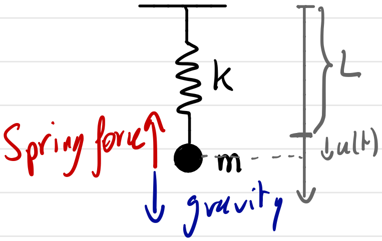
§ 2.7 Mechanical and Electrical Vibrations



Mass - spring system.

Hooke's law : $F_{spring} = -k(L+u)$

Gravity : $F_{gravity} = mg$

Friction : $F_{friction} = -\gamma u'$

- Equilibrium : $mg - kL = 0$
- Dynamical problem:

$$m\frac{d^2u}{dt^2} = mu''(t) = mg - k(L+u) - \gamma u' + F(t)$$

$$mu'' + \gamma u' + ku = F(t)$$

Balance of energy

$$mu''.u' + ku.u'$$

$$\frac{d}{dt}\Big(\underbrace{\tfrac{1}{2}m(u')^2}_{\text{Kinetic energy}} + \underbrace{\tfrac{1}{2}ku^2}_{\text{Potential Energy}}\Big) = \underbrace{-\gamma(u')^2}_{\text{Viscous losses}} + \underbrace{F(t).u'}_{\text{External work}}$$

I) Undamped Free Vibrations.

$F(t) = 0 \qquad \gamma = 0$

Harmonic oscillator: $mu'' + ku = 0$

Solution: characteristic equation $mr^2 + k = 0$

$$r = \pm i\sqrt{\frac{k}{m}}$$

General solution: $u(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$

where A, B arbitrary constants and

$\omega_0 = \sqrt{k/m}$ is the natural frequency.

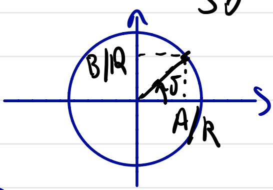
Better form for the solution:

$$u(t) = R\cos(\omega_0 t - \delta)$$

where R, δ are new constants:

$$u(t) = \underbrace{R\cos(\delta)}_{=A}\cos(\omega_0 t) + \underbrace{R\sin(\delta)}_{=B}\sin(\omega_0 t)$$

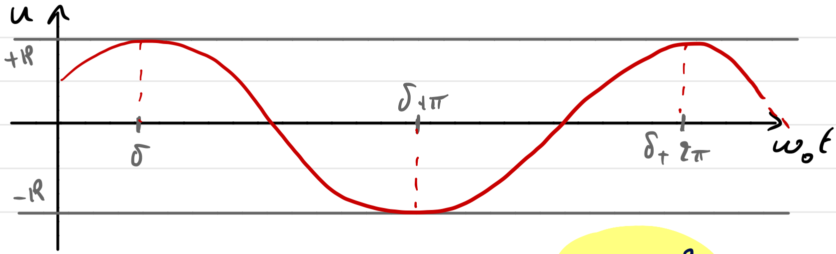
So



$$\boxed{\begin{array}{ll} A = R\cos(\delta) & B = R\sin(\delta) \\ R = \sqrt{A^2 + B^2} & \tan\delta = B/A \end{array}}$$

⚠ Check signs to find δ quadrant!

This is harmonic motion:



The period of the motion is $T_0 = \frac{2\pi}{\omega_0}$

* R: amplitude
* δ: phase angle

Note that $T_0 = 2\pi\sqrt{\frac{m}{k}}$

↳ period ↗ with mass, ↘ with spring stiffness

II) Damped free vibrations $F(t) = 0, \gamma > 0$

$$mu'' + \gamma u' + ku = 0$$

↳ Physical effect of damping.

Characteristic equation: $mr^2 + \gamma r + k = 0$

Three cases depending on the sign of $\Delta = \gamma^2 - 4mk$

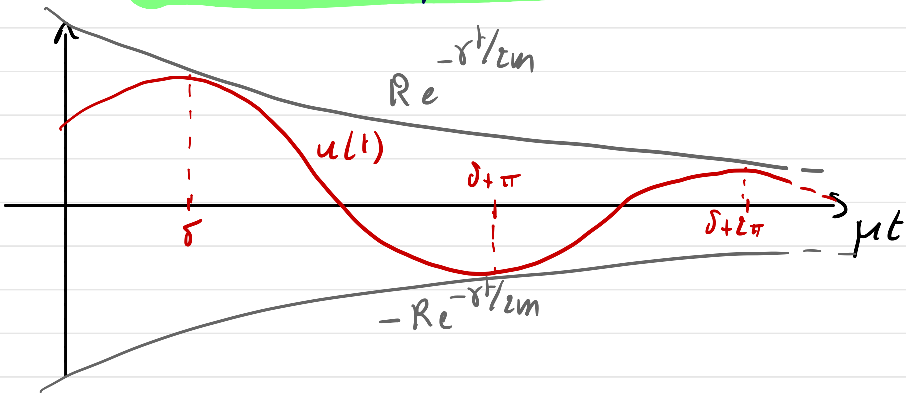
(a) $\Delta < 0$, $\gamma < 2\sqrt{mk}$: **UNDERDAMPED MOTION**

$$u(t) = \left(A\cos(\mu t) + B\sin(\mu t)\right)e^{-\frac{\gamma t}{2m}}$$

where $\mu = \sqrt{\dfrac{4mk-\gamma^2}{2m}}$ is the QUASI-FREQUENCY.

As before, we may rewrite $A = R\cos(\delta)$, $B = R\sin(\delta)$ to have

$$u(t) = R\cos(\mu t - \delta)\,e^{-\gamma t/2m}$$



Again, we define $T = 2\pi/\mu$: the QUASI-PERIOD

Note, as $\gamma \to 0$ we recover the harmonic free motion. In fact,

$$\frac{\mu}{\omega_0} = \sqrt{1 - \frac{\gamma^2}{4km}} < 1$$

or $T/T_0 > 1$.

(b) $\Delta = 0$, $\gamma = 2\sqrt{mk}$: **CRITICALLY DAMPED MOTION**

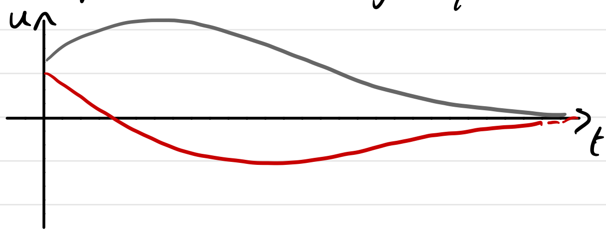
Then $$u(t) = (A + Bt)\, e^{-\gamma t/2m}$$

(c) $\Delta > 0$, $\gamma > 2\sqrt{mk}$: **OVERDAMPED MOTION**

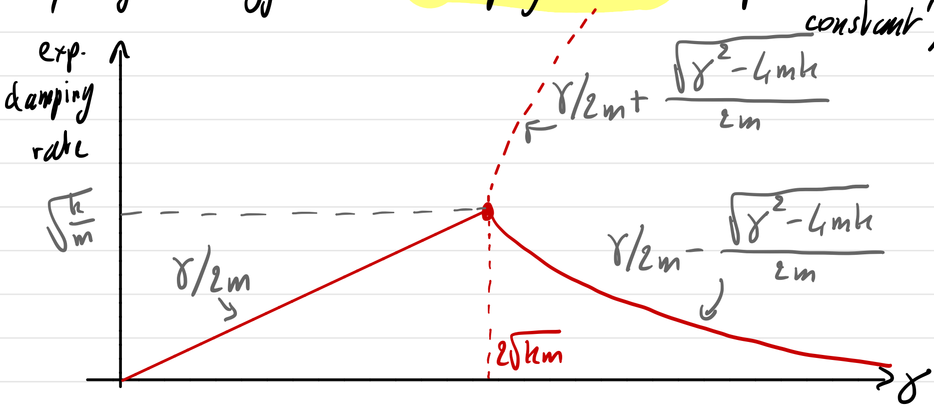
$$r_{1,2} = -\gamma/2m \pm \frac{\sqrt{\gamma^2 - 4mk}}{2m} < 0$$

Then $$u(t) = \left(A e^{\frac{\sqrt{\gamma^2 - 4mk}\, t}{2m}} + B e^{-\frac{\sqrt{\gamma^2 - 4mk}\, t}{2m}} \right) e^{-\gamma t/2m}$$

There are no oscillations in either case. At most, the mass passes once through equilibrium point:



Study of the effective damping rate (exponential decay constant)



↳ Critical damping yields the highest possible rate of return to the equilibrium position.

Numerical word example:

(a) A mass of 1 kg stretches a spring by 20 cm. The mass is set in motion from equilibrium with an upwards velocity of 5 cm/s and there is no damping.

First: determine the spring constant, k:

$mg = kL$ ↳ $k = \frac{mg}{L} = \frac{(1\,kg)(9.81\,m.s^{-2})}{0.2\,m}$

$k \approx 49\ kg.s^{-2}$

Then we have

$mu'' + ku = 0$ or $u'' + 49u = 0$

so $\omega_0 = \sqrt{k/m} = \sqrt{49} = 7$

$u(t) = A\cos(7t) + B\sin(7t)$

Initial conditions: $u(0) = 0 = A$ | $A = 0$

$u'(0) = 5 = -7B$ | $B = -5/7$

$u(t) = \frac{-5}{7}\sin(7t)$ or $u(t) = \frac{5}{7}\cos(7t + \frac{\pi}{2})$

(b) A viscous dampener is attached to the previous mass with a damping constant $\gamma = 10$ kg/s.
What is the new trajectory?

$$mu'' + \gamma u' + ku = 0 \quad \text{or} \quad u'' + 10u' + 49u = 0$$

Characteristic equation: $r^2 + 10r + 49 = 0$

$$\Delta = 100 - 4 \cdot 49 = -96 < 0$$

↳ Underdamped motion: $r_{1,2} = \underbrace{\frac{-10}{2}}_{\lambda = -5} \pm i \underbrace{\frac{4\sqrt{6}}{2}}_{\mu = 2\sqrt{6}}$

so $$u(t) = \left(A\cos(2\sqrt{6}t) + B\sin(2\sqrt{6}t)\right)e^{-5t}$$

Initial conditions: $u(0) = 0 = A$

$$u'(t) = 2\sqrt{6}\left(-A\sin(2\sqrt{6}t) + B\cos(2\sqrt{6}t)\right)e^{-5t} - 5\left(A\cos(2\sqrt{6}t) + B\sin(2\sqrt{6}t)\right)e^{-5t}$$

Thus $u'(0) = -5 = 2\sqrt{6}B - 5A$:

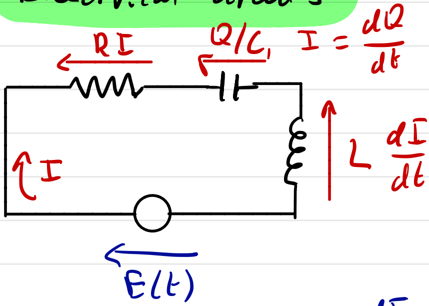
$$A = 0 \qquad B = \frac{-5}{2\sqrt{6}} = \frac{-5\sqrt{6}}{12}$$

Finally, $$u(t) = -\frac{5\sqrt{6}}{12}\sin(2\sqrt{6}t)e^{-5t}$$

(c) What would be the critical damping constant?

$$\gamma = 2\sqrt{km} = 2\sqrt{49} = 14 \text{ kg/s}$$

Electrical circuits



Kirchoff's law: $$L\frac{dI}{dt} + RI + \frac{Q}{C} = E(t)$$

or $$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$

Correspondance between mechanical / electrical components:

Mass/Spring system	Electrical circuit
MASS	INDUCTANCE
DAMPING	RESISTANCE
SPRING	1/CAPACITANCE

§3.8 Forced Periodic Vibrations

I With damping

Recall $mu'' + \gamma u' + ku = F(t)$

m: mass,
γ: damping coefficient,
k: spring constant,
$F(t)$: external force, here $F(t) = F_0 \cos(\omega t)$

↳ A periodic force is applied to the system.

Example: a car on a bumpy road;
a building in an earthquake;
a bridge in high winds.

↳ Inhomogeneous equation,

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t)$$

General solution:

$$u(t) = \underbrace{C_1 u_1(t) + C_2 u_2(t)}_{\substack{\text{homogeneous solution} \\ \text{as before.} \\ \to 0 \text{ as } t \to \infty}} + \underbrace{A \cos(\omega t) + B \sin(\omega t)}_{\substack{\text{Particular solution} \\ \text{Undetermined coefficients} \\ \text{Periodic}}}$$

Example

$$\begin{cases} u'' + 2u + u = 2\cos(5t) \\ \text{Initial conditions} \quad u(0)=1, \quad u'(0)=-1 \end{cases}$$

- Homogeneous solution: $r^2 + 2r + 1 = 0$

$\Delta = 4 - 4 = 0$

↳ Double root $\underline{r = -1}$

$$y_h(t) = (C_1 + C_2 t)e^{-t}$$

- Particular solution:

Undetermined coefficients:

$$y_p(t) = A\cos(5t) + B\sin(5t)$$

So

$$\begin{aligned} y_p'' + 2y_p' + y_p &= (-25A + 10B + A)\cos(5t) \\ &\quad + (-25B - 10A + B)\sin(5t) \\ &\overset{?}{=} 2\cos(5t) \end{aligned}$$

Identifying coefficients:

$$\begin{cases} 10B - 24A = 2 \\ -10A - 24B = 0 \end{cases} \longrightarrow A = \frac{-12}{5}B \longrightarrow \left(10 + \frac{12 \cdot 24}{5}\right)B = 2$$

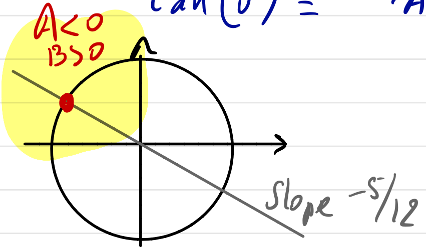
$$B = \frac{5}{169} \quad \text{and} \quad A = \frac{-12}{169}$$

$$y_p(t) = \frac{-12}{169}\cos(5t) + \frac{5}{169}\sin(5t)$$

Radial form: $y_p(t) = R\cos(5t - \delta)$

Then $R = \dfrac{\sqrt{12^2+5^2}}{169} = \dfrac{\sqrt{169}}{169} = \dfrac{1}{13}$

$\tan(\delta) = B/A = -5/12$



$\rightarrow \delta = \operatorname{atan}\left(\dfrac{-5}{12}\right) + \pi$

$\boxed{\delta \approx 2.75}$

So $y_p(t) \approx \dfrac{1}{13}\cos(5t - 2.75)$

- General solution:

$$u(t) = (C_1 + C_2 t)e^{-t} + \frac{1}{13}\cos(5t - \delta)$$

- Initial conditions:

$$u(0) = C_1 + \frac{1}{13}\cos(\delta) = C_1 - \frac{12}{169} = 1$$

so $\boxed{C_1 = \dfrac{181}{169}}$

$$u'(0) = C_2 - C_1 + \frac{5}{13}\sin(\delta) = -1$$

$$C_2 = \frac{181 - 25 - 169}{169} = \frac{-1}{13}$$

$$u(t) = \underbrace{\left(\frac{181}{169} - \frac{1}{13}t\right)e^{-t}}_{\text{TRANSIENT}} + \underbrace{\frac{1}{13}\cos(5t - 2.75)}_{\text{STEADY-STATE}}$$

General case: Question – What is the amplitude of the steady state response?

↳ Determine particular solution

$$u(t) = A\cos(\omega t) + B\sin(\omega t) = R\cos(\omega t - \delta)$$

to diff. eq. $mu'' + \gamma u' + ku = F_0 \cos(\omega t)$.

Some computations lead to

$$A = \frac{k - m\omega^2}{(k-m\omega^2)^2 + \gamma^2\omega^2} F_0$$

$$B = \frac{\gamma\omega}{(k-m\omega^2)^2 + \gamma^2\omega^2} F_0$$

So the amplitude of the response is

$$R = \frac{|F_0|}{\sqrt{(k-m\omega^2)^2 + \gamma^2\omega^2}}$$

Adimensionalization: introduce

$\omega_0 = \sqrt{\frac{k}{m}}$	$R_0 = \frac{\lvert F_0\rvert}{k}$	$\gamma_c = 2\sqrt{mk}$
Natural frequency	Response to constant force F_0	Critical damping
$\rightarrow \tilde{\omega} = \omega/\omega_0$	$\tilde{R} = R/R_0$	$\Gamma = \gamma^2/mk$

Then after some computations:

$$\tilde{R}(\omega) = \frac{1}{\sqrt{\dfrac{(k-m\omega^2)^2+\gamma^2\omega^2}{k^2}}}$$

$$= \frac{1}{\sqrt{\left(1-\frac{m}{k}\omega^2\right)^2+\frac{\gamma^2}{mk}\frac{m}{k}\omega^2}}$$

$$= \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2+\Gamma\left(\frac{\omega}{\omega_0}\right)^2}}$$

The adimensionalized response amplitude is

$$\tilde{R}(\tilde{\omega}) = \frac{1}{\sqrt{(1-\tilde{\omega}^2)^2+\Gamma\tilde{\omega}^2}}$$

↳ Only parameter: Γ.

Parametric study

$$\begin{cases} \tilde{R} \to 1 & \text{as } \tilde{\omega} \to 0 \\ \tilde{R} \to 0 & \text{as } \tilde{\omega} \to \infty \end{cases} \quad \text{or} \quad \begin{cases} R \to R_0 = \dfrac{|F_0|}{k} & \text{as } \omega \to 0 \\ R \to 0 & \text{as } \omega \to \infty \end{cases}$$

In between 0 and ∞, maybe a maximum?
This would correspond to a minimum of the denominator,
$(1-\tilde{\omega}^2)^2+\Gamma\tilde{\omega}^2$.

Changing variables: $x = \tilde{\omega}^2 \geq 0$, we note that a minimum for $(1-x)^2 + \Gamma x$ corresponds to a zero for the derivative,

$$2(x-1) + \Gamma = 0 \quad \text{or} \quad x_{max} = 1 - \frac{\Gamma}{2}.$$

Two cases:

* if $\Gamma < 2$, then $x_{max} = 1 - \frac{\Gamma}{2} \geq 0$ corresponds to a maximum amplitude frequency

$$\tilde{\omega}_{max} = \sqrt{1 - \frac{\Gamma}{2}} \quad \text{or} \quad \omega_{max} = \omega_0 \sqrt{1 - \frac{\Gamma}{2}}$$

The corresponding maximum amplitude is

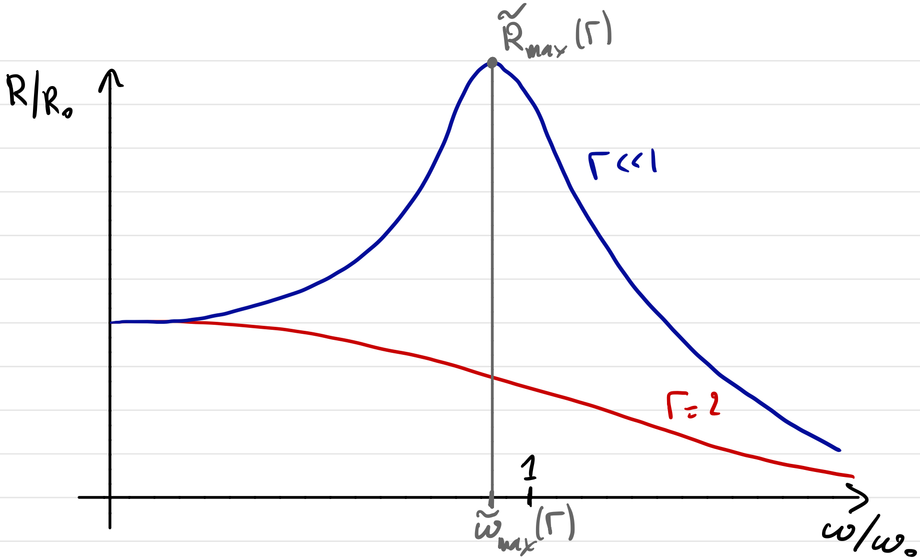
$$\tilde{R}(\tilde{\omega}_{max}) = \frac{1}{\sqrt{\Gamma(1 - \Gamma/4)}}$$

or

$$R(\omega_{max}) = \frac{F_0}{\gamma \omega_0 \sqrt{1 - \frac{\gamma^2}{4mk}}}$$

This maximum is the RESONANCE PEAK.

* If $\Gamma \geq 2$, i.e. $\gamma \geq \sqrt{2mk} = \gamma_c/\sqrt{2}$ there is no maximum: $R(\omega)$ is decreasing from $\omega = 0$ to ∞.



II) Case without damping.

Now dynamics are governed by the D.E.:

$$m u'' + k u = F_0 \cos(\omega t).$$

For $\omega \neq \omega_0$ the solution writes in general

$$u(t) = \underbrace{C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)}_{\text{homogeneous solution}} + \underbrace{\frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)}_{\text{particular solution}}$$

This time there is no transient: this is a sum of two signals with different frequencies and no decay.

In particular, the case $\omega \approx \omega_0$ is of interest.

Let us investigate the motion starting from rest:

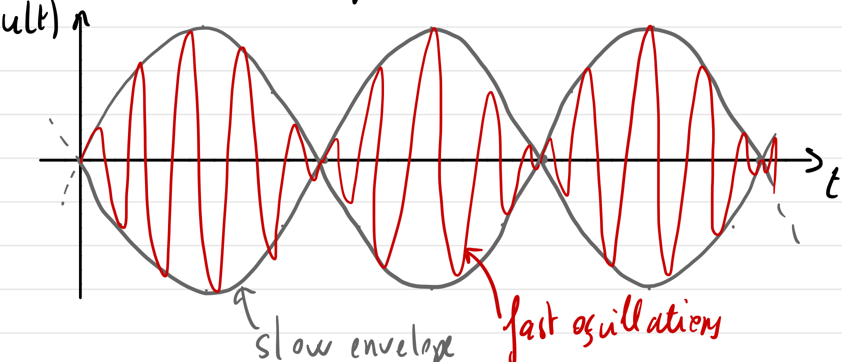
$$u(0) = u'(0) = 0.$$

Then we compute c_1, c_2 and get

$$u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \left(\cos(\omega t) - \cos(\omega_0 t) \right)$$

$$= \frac{2F_0}{m(\omega_0^2 - \omega^2)} \underbrace{\sin\left(\frac{\omega_0 - \omega}{2} t \right)}_{\substack{\text{slow} \\ \text{since } \frac{\omega_0 - \omega}{2} \ll 1}} \underbrace{\sin\left(\frac{\omega_0 + \omega}{2} t \right)}_{\substack{\text{fast} \\ \frac{\omega_0 + \omega}{2} \approx \omega_0}}$$

This is the BEATS phenomenon:

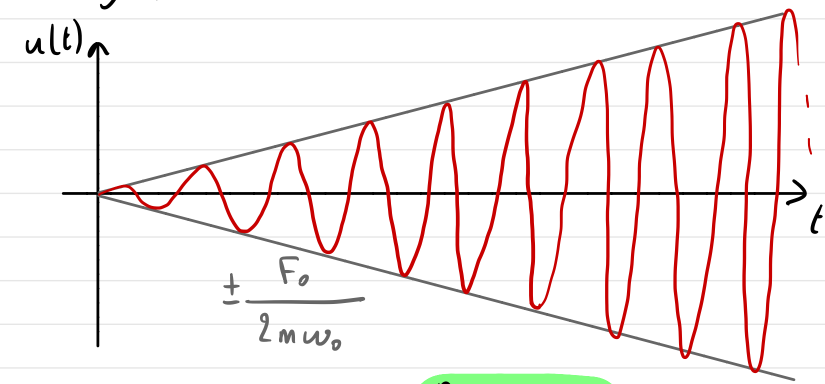


$$\pm \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{\omega_0 - \omega}{2} t \right)$$

When $\omega = \omega_0$ (RESONANT case) then the solution takes the form

$$u(t) = \underbrace{C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)}_{\text{constant amplitude osillations.}} + \underbrace{\frac{F_0}{2m\omega_0} t \sin(\omega_0 t)}_{\text{osillations growing without bound}}$$

Starting from rest, $u(0) = u'(0) = 0$, we find $C_1 = C_2 = 0$



RESONANCE