§3.4 Repeated Roots

Consider the O.E. $ay'' + by' + cy = 0$
with $a \neq 0$ and $\Delta = b^2 - 4ac = 0$.

↳ Difficulty: only ONE root of the characteristic equation:

$$r_1 = r_2 = -b/2a.$$

Then we have only one solution of the form $y(t) = e^{rt}$ which is

$$y_1(t) = e^{-bt/2a}.$$

<u>How to get a second, different solution?</u>

There is a variety of ideas on how to do it. Here idea by D'Alembert (related to "variation of the constant" for 1^{st} order ODEs)

$$y(t) = C\,y_1(t) \xrightarrow[\text{Try}]{} y(t) = v(t)\,y_1(t).$$

Then: $y_1(t) = v(t)\,e^{-bt/2a}$

$$y_1'(t) = \left(v'(t) - \frac{b}{2a}v(t)\right)e^{-bt/2a}$$

$$y_1''(t) = \left(v''(t) - \frac{b}{a}v'(t) + \left(\frac{b}{2a}\right)^2 v(t)\right)e^{-bt/2a}$$

Plug into the D.E. :

$$a\left(v'' - \frac{b}{a}v' + \left(\frac{b}{2a}\right)^2 v\right)e^{-bt/2a} + b\left(v' - \frac{b}{2a}v\right)e^{-\frac{bt}{2a}}$$

$$+ c\, v\, e^{-bt/2a} = 0$$

or

$$a v'' + \left(b - \cancel{a\frac{b}{a}}\right)v' + \left(\cancel{\frac{b^2}{4a} - \frac{b^2}{2a}} + c\right)v = 0$$

$\frac{b^2 - 2b^2 + 4ac}{4a} = 0$ $\quad$ 0

or $\quad v''(t) = 0.$

Then we have $\quad v(t) = c_1 + c_2 t$

$$y(t) = c_1 e^{-bt/2a} + c_2 t e^{-bt/2a}$$

THE GENERAL SOLUTION.

We extract the fundamental set of solutions:

$$y_1(t) = e^{-\frac{bt}{2a}} \qquad y_2(t) = t e^{-bt/2a}.$$

To check, we compute the Wronskian:

$$W[y_1, y_2](t) = \begin{vmatrix} e^{-bt/2a} & t e^{-bt/2a} \\ -b/2a\, e^{-bt/2a} & (1 - bt/2a)e^{-bt/2a} \end{vmatrix}$$

③

so $$W(t) = \left(1 - \frac{bt}{2a}\right)e^{-bt/a} - \left(-\frac{b}{2a}\right)t e^{-bt/a}$$

$$W(t) = e^{-bt/a}$$

Note: check Abel's theorem prediction.

Also: again, the solutions are different from previous cases ($\Delta > 0$, $\Delta < 0$).

Example $$\begin{cases} y'' + 6y' + 9y = 0, \\ y(0) = 1, \quad y'(0) = 1 \end{cases}$$

* Characteristic equation:

$$r^2 + 6r + 9 = 0$$

$$\Delta = 6^2 - 4 \times 9 = 0 \quad ; \quad r = \frac{-6}{2} = -3.$$

* Repeated roots!

$$y_1(t) = e^{-3t} \qquad y_2(t) = t e^{-3t}$$

* General solution:

$$y(t) = C_1 e^{-3t} + C_2 t e^{-3t}$$

③

④

<u>Wronskian</u>: $W(t) = \begin{vmatrix} e^{-3t} & te^{-3t} \\ -3e^{-3t} & (1-3t)e^{-3t} \end{vmatrix}$

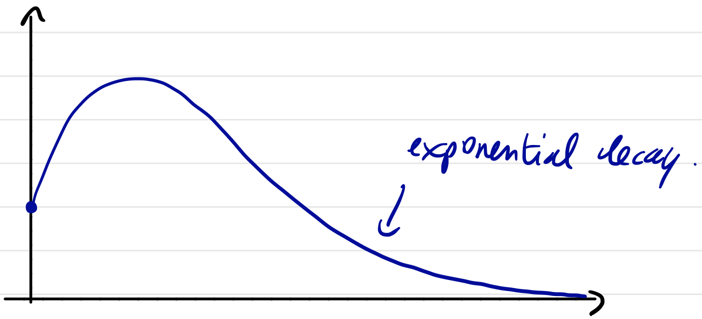
$$= (1-\cancel{3t})e^{-6t} + \cancel{3te^{-6t}}$$

$$W(0) = e^{-6.0} = 1$$

* Next, initial conditions:

$$\begin{cases} y(0) = c_1 = 1 \\ y'(0) = -3c_1 + c_2 = 1 \end{cases} \rightarrow c_2 = 4$$

$$y(t) = (1+4t)e^{-3t}$$



④

Note: technique of reduction of order.

Procedure to find a second solution.

Consider the D.E. $\quad y'' + p(t)\,y' + q(t)\,y = 0.$

Assume we know one solution $y_1(t)$.

To find a 2^{nd} solution, write $\quad y(t) = v(t)\,y_1(t).$

Then
$$y'(t) = v'(t)\,y_1(t) + v(t)\,y_1'(t)$$
$$y_1''(t) = v''(t)\,y_1(t) + 2v'(t)\,y_1'(t) + v(t)\,y_1''(t)$$

so that

$$\begin{aligned} y'' + p(t)y' + q(t)y = \; & y_1(t)\,v''(t) \\ & + \big(2y_1'(t) + p(t)\,y_1(t)\big)\,v'(t) \\ & + \underbrace{\big(y_1''(t) + p(t)\,y_1'(t) + q(t)\,y_1(t)\big)}_{0}\,v(t) \end{aligned}$$

so for $y(t)$ to be a solution, v satisfies

$$y_1(t)\,v''(t) + \big(2y_1'(t) + p(t)\,y_1(t)\big)\,v'(t) = 0$$

↳ 1^{st} order ODE in $v'(t)$

Example (1) $\begin{cases} 2t^2 y'' - t y' + y = 0 \\ y(1) = 0, \quad y'(1) = 1 \end{cases}$

We note that $y_1(t) = t$ is a solution.
Then we propose to find solutions of the form

$$y(t) = v(t)\, t.$$

Then
$$y'(t) = v'(t)\, t + v(t)$$
$$y''(t) = v''(t)\, t + 2v'(t)$$
so

$$(1) \Leftrightarrow 2t^2\left(t v'' + 2v'\right) - t\left(t v' + \cancel{v}\right) + \cancel{t v} = 0$$

$$2t^3 v'' + (4t^2 - t^2) v' = 0$$

$$\frac{d}{dt}(v') + \frac{3}{2t} v' = 0$$

Integrating factor:
$$v'(t) = C \exp\left(-\int \frac{3}{2r}\, dr\right)$$
$$= C \exp\left(-\frac{3}{2} \ln t\right)$$

$$v'(t) = \frac{C}{t^{3/2}} \quad \text{and} \quad v(t) = C_1 + \frac{C_2}{\sqrt{t}}.$$

$$y(t) = C_1 t + C_2 \sqrt{t}$$

Finally: initial conditions

$$y(1) = C_1 + C_2 = 0$$
$$y'(1) = C_1 + \frac{C_2}{2} = 1$$
$$\Rightarrow \begin{cases} C_2 = -2 \\ C_1 = 2 \end{cases}$$

$$y(t) = 2\left(t - \sqrt{t}\right)$$

Summary: $ay'' + by' + cy = 0$

Characteristic Equation: $ar^2 + br + c = 0.$

Case 1) $b^2 - 4ac > 0.$ Two real different roots.

$$r_1 = \frac{-b}{2a} + \frac{\sqrt{b^2-4ac}}{2a}, \quad r_2 = \frac{-b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}$$

Gen. solution: $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

Linear combination of two exponentials. No oscillations; behavior at ∞ depends on sign of r_1, r_2

Case 2) $b^2 - 4ac = 0.$ One real repeated root.

$r = -b/2a$, Gen. solution: $y(t) = (c_1 + c_2 t)e^{-bt/2a}.$

Behavior at ∞ depends on sign of r. Changes sign if $c_2 t$

Case 3) $b^2 - 4ac < 0.$ Two complex roots.

$$r_{1,2} = \lambda \pm i\mu, \quad \lambda = \frac{-b}{2a}, \quad \mu = \frac{\sqrt{4ac-b^2}}{2a}.$$

Gen. solution: $y(t) = \left(c_1 \cos(\mu t) + c_2 \sin(\mu t)\right)e^{\lambda t}.$

Oscillations. Either damped or growing. Behavior at ∞ depends on sign of $\lambda = -b/2a.$

Wed. Oct 10 — Non-homogeneous Equations: Method of Undetermined Coefficients.

Let's go beyond the homogeneous case:

$$L[y] = y'' + p(t)y' + q(t)y = g(t)$$

with p, q, g given continuous functions on the open interval I.

First Observation: If $Y_1(t)$ and $Y_2(t)$ are two solutions of the linear D.E.

$$L[y] = g,$$

then $Y_1 - Y_2$ solves the corresponding homogeneous eqn,

$$L[y] = 0.$$

As a result, given a fundamental set of solutions y_1, y_2 of this homogeneous eq. then

$$Y_1(t) - Y_2(t) = C_1 y_1(t) + C_2 y_2(t)$$

for some constants C_1 and C_2.

Check: $L[Y_1 - Y_2] = L[Y_1] - L[Y_2] = g - g = 0.$

Implication: all solutions of the D.E. $L[y]=g$ can be written under the form,

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$$

where
- y_1, y_2 form a fundamental set of solutions for the homogeneous equation, $L[y]=0$
- c_1, c_2 are arbitrary constants,
- y_p is a particular solution of the non-homogeneous equation!

We can also write $y(t) = y_h(t) + y_p(t)$

solution of homogeneous eqn

particular solution of non-homogeneous.

Example 1

$$\begin{cases} y'' + y = 1 \\ y(0) = 0, \quad y'(0) = 0 \end{cases}$$

Particular solution: $y_p(t) = 1$

Homogeneous solution: $y'' + y = 0$

$r^2 + 1 = 0 \to r = \lambda \pm i\mu, \quad \lambda = 0, \quad \mu = 1$

$$y_h(t) = c_1 \cos(t) + c_2 \sin(t)$$

Conclusion: $y(t) = C_1 \cos(t) + C_2 \sin(t) + 1$

$$\left.\begin{array}{l} y(0) = C_1 + 1 = 0 \\ y'(0) = C_2 = 0 \end{array}\right\} \quad y(t) = 1 - \cos(t)$$

Summary: Linear non-homogeneous equation:

We have to solve two separate problems:

① Homogeneous equation $L[y] = 0 \to y_h(t)$

⇝ See previous sections.

② Particular solution $y_p(t)$

⇝ In general: hard problem.

Sometime (special cases): technique of undetermined coefficients.

③ Final result: $y(t) = y_h(t) + y_p(t)$

Now, use initial conditions to obtain the values of coefficients C_1, C_2 in $y_h(t)$.

Method of Undetermined Coefficients.

Main idea: some "families" of functions form closed ensembles under differentiation.

Examples:

* Exponentials: $f(t) = C e^{at}$

  ↳ $f'(t) = \underline{(aC)} e^{at}$

  Only modification is the value of the coefficient.

* Polynomials: $p(t) = a + bt + ct^2 + \dots$

  ↳ $p'(t) = b + 2ct + \dots$

  Also a polynomial!

* Linear combinations of Sines and cosines: $f(t) = C_1 \cos(at) + C_2 \sin(at)$

  ↳ $f'(t) = a C_2 \cos(at) + (-aC_1) \sin(at)$

  Note: need <u>both</u> sine and cosines!

In each case, if $f(t)$ has the given type then $L[f]$ also does

↳ Idea: match the type of RHS $g(t)$ and find the right coefficients.

Notes: Usually restricted to cases with...
- $L[y]$ has constant coefficients,
- $g(t)$ is fairly simple.

Example 1 $y'' - 2y' + y = e^{2t}$.

Let us find a particular solution $y_p(t)$.

Plausible: $y_p(t)$ has the form Ae^{2t}.

$$y_p'(t) = 2Ae^{2t}; \quad y_p''(t) = 4Ae^{2t}.$$

So we want $(4A - 4A + A)e^{2t} = e^{2t}$

or $A = 1$ and $y_p(t) = e^{2t}$.

Example 2 $y'' - 2y' + y = t^3$

Idea: $y_p(t) = A + Bt + Ct^2 + Dt^3$

$$\hookrightarrow \begin{aligned} y_p'(t) &= B + 2Ct + 3Dt^2 \\ y_p''(t) &= 2C + 6Dt \end{aligned}$$

So

$$\begin{aligned} &2C + 6Dt \\ &- 2B - 4Ct - 6Dt^2 \\ &+ A + Bt + Ct^2 + Dt^3 = t^3 \end{aligned}$$

Matching the coefficients:

$$\begin{cases} 2C - 2B + A = 0 \\ 6D - 4C + B = 0 \\ -6D + C = 0 \\ D = 1 \\ C = 6 \\ B = 4\cdot 6 - 6\cdot 1 = 18 \\ A = 2\cdot 18 - 2\cdot 6 = 24 \end{cases} \quad y_p(t) = 24 + 18t + 6t^2 + t^3$$

Example 3 $\quad y'' - 2y' + y = 3\cos(2t)$

Pb : Find a particular solution $y_p(t)$.

Let us try the family of sines/cosines above:

$$y_p(t) = A\cos(2t) + B\sin(2t).$$

$$\hookrightarrow \begin{aligned} y_p'(t) &= 2B\cos(2t) - 2A\sin(2t) \\ y_p''(t) &= -4A\cos(2t) - 4B\sin(2t) \end{aligned}$$

$$\begin{aligned} y_p'' - 2y_p' + y_p &= (-4A - 4B + A)\cos(2t) + (-4B + 4A + B)\sin(2t) \\ &= (-3A - 4B)\cos(2t) + (4A - 3B)\sin(2t) \\ &= 3\cos(2t) \end{aligned}$$

Matching the coefficients we find

$$\begin{cases} -3A - 4B = 3 \\ 4A - 3B = 0 \end{cases} \quad \text{so} \quad B = 4/3\, A$$

$$\left(-3 - \frac{16}{3}\right)A = 3 \quad \text{and} \quad A = \frac{-9}{25},\ B = \frac{-12}{25}$$

Finally, $\quad y_p(t) = -\dfrac{3}{25}\left(3\cos(2t) + 4\sin(2t)\right)$

Example 4 Mix and Match

"Product rule"

$$y'' - 2y' + y = \underbrace{t}_{\text{polynomials of degree 2}} \underbrace{\sin(t)}_{\text{sine/cosine family}} + 1$$

Sum: Treat separately and add results.

For products: use products of the same families

$$g_1(t) = t \sin(t) \longrightarrow y_{p,1}(t) = (A + Bt)\cos(t) + (C + Dt)\sin(t)$$

Hang on to your hats!

$$y'_{p,1}(t) = B\cos(t) - (A + Bt)\sin(t) + D\sin(t) + (C + Dt)\cos(t)$$
$$= (B + C + Dt)\cos(t) + (D - A - Bt)\sin(t)$$

$$y''_{p,1}(t) = D\cos(t) - (B + C + Dt)\sin(t) - B\sin(t) + (D - A - Bt)\cos(t)$$
$$= (2D - A - Bt)\cos(t) - (2B + C + Dt)\sin(t)$$

Now,

$$y''_{p,1} - 2y'_{p,1} + y_{p,1} = \left(2D \cancel{-A} \cancel{-Bt} - 2(B + C + Dt) \cancel{+A} \cancel{+Bt}\right)\cos(t)$$
$$+ \left(-2B \cancel{-C} \cancel{-Dt} - 2(D - A - Bt) \cancel{+C} \cancel{+Dt}\right)\sin(t)$$
$$= \left(2D - 2B - 2C - 2Dt\right)\cos(t) + \left(2A - 2B - 2D + 2Bt\right)\sin(t)$$
$$\stackrel{?}{=} t\sin(t)$$

Example 5 Mix and Match

product: try similar product

$$y'' - 2y' + y = e^{2t} + t \sin(t) e^{-t}$$

Sum: treat separately and add solutions later

* First: $g_1(t) = e^{2t} \longrightarrow$ above computation: $y_{p,1}(t) = e^{2t}$

* Next: do $g_2(t) = t \sin(t) e^{-t}$

First-order poly: $A + Bt$

Sines/Cosines: $C\cos(t) + D\sin(t)$

Exponential: Ee^{-t}

Combine

$$y_p(t) = (A + Bt)\cos(t)e^{-t} + (C + Dt)\sin(t)e^{-t}.$$

Hang on to your hats!

$$y_p'(t) = B\cos(t)e^{-t} + (A+Bt)(-\sin(t) - \cos(t))e^{-t}$$
$$+ D\sin(t)e^{-t} + (C+Dt)(\cos(t) - \sin(t))e^{-t}$$
$$= (B - A + C + (D-B)t)\cos(t)e^{-t}$$
$$+ (D - C - A - (D+B)t)\sin(t)e^{-t}$$

$$
\begin{aligned}
y_p''(t) &= (D-B)\cos(t)e^{-t} \\
&\quad + \big(B-A+C+(D-B)t\big)\big(-\sin(t)-\cos(t)\big)e^{-t} \\
&\quad - (D+B)\sin(t)e^{-t} \\
&\quad + \big(D-C-A-(D+B)t\big)\big(\cos(t)-\sin(t)\big)e^{-t} \\
&= \big(D-B-B+\cancel{A}-C+D-C-\cancel{A} \\
&\qquad + (\cancel{B}-D-D-\cancel{B})t\big)\cos(t)e^{-t} \\
&\quad + \big(A-B-\cancel{C}-D-B-D+\cancel{C}+A \\
&\qquad + (B-\cancel{D}+\cancel{D}+B)t\big)\sin(t)e^{-t} \\
&= 2(D-B-C-Dt)\cos(t)e^{-t} \\
&\quad + 2(A-B-D+Bt)\sin(t)e^{-t}
\end{aligned}
$$

Then:

$$
\begin{aligned}
y_p''-2y_p'+y_p &= \big[2(D-B-C)-2(B-A+C)+A\big]\cos(t)e^{-t} \\
&\quad + \big[-2D-2(D-B)+B\big]t\cos(t)e^{-t} \\
&\quad + \big[2(A-B-D)-2(D-A-C)+C\big]\sin(t)e^{-t} \\
&\quad + \big[2B+2(B+D)+D\big]t\sin(t)e^{-t} \\
&= (3A-4B-4C+2D)\cos(t)e^{-t} \\
&\quad + (3B-4D)t\cos(t)e^{-t} \\
&\quad + (4A-2B+3C-4D)\sin(t)e^{-t} \\
&\quad + (4B+3D)t\sin(t)e^{-t} \overset{?}{=} t\sin(t)e^{-t}.
\end{aligned}
$$

Let us identify the coefficients:

$$\begin{cases} 3B - 4D = 0 \\ 4B + 3D = 1 \end{cases} \quad ; \quad \begin{cases} 3A - 4B - 4C + 2D = 0 \\ 4A - 2B + 3C - 4D = 0 \end{cases}$$

$$\Downarrow$$

$$B = \frac{4}{3}D \; ; \; \left(\frac{16}{3} + 3\right)D = 1$$

$$D = \frac{3}{25} \qquad B = \frac{4}{25}$$

$$\begin{cases} 3A - 4C = 4B - 2D = \frac{10}{25} = \frac{2}{5} \\ 4A + 3C = 2B + 4D = \frac{20}{25} = \frac{4}{5} \end{cases}$$

Add 3 x first line + 4 x second one:

$$(3^2 + 4^2)A = \frac{6}{5} + \frac{16}{5} = \frac{22}{5}$$

$$A = \frac{22}{125}$$

and similarly,

$$(4^2 + 3^2)C = \frac{-8}{5} + \frac{12}{5} = \frac{4}{5}$$

$$C = \frac{4}{125}$$

So finally, $$y_p(t) = \left(\frac{22}{125} + \frac{3}{25}t\right)\cos(t)\,e^{-t} + \left(\frac{4}{125} + \frac{3}{25}t\right)\sin(t)\,e^{-t}.$$

Example 6 Breakdown

$$y'' - 2y' + y = e^t$$

Following the previous idea, we try

$$y_p(t) = A e^t \quad (= y_p'(t) = y_p''(t))$$

Plug into the equation:

$$Ae^t - 2Ae^t + Ae^t = 0 = e^t$$ Won't work!

The problem: Ae^t is a solution of the homogeneous equation...

The solution: multiply by t until it works.

$$y_p(t) = Ate^t \quad \text{so} \quad \begin{aligned} y_p'(t) &= A(t+1)e^t \\ y_p''(t) &= A(t+2)e^t \end{aligned}$$

$$A(t+2)e^t - 2A(t+1)e^t + Ate^t = 0$$

Pb: te^t is also a solution of the eqn!

Again! $$y_p(t) = At^2e^t \quad \text{so} \quad \begin{aligned} y_p'(t) &= A(t^2+2t)e^t \\ y_p''(t) &= A(t^2+4t+2)e^t \end{aligned}$$

$$A(\cancel{t^2} + \cancel{4t} + 2 - 2(\cancel{t^2} + \cancel{2t}) + \cancel{t^2})e^t = 2Ae^t \stackrel{?}{=} e^t$$

$\hookrightarrow$ $A = 1/2$ $$y_p(t) = \frac{1}{2}t^2e^t$$

Summary

$$g(t) = g_1(t) + g_2(t) + \dots$$

* Polynomials:

$$g(t) = a_0 t^n + a_1 t^{n-1} + \dots + a_n = P_n(t)$$

↳ Try $y_p(t) = A_0 t^n + A_1 t^{n-1} + \dots + A_n$

* Exponentials:

$$g(t) = P_n(t) e^{rt}$$

↳ Try $y_p(t) = (A_0 t^n + A_1 t^{n-1} + \dots + A_n) e^{rt}$

Particular case: $n = 0$, $g_i(t) = a e^{rt}$

↳ $y_{p,i}(t) = A e^{rt}$

* Linear combinations of sines/cosines:

$$g(t) = P_n(t) \cos(t) e^{rt} + Q_n(t) \sin(t) e^{rt}$$

↳ Try
$$y_p(t) = (A_0 t^n + A_1 t^{n-1} + \dots + A_n) \cos(t) e^{rt} + (B_0 t^n + B_1 t^{n-1} + \dots + B_n) \sin(t) e^{rt}$$

If BREAKDOWN: multiply by t until it works.

↳ it doesn't work!

Last example $\quad y'' + y' = t^2 e^{-t} + \cos(t)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ pb 1 $\qquad\qquad$ pb 2

* First deal with $\quad g_1(t) = t^2 e^{-t}$
  ↳ Try $\quad y_p(t) = (A + Bt + Ct^2)e^{-t}$

$$y_p'(t) = (B - A + (2C - B)t - Ct^2)e^{-t}$$

$$y_p''(t) = (2C - B - 2Ct - (B - A + (2C - B)t - Ct^2))e^{-t}$$

$$= (2C - 2B + A + (B - 4C)t + Ct^2)e^{-t}$$

Then $\quad y_p'' + y_p' = (2C - B - 2Ct)e^{-t} =? \; t^2 e^{-t}$

Doesn't work!

Multiply by t: $\quad y_p(t) = (At + Bt^2 + Ct^3)e^{-t}$

$$y_p'(t) = (A + (2B - A)t + (3C - B)t^2 - Ct^3)e^{-t}$$

$$y_p''(t) = (2B - A + 2(3C - B)t - 3Ct^2 - A - (2B - A)t - (3C - B)t^2 + Ct^3)e^{-t}$$

$$= (2B - 2A + (A - 4B + 6C)t + (B - 6C)t^2 + Ct^3)e^{-t}$$

Then $\quad y_p'' + y_p' = (2B - A + (6C - 2B)t - 3Ct^2)e^{-t}$

$$=? \; (0 + 0 \cdot t + 1 \cdot t^2)e^{-t}$$

Coefficient match: $\begin{cases} -3C = 1 \\ 6C - 2B = 0 \\ 2B - A = 0 \end{cases} \longrightarrow \begin{cases} C = -1/3 \\ B = -1 \\ A = -2 \end{cases}$

So $$y_{p,1}(t) = -\left(2t + t^2 + t^3/3\right)e^{-t}$$

* Now deal with $g_2(t) = \cos(t)$

$$y_p(t) = A\cos(t) + B\sin(t)$$

$$y_p'(t) = B\cos(t) - A\sin(t)$$

$$y_p''(t) = -A\cos(t) - B\sin(t)$$

So $$y''_p + y'_p = (B-A)\cos(t) - (A+B)\sin(t) \overset{?}{=} 1\cdot\cos(t) + 0\cdot\sin(t)$$

$B - A = 1$ and $A + B = 0$ $\rightarrow$ $A = -1/2$, $B = 1/2$

$$y_{p,2}(t) = \frac{1}{2}\left(\sin(t) - \cos(t)\right)$$

Finally the required particular solution is:

$$y_p(t) = \frac{1}{2}\left(\sin(t) - \cos(t)\right) - \left(2t + t^2 - t^3/3\right)e^{-t}$$

General solution:

$$y(t) = C_1 + C_2 e^{-t} + \frac{1}{2}\left(\sin t - \cos t\right) - \left(2t + t^2 - t^3/3\right)e^{-t}$$