Math 2373: Linear Algebra and Differential Equations

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Contents

Why study linear algebra?

While not as flashy as differential equations in its examples and uses (pendulum, projectile trajectories, chemical reactions), linear algebra is a *vital* tool for modern science and engineering. Examples abound and are too many to draw a comprehensive list:

- Linear elasticity, relations between stresses and strains;
- Vibrations of structures;
- Electrical and models of fluid flows;
- Optimization problems;
- Big data, machine learning, Google search algorithms

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- Image processing...

All these problems can be framed at some level into two basic equations:

1. A *linear system:* find $\vec{x}$ such that

$$A\vec{x} = \vec{b}. \tag{1}$$

2. An *eigenvalue equation:* find λ and $\vec{x}$ such that

$$A\vec{x} = \lambda\vec{x}. \tag{2}$$

Back in 170CE, the Chinese were already solving linear systems: *One pint of good wine costs 50 gold pieces, while one pint of poor wine costs 10. I bought two pints of wine for 30 gold pieces. How much of each kind of wine did I buy?* If x is the amount of good wine I bought (in pints), and y the amount of bad wine, then this problem writes in the modern notation:

$$\begin{cases} x + y = 2, \\ 50x + 10y = 30. \end{cases} \tag{3}$$

1 Gaussian Elimination

The formalism for a particular method to solve linear system, Gauss-Jordan elimination[1], was developed by Karl Friedrich Gauß(1777-1855) and Wilhelm Jordan (1842-1899) Note that this is not the first approach developed for the solution of linear systems, as Newton (1670) already discussed such methods he knew from much older Chinese books of mathematics.

Gaussian elimination is an effective strategy for solving linear systems, based on the idea of *equivalent systems*: the solution is unchanged by

1. Multiplication of one equation by a nonzero number,
2. Switching order of the equations,
3. Combining equations together by addition.

All of these operations *preserve the information in the system*.

Our particular motivation is to answer the following questions about a system:

- Does it have any solutions? How many solutions?
- How can I compute these solutions?

The particular notation used for this computation is the *augmented matrix:*

$$\begin{cases} x + y = 2, \\ 50x + 10y = 30, \end{cases} \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 50 & 10 & 30 \end{array}\right]. \tag{4}$$

[1] Textbook: section 3.2

The operations introduced above have a natural interpretation in the augmented matrix framework.

They form the **elementary row operations:**

1. Multiplication of all the entries in one row by a nonzero number,
2. Exchange of two rows,
3. Entry-wise addition of some multiple of one row to another row.

These are the three operations that enter into the Gauss-Jordan elimination process, which writes as the following algorithm:

Gauß-Jordan elimination algorithm.

Step 1 Assemble the augmented matrix:

$$\begin{cases} x + y = 2, \\ 50x + 10y = 30, \end{cases} \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 50 & 10 & 30 \end{array}\right].$$

Step 2 Using elementary row operations, work *left to right* to obtain zeros below the diagonal and ones on the diagonal:

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 50 & 10 & 30 \end{array}\right] \longrightarrow \begin{bmatrix} 1 & 1 & 2 \\ 5 & 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & -4 & -7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 7/4 \end{bmatrix}$$

Step 3 Using elementary row operations, work *right to left* to obtain zeros above the diagonal:

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 7/4 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 1/4 \\ 0 & 1 & 7/4 \end{array}\right].$$

This leads to the solution:

$$\begin{cases} x = 1/4, \\ y = 7/4. \end{cases}$$

Note that it is not always possible to obtain a diagonal of ones with zeros everywhere. In any case, the goal is always to select successive *pivots* and obtain a matrix which has the RREF form:

Reduced row echelon form or **RREF**

1. Zero rows (if there are any) are at the bottom of the array;
2. The leftmost nonzero entry in a nonzero row is 1; it is called the **pivot**.
3. Each pivot is further to the right than the pivot in the row above it.
4. Each pivot is the only nonzero entry in its column.

Let us work out another example:

Step 1

$$\begin{cases} x + 2y - z = -1, \\ 2x + 4y - z = 4, \\ -x - 2y + 3z = 5 \end{cases} \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 4 & -1 & 4 \\ -1 & -2 & 3 & 5 \end{array}\right].$$

Step 2

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 4 & -1 & 4 \\ -1 & -2 & 3 & 5 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 0 & 1 & 6 \\ -1 & -2 & 3 & 5 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 + R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & -8 \end{array}\right]$$

Step 3

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & -8 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & -8 \end{array}\right]$$

We have obtained the RREF. Note that the last equation does not have a solution:

$$0x + 0y + 0z = -8!$$

We say that the system is *inconsistent*.

Existence and uniqueness of solutions:

- A system is **inconsistent** when it has *no solution*,
- A system is **consistent** when it has *one or more solutions*. In this case,
    1. If every column in the RREF is a *pivot* column, then there is *only one* solution: it is unique.
    2. If one or more columns are non-pivot columns, then there is an *infinity* of solutions. The system is underdetermined.

Not all consistent systems have as many equations as unknowns: for example, let us reduce the following system:

STEP 1

$$\begin{cases} 2x + y = 6, \\ -x - y = -2, \\ 3x + 4y = 4, \\ 3x + 5y = 2, \end{cases} \longrightarrow \left[\begin{array}{cc|c} 2 & 1 & 6 \\ -1 & -1 & -2 \\ 3 & 4 & 4 \\ 3 & 5 & 2 \end{array}\right].$$

STEP 2

$$\left[\begin{array}{cc|c} 2 & 1 & 6 \\ -1 & -1 & -2 \\ 3 & 4 & 4 \\ 3 & 5 & 2 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{cc|c} -1 & -1 & -2 \\ 2 & 1 & 6 \\ 3 & 4 & 4 \\ 3 & 5 & 2 \end{array}\right] \xrightarrow{R_1 \leftarrow -R_1} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 1 & 6 \\ 3 & 4 & 4 \\ 3 & 5 & 2 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -1 & 2 \\ 3 & 4 & 4 \\ 3 & 5 & 2 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 3R_1} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -1 & 2 \\ 0 & 1 & -2 \\ 3 & 5 & 2 \end{array}\right]$$

$$\xrightarrow{R_4 \leftarrow R_4 - 3R_1} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 2 & -4 \end{array}\right] \xrightarrow{R_2 \leftarrow -R_2} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \\ 0 & 2 & -4 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 + R_2} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 2 & -4 \end{array}\right] \xrightarrow{R_4 \leftarrow R_4 + 2R_2} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Step 3

$$\left[\begin{array}{cc|c}1 & 1 & 2\\0 & 1 & -2\\0 & 0 & 0\\0 & 0 & 0\end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{cc|c}1 & 0 & 4\\0 & 1 & -2\\0 & 0 & 0\\0 & 0 & 0\end{array}\right]$$

This last form is the RREF. The system is consistent, and every column is a pivot column: there is a unique solution.

Here is another example:

Step 1

$$\begin{cases} x - y + 2z = 1, \\ 2x + y + z = 8, \\ x + y = 5, \end{cases} \longrightarrow \left[\begin{array}{ccc|c}1 & -1 & 2 & 1\\2 & 1 & 1 & 8\\1 & 1 & 0 & 5\end{array}\right].$$

Step 2

$$\left[\begin{array}{ccc|c}1 & -1 & 2 & 1\\2 & 1 & 1 & 8\\1 & 1 & 0 & 5\end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{ccc|c}1 & -1 & 2 & 1\\0 & 3 & -3 & 6\\1 & 1 & 0 & 5\end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|c}1 & -1 & 2 & 1\\0 & 3 & -3 & 6\\0 & 2 & -2 & 4\end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow R_2/3} \left[\begin{array}{ccc|c}1 & -1 & 2 & 1\\0 & 1 & -1 & 2\\0 & 2 & -2 & 4\end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - 2R2} \left[\begin{array}{ccc|c}1 & -1 & 2 & 1\\0 & 1 & -1 & 2\\0 & 0 & 0 & 0\end{array}\right]$$

Step 3

$$\left[\begin{array}{ccc|c}1 & -1 & 2 & 1\\0 & 1 & -1 & 2\\0 & 0 & 0 & 0\end{array}\right] \xrightarrow{R_3 \leftarrow R_3/2} \left[\begin{array}{ccc|c}1 & 0 & 2 & 3\\0 & 1 & -1 & 2\\0 & 0 & 0 & 0\end{array}\right]$$

- The system is consistent, there are an infinity of solutions (the third column is not a pivot column).
- z is a free variable: we apply the *principle of superposition*.

Particular solution: we choose $z = 0$, so that a solution is

$$X_p = \begin{bmatrix}3\\2\\0\end{bmatrix}.$$

Homogeneous system: we solve with zero right-hand side,

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{cases} x = -t, \\ y = t, \\ z = t, \end{cases}$$

where t is a free parameter. Thus

$$X_h = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Superposition: all the solutions have the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = X_h + X_p = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3-t \\ 2+t \\ t \end{bmatrix}.$$

2 Matrices

Our goal in this section is to provide a *formalization* of the previous problems[2].

2.1 *Introduction*

Definition: A matrix is a rectangular array of numbers.
If the matrix has m rows and n columns, then we say that it has order, or dimensions, $m \times n$.

In the Chinese problem example from the first lecture, we find that the **coefficients of the system** form a 2×2 matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 50 & 10 \end{bmatrix}.$$

The notation for a matrix is with square brackets $[\cdots]$ or occasionally with parenthesis $(\cdots)$. Usually they are noted with capital letters. When both of the dimensions are equal to n (square $n \times n$ matrix), we may abbreviate and say that it has order n.

Definition: When one of the dimensions is 1, then we obtain a *row* or *column* **vector**.

An example of column vector is the right-hand side of the problem:

$$\vec{b} = \begin{bmatrix} 2 \\ 30 \end{bmatrix},$$

[2] Textbook: Section 3.1

as well as the vector of unknowns:

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

A vector is occasionally noted with an arrow on top as here $\vec{x}$ or $\vec{b}$, but this is not mandatory.

Our goal here is to make sense of the matrix notation for linear systems,

$$A\vec{x} = \vec{b},$$

where we see a matrix-vector product and an equality. Let us proceed slowly.

DEFINITION: Two matrices are **equal** if

- They have the same order,
- All corresponding entries are equal.

Special matrices

- Zero matrix: all entries are equal to zero. It is denoted by 0, while keeping track of its dimensions! For example,

  $$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

  is a 2×3 zero matrix.

- Diagonal matrices: those are *square* matrices where only *diagonal* entries are different from zero. For example,

  $$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

  is a diagonal matrix of order 3.

- Identity matrices: this is a *square* diagonal matrix where *diagonal* entries are all equal to 1. For example,

  $$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  is an identity matrix of order 4. It is usually noted by the letter I indexed by the order n: I_n.

2.2 *Operations on matrices*

Matrices can be manipulated like simple numbers: they can be added, substracted, multiplied. However the rules are more complicated! Let us begin with the simple addition and substraction.

> Definition: **Addition**.
>
> - Two matrices A and B *of the same order* can be added.
> - $A + B$ denotes the matrix obtained by adding the corresponding entries of A and B.

For example,

$$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1+(-1) & 0+2 \\ 2+0 & 3+4 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 7 \end{bmatrix}.$$

> Definition: **Substraction**.
>
> - Two matrices A and B *of the same order* can be substracted.
> - $A - B$ denotes the matrix obtained by substracting the corresponding entries of A and B.

For example,

$$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1-(-1) & 0-2 \\ 2-0 & 3-4 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2 & -1 \end{bmatrix}.$$

Another possibility is to multiply a matrix by a number:

> Definition: **Multiplication by a number**.
>
> - Any matrix A can be multiplied by a real or complex number c.
> - cA denotes the matrix obtained by multiplying each entry of that matrix by the number c.

For example,

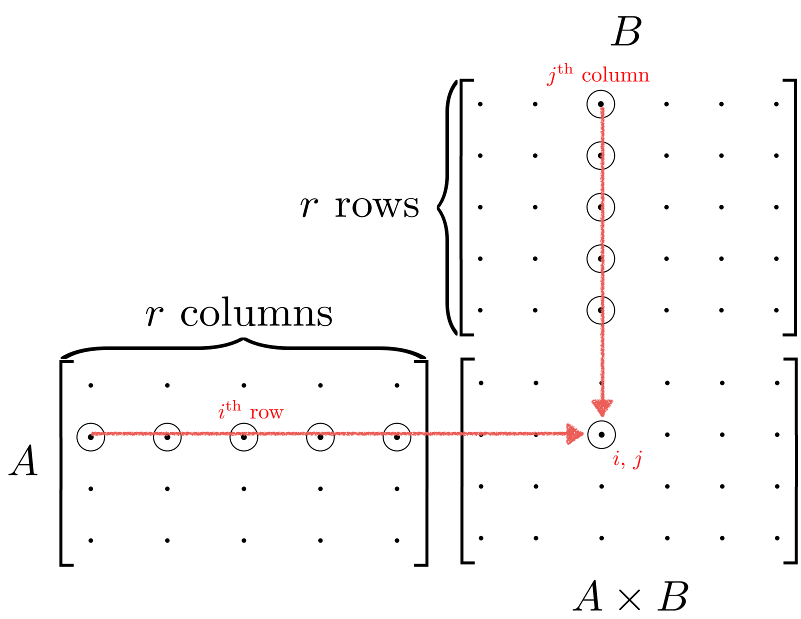
$$2 \cdot \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 & 2\cdot 0 \\ 2\cdot 2 & 2\cdot 3 \end{bmatrix} = \begin{bmatrix} 2 & 9 \\ 4 & 6 \end{bmatrix}.$$

The multiplication of matrices is not as straightforward.

Definition: **Multiplication of matrices.**

- One can multiply two matrices A and B *only* if the *number of columns of* A is the same as the *number of rows of* B.
- If A has order $m \times r$ and B has dimension $r \times n$, then the product $C = AB$ is a $m \times n$ matrix where the entry in the i^{th} row and j^{th} column is the **scalar product** of the i^{th} row of A and j^{th} row of B.

To remember this definition, it is useful to keep in mind the following diagram:



For example,

$$\begin{bmatrix} 3 & 1 \\ 2 & -4 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 \times 2 + 1 \times 0 & 3 \times (-1) + 1 \times 4 \\ 2 \times 2 + (-4) \times 0 & 2 \times (-1) + (-4) \times 4 \\ -1 \times 2 + 0 \times 0 & -1 \times (-1) + 0 \times 4 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 4 & -18 \\ -2 & 1 \end{bmatrix}.$$

2.3 *Properties of the matrix operations*

Let A, B, C be three matrices of the same order $m \times n$, d and e two numbers. Then,

- $A + B = B + A$: the matrix addition is *commutative*,
- $(A + B) + C = A + (B + C)$: the matrix addition is *associative*,
- $A + 0 = A$: the zero matrix is a *neutral element* for the addition,
- $A + (-A) = 0$: *opposite element* for the addition,
- $(d + e) \cdot A = dA + eA$: multiplication by a number is *distributive*,
- $e \cdot (A + B) = dA + dB$: ________ ,
- $I_m \cdot A = A \cdot I_n = A$: the identity matrix is a *neutral element* for the multiplication,
- $0 \cdot A = A \cdot 0 = 0$: the zero matrix is an *absorbing element* for the multiplication,

Now if A_1 and A_2 are of order $m \times r$ and B is of order $r \times n$,

- $(A_1 + A_2) \cdot B = A_1 \cdot B + A_2 \cdot B$: multiplication of matrices is *distributive*,
- $d \cdot (A_1 \cdot B) = (d \cdot A_1) \cdot B = A_1 \cdot (d \cdot B)$: the multiplication by a number is compatible with the matrix product.

As a conclusion, we note that the definition of the matrix product gives a well-defined meaning to the usual notation for linear systems:

$$A\vec{x} = \vec{b},$$

where A is the matrix of coefficients of the system, $\vec{x}$ is the column vector of unknowns, and $\vec{b}$ is the column vector collecting the right-hand side data. For example, the system

$$\begin{cases} x + y = 2, \\ 50x + 10y = 30. \end{cases}$$

corresponds to the matrix notation

$$\begin{bmatrix} 1 & 1 \\ 50 & 10 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 30 \end{bmatrix}.$$

3 Determinants

Motivation

We have seen that square systems of linear equations can have either:

- zero solutions (inconsistent system),
- a unique solution (consistent system),
- an infinity of solutions (consistent underdetermined system).

A first criterion to know which category a given system belongs to is the RREF obtained by Gauß-Jordan reduction. In particular, we have seen that if the RREF is an identity matrix, then the system is consistent. This indicates that this property depends only on the coefficients of the system: the entries of the coefficient matrix A.

We will see in this section a second criterion: the determinant[3]. This is a *number* associated to any square matrix such that in particular,

$$\det(A) \neq 0$$

is equivalent to the fact that associated systems always have a unique solution.

3.1 *Definition*

First, we may look at the simplest possible system: a 1×1 problem,

$$a \cdot x = b.$$

The matrix of coefficients is then

$$A = \begin{bmatrix} a \end{bmatrix}$$

and we define its determinant as

$$\det(A) = \left| A \right| = a.$$

We can check that this system will have a unique solution if $a \neq 0$.

Next, we may look at a 2×2 system:

$$\begin{cases} ax + by = 0, \\ cx + dy = 0. \end{cases}$$

The matrix associated with this system is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

[3]Textbook: Section 3.4

We define its determinant as

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

As an example, we can compute the 2×2 determinant,

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2.$$

Since this determinant is not zero, any system described by these coefficients is consistent.

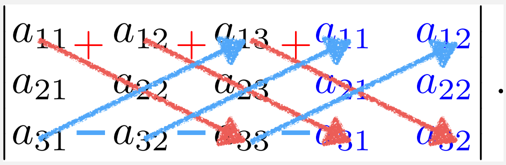
Next, we examine the case of a 3×3 system. This time, the evaluation of the determinant is a bit more complicated, and can be executed using the so-called basketweave method:

Basketweave method

STEP 1. Write the matrix and repeat the first two columns to the right of the matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \longrightarrow \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} & {\color{blue}a_{11}} & {\color{blue}a_{12}} \\ a_{21} & a_{22} & a_{23} & {\color{blue}a_{21}} & {\color{blue}a_{22}} \\ a_{31} & a_{32} & a_{33} & {\color{blue}a_{31}} & {\color{blue}a_{32}} \end{vmatrix}.$$

STEP 2. Compute the determinant by adding the product of the entries on each of the three downwards diagonals and substracting the product of the entries on each of the upwards diagonals:



$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{cases} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ \qquad -a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}. \end{cases}$$

We may use this method to compute the following 3×3 determinant:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{cases} 1 \cdot 5 \cdot 9 + 2 \cdot 6 \cdot 7 + 3 \cdot 4 \cdot 8 \\ \qquad -7 \cdot 5 \cdot 3 - 8 \cdot 6 \cdot 1 - 9 \cdot 4 \cdot 2 \end{cases} = 0.$$

GOING TO HIGHER ORDERS, we need to find a *systematic* method to compute the determinant. Note that we do not expand here on the theory behind the determinant, but give a practical method that allows to compute it. You have to believe that there are good reasons why this is the right thing to do!

The idea is to apply a *recursive* method: large $n \times n$ determinants are expressed in terms of slightly smaller $(n-1) \times (n-1)$ determinants; these can in turn be expressed in terms of $(n-2) \times (n-2)$ determinants, and so on until we reach determinants we already know how to compute (2×2 or 3×3).

First, we introduce two notions. Let A be a $n \times n$ matrix.

DEFINITION: The minor M_{ij} of a coefficient a_{ij} of A is an $(n-1) \times (n-1)$ **matrix** obtained by deleting the i^{th} row and j^{th} column of A.

$$A = \begin{bmatrix} \cdot & \cdot & \odot & \cdot \\ \cdot & \cdot & \odot & \cdot \\ \odot & \odot & \odot & \odot \\ \cdot & \cdot & \odot & \cdot \end{bmatrix} \longrightarrow M_{33} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

DEFINITION: The cofactor C_{ij} of a coefficient a_{ij} of A is a **number** obtained by computing the determinant of the minor:

$$C_{ij} = (-1)^{i+j} \left| M_{ij} \right| .$$

Note the $(-1)^{i+j}$ sign in front of the cofactor, which depends on the row and column index. These follow a checkerboard pattern:

	1	2	3	4	$\cdots$
1	$+$	$-$	$+$	$-$	$\cdots$
2	$-$	$+$	$-$	$+$	$\cdots$
3	$+$	$-$	$+$	$-$	$\cdots$
4	$-$	$+$	$-$	$+$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Finally, we can write any determinant by using the cofactors. More precisely, we can **choose** any column or row of the matrix.

Development along a column or row.

- Let i be a row number, then we develop along the i^{th} row as

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \left| M_{ij} \right| .$$

- Let j be a column number, then we develop along the j^{th} column as

$$\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \left| M_{ij} \right| .$$

PROPERTY: Any choice leads to the same number in the end.

In this process, we obtain *smaller matrices* at each step!

Let us compute our favorite 3×3 determinant using this approach. Developing along the first column, we have

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = a_{11} C_{11} + a_{21} C_{21} + a_{31} C_{31} = 1 \cdot C_{11} + 4 \cdot C_{21} + 7 \cdot C_{31},$$

where we compute the cofactors:

$$M_{11} = \begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ \cancel{4} & 5 & 6 \\ \cancel{7} & 8 & 9 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = -3, \qquad C_{11} = (-1)^{1+1} \left| M_{11} \right| = -3;$$

$$M_{21} = \begin{vmatrix} \cancel{1} & 2 & 3 \\ \cancel{4} & \cancel{5} & \cancel{6} \\ \cancel{7} & 8 & 9 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = -6, \qquad C_{21} = (-1)^{2+1} \left| M_{21} \right| = +6;$$

$$M_{31} = \begin{vmatrix} \cancel{1} & 2 & 3 \\ \cancel{4} & 5 & 6 \\ \cancel{7} & \cancel{8} & \cancel{9} \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = -3, \qquad C_{31} = (-1)^{3+1} \left| M_{31} \right| = -3.$$

The result is then:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 \cdot (-3) + 4 \cdot 6 + 7 \cdot (-3) = 0.$$

The result is the same as the basketweave method!

Next, we compute a 4×4 determinant:

$$\begin{vmatrix} 7 & 6 & 0 & 1 \\ 9 & -3 & 2 & -1 \\ 4 & 5 & 0 & 1 \\ 2 & 3 & 0 & -1 \end{vmatrix}$$

Due to the 3rd column having almost only zeros, it is advantageous to develop along it.

$$\begin{aligned} \begin{vmatrix} 7 & 6 & 0 & 1 \\ 9 & -3 & 2 & -1 \\ 4 & 5 & 0 & 1 \\ 2 & 3 & 0 & -1 \end{vmatrix} &= 0 \cdot \begin{vmatrix} 9 & -3 & -1 \\ 4 & 5 & 1 \\ 2 & 3 & -1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 7 & 6 & 1 \\ 4 & 5 & 1 \\ 2 & 3 & -1 \end{vmatrix} \\ &\qquad + 0 \cdot \begin{vmatrix} 7 & 6 & 1 \\ 9 & -3 & -1 \\ 2 & 3 & -1 \end{vmatrix} + 0 \cdot \begin{vmatrix} 7 & 6 & 1 \\ 9 & -3 & -1 \\ 4 & 5 & 1 \end{vmatrix} \\ &= -2 \cdot \begin{vmatrix} 7 & 6 & 1 \\ 4 & 5 & 1 \\ 2 & 3 & -1 \end{vmatrix}. \end{aligned}$$

We continue by computing the remaining 3×3 determinant. We develop it along the first row:

$$\begin{aligned} \begin{vmatrix} 7 & 6 & 1 \\ 4 & 5 & 1 \\ 2 & 3 & -1 \end{vmatrix} &= 7 \cdot \begin{vmatrix} 5 & 1 \\ 3 & -1 \end{vmatrix} - 6 \cdot \begin{vmatrix} 4 & 1 \\ 2 & -1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix} \\ &= 7 \cdot (-8) - 6 \cdot (-6) + 1 \cdot 2 \\ &= -10. \end{aligned}$$

Thus we conclude

$$\begin{vmatrix} 7 & 6 & 0 & 1 \\ 9 & -3 & 2 & -1 \\ 4 & 5 & 0 & 1 \\ 2 & 3 & 0 & -1 \end{vmatrix} = 20.$$

3.2 *Row and column properties: elementary operations*

Multiplication by a number.

Let k be a real or complex number and i a row number. Suppose we have two matrices A, B of the same size n, that we write as a vertical stack of rows $R_1, \ldots, R_n$:

$$A = \begin{bmatrix} - & R_1 & - \\ & \vdots & \\ - & R_i & - \\ & \vdots & \\ - & R_n & - \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} - & R_1 & - \\ & \vdots & \\ - & kR_i & - \\ & \vdots & \\ - & R_n & - \end{bmatrix}$$

THE ONLY DIFFERENCE BETWEEN THESE MATRICES IS THE i^{th} ROW, FOR WHICH ALL ENTRIES ARE MULTIPLIED BY k IN B COMPARED TO A.

Then, we have the rule:

$$\det(B) = k \det(A),$$

i.e. we can take the factor k out of the determinant.

Exchange of rows.

Let i, j be two *different* row numbers. Suppose we have two matrices A, B of the same size n:

$$A = \begin{bmatrix} & \vdots & \\ - & R_i & - \\ & \vdots & \\ - & R_j & - \\ & \vdots & \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} & \vdots & \\ - & R_j & - \\ & \vdots & \\ - & R_i & - \\ & \vdots & \end{bmatrix}$$

THE ONLY DIFFERENCE BETWEEN THESE MATRICES IS THE i^{th} and j^{th} ROWS, WHICH ARE EXCHANGED.

Then, we have the rule:

$$\det(B) = -\det(A),$$

i.e. exchanging two rows changes the sign of the determinant.

Addition.

Let i be a row number. Suppose we have three matrices A_1, A_2 and B of the same size n,

$$A_1 = \begin{bmatrix} & \vdots & \\ - & R_i & - \\ & \vdots & \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} & \vdots & \\ - & S_i & - \\ & \vdots & \end{bmatrix},$$

$$B = \begin{bmatrix} & \vdots & \\ - & R_i + S_i & - \\ & \vdots & \end{bmatrix}.$$

THE ONLY DIFFERENCE BETWEEN THE A_1, A_2 AND B MATRICES IS THE i^{th} ROW.
THE i^{th} ROW OF B IS THE ENTRYWISE SUM OF THE i^{th} ROWS OF A_1 AND A_2.

Then, AND ONLY THEN, we have the rule:

$$\det(B) = \det(A_1) + \det(A_2),$$

i.e. exchanging two rows changes the sign of the determinant.

Consequences:

1. If two rows of A are identical, then we have

$$\det(A) = 0.$$

2. We can use addition of a multiple of a row to another: if we have two matrices A, B, two row numbers i, j and a real or complex number k:

$$A = \begin{bmatrix} & \vdots & \\ - & R_i & - \\ & \vdots & \\ - & R_j & - \\ & \vdots & \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} & \vdots & \\ - & R_i & - \\ & \vdots & \\ - & R_j + kR_i & - \\ & \vdots & \end{bmatrix}$$

then we have the rule,

$$\det(A) = \det(B).$$

Here is an example:

$$\begin{vmatrix}1&2&3\\4&5&6\\7&8&9\end{vmatrix} \xrightarrow{R_2 \leftarrow R_2 - 4R_1} \begin{vmatrix}1&2&3\\0&-3&-6\\7&8&9\end{vmatrix} \xrightarrow{R_3 \leftarrow R_3 - 7R_1} \begin{vmatrix}1&2&3\\0&-3&-6\\0&-6&-12\end{vmatrix}$$

$$\xrightarrow{R_2 \leftarrow R_2/(-3)} (-3)\begin{vmatrix}1&2&3\\0&1&2\\0&-6&-12\end{vmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3/(-6)} (-3)(-6)\begin{vmatrix}1&2&3\\0&1&2\\0&1&2\end{vmatrix}$$

Now this last determinant has two identical rows: it is zero.

3.3 *Special properties of the determinant.*

Zero determinant: if $\det(A) = 0$, then the homogeneous system

$$A\vec{x} = \vec{0}$$

has infinitely many solutions (consistent, underdetermined system).

Matrix product and determinant. If A, B are of the same order n, then

$$\det(AB) = \det(A)\det(B).$$

Triangular matrices. If A is upper or lower triangular[4] then the determinant is the product of the diagonal elements:

$$\begin{vmatrix}1&6&10\\0&5&8\\0&0&7\end{vmatrix} = 1 \cdot 5 \cdot 7 = 35,$$

$$\begin{vmatrix}1&0&0\\3&7&0\\4&6&6\end{vmatrix} = 1 \cdot 7 \cdot 6 = 42.$$

[4] A matrix is upper (resp. lower) triangular if all elements below (resp above) the diagonal are zero.

CRAMER'S RULE. Let A be a matrix of order n with $\det(A) \neq 0$, and consider a linear system

$$A\vec{x} = \vec{b}.$$

Let A_i be the matrix obtained by replacing the i^{th} column by the column vector $\vec{b}$. Then, we obtain the i^{th} component of the solution of the linear system $A\vec{x} = \vec{b}$ by the formula,

$$x_i = \frac{\det(A_i)}{\det(A)}.$$

Here is an example: we solve the system,

$$\begin{cases} x + 2y = 5 \\ 3x + 4y = 6. \end{cases}$$

Then the corresponding matrix and vector are:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad \text{and} \qquad \vec{b} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}.$$

The determinant of A is:

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2 \neq 0,$$

so we can apply Cramer's rule:

$$x = \frac{\begin{vmatrix} 5 & 2 \\ 6 & 4 \end{vmatrix}}{-2} = -4, \qquad \text{and} \qquad y = \frac{\begin{vmatrix} 1 & 5 \\ 3 & 6 \end{vmatrix}}{-2} = 9/2.$$

4 Matrix inverse

4.1 *Definition.*

QUESTIONS:

1. What property must A fulfill so that

$$AB = AC \qquad \text{implies} \qquad B = C?$$

2. Is there a connexion with the determinant and linear systems with unique solutions,

$$\det(A) \neq 0?$$

> DEFINITION: A square matrix A is called *invertible* if there exists another matrix B such that
>
> $$AB = BA = I_n,$$
>
> where I_n is the $n \times n$ identity matrix.

Such a matrix B is then called the *inverse*[5] of A and noted A^{-1}.

There is only one such inverse!

ANSWER TO OUR PREVIOUS QUESTION: If A is invertible, then

$$\begin{aligned} AB = AC \qquad &\Longrightarrow \qquad A^{-1}AB = A^{-1}AC \\ &\Longrightarrow \qquad I_nB = I_nC \\ &\Longrightarrow \qquad B = C. \end{aligned}$$

4.2 *How to compute the inverse?*

1×1 MATRICES. If $A = [a]$, then $A^{-1} = [1/a]$.

- This is possible only if $a \neq 0$.

2×2 MATRICES. For a 2×2 matrix, we have

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \Longrightarrow \qquad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

- This is possible only if $\det(A) = ad - bc \neq 0$.

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{(-2)} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

[5] Textbook: Section 3.3

General $n \times n$ matrices.

- A is invertible if, and only if, its Reduced Row Equivalent Form is the identity matrix I_n.
- Algorithm: we apply the Gauß-Jordan elimination procedure!

$$\left[A \mid I_n\right] \qquad \longrightarrow \qquad \left[I_n \mid A^{-1}\right].$$

Example: find the inverse of

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{bmatrix}.$$

Step 1: Write the augmented matrix

$$\left[\begin{array}{ccc|ccc} 4 & 3 & 2 & 1 & 0 & 0 \\ 5 & 6 & 3 & 0 & 1 & 0 \\ 3 & 5 & 2 & 0 & 0 & 1 \end{array}\right].$$

Step 2: Work left to right to obtain zeros under the diagonal and ones on the diagonal.

$$\left[\begin{array}{ccc|ccc} 4 & 3 & 2 & 1 & 0 & 0 \\ 5 & 6 & 3 & 0 & 1 & 0 \\ 3 & 5 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 5 & 6 & 3 & 0 & 1 & 0 \\ 3 & 5 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow R_2 - 5R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 16 & 3 & -5 & 1 & 5 \\ 3 & 5 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - 3R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 16 & 3 & -5 & 1 & 5 \\ 0 & 11 & 2 & -3 & 0 & 4 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 5 & 1 & -2 & 1 & 1 \\ 0 & 11 & 2 & -3 & 0 & 4 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 5 & 1 & -2 & 1 & 1 \\ 0 & 1 & 0 & 1 & -2 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -2 & 2 \\ 0 & 5 & 1 & -2 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - 5R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -7 & 11 & -9 \end{array}\right]$$

Step 3: Work right to left to obtain zeros above the diagonal.

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -7 & 11 & -9 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 3 \\ 0 & 1 & 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -7 & 11 & -9 \end{array}\right]$$

Conclusion:

- A is row equivalent to I_n (they share the same RREF), thus A is invertible.
- We have obtained its inverse:

$$A^{-1} = \begin{bmatrix} 3 & -4 & 3 \\ 1 & -2 & 2 \\ -7 & 11 & -9 \end{bmatrix}.$$

- Check: compute the matrix product,

$$AA^{-1} = \begin{bmatrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & -4 & 3 \\ 1 & -2 & 2 \\ -7 & 11 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \checkmark$$