to a spring. The restoring force of the spring is

$$F_{spring} = -\kappa x,$$

where $\kappa > 0$ is the spring constant. When friction is negligible and a mass m is attached to the spring, Newton's First Law gives us the equation of motion,

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\kappa x.$$

This is a **2^{nd} order ordinary differential equation**.

It is important to notice that this problem can be transposed into an equivalent system of first-order equations! We introduce another dependent variable

$$y = \frac{\mathrm{d}x}{\mathrm{d}t},$$

the velocity of the mass, and we can convert the above problem into a first-order system of two equations:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -\kappa x. \end{cases}$$

2 Direction fields and solution of first-order ordinary differential equations.

A very general form for a first-order ODE is

$$\frac{dy}{dt} = f(t,y) \qquad \text{for} \qquad y' = f(t,y), \tag{1}$$

where t is the independent variable, y the dependent variable, and f is a given function describing the relation between the rate of change of y and t, y.

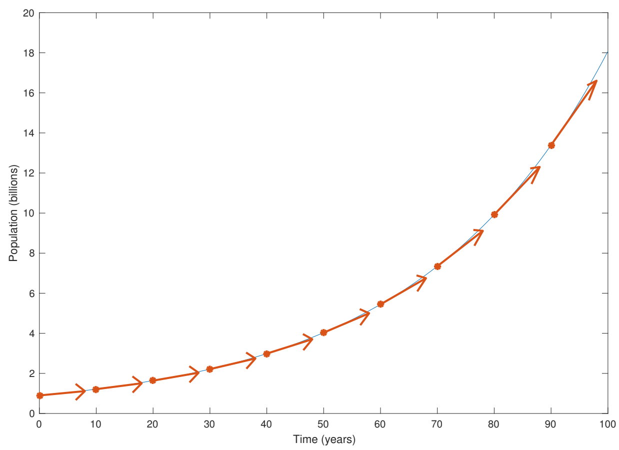
Definition: A solution of a differential equation is a function $y(t)$ such that substituting $y(t)$ for y in (1) satisfies the identity on some appropriate domain for t.

Examples:

1. The Malthus problem,

   $$y' = .03y.$$

   The solution is $y(t) = Ce^{\kappa t}$ with a fixed constant C:

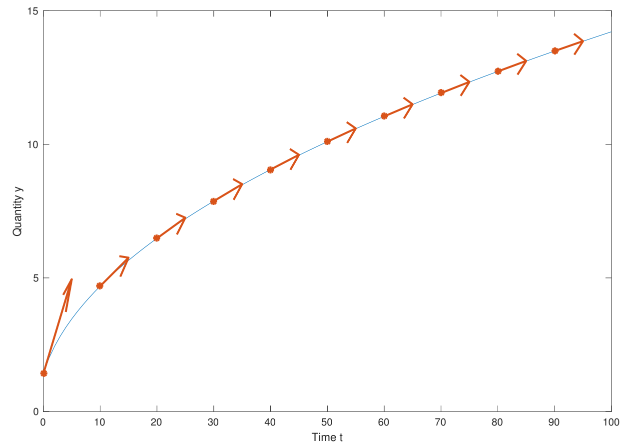


   The **slope of the tangent**, i.e. the derivative, is prescribed by the differential equation at any point of the curve representing the solution! This can be seen on the graph with the arrows that are tangent to the curve.

2.

$$y' = 1/y.$$

The solution is $y(t) = \pm\sqrt{2t + C}$ for any constant C and $t \geq -C/2$:



Again, arrows describing tangent lines can be drawn at each point of the curve thanks to the differential equation.

3.

$$y' = 2\sqrt{y}\,:$$

A family of solutions is $y(t) = (t + C)^2$ for any constant C. However this problem has in fact multiple solutions satisfying $y(0) = 0$:

$$y_1(t) = 0, \qquad y_2(t) = t^2.$$