§3.3 Crash course: Complex numbers.

Fabulously important!

G. Cardano (Italy, 1545) observes that solutions of cubic equations contain roots of negative numbers! "fictitious numbers"

Problem: $x^2 = -1$ has no solution.

↳ Introduce the "number" i such that $i^2 = -1$.

A **complex number** writes as a linear combination, $z = a + ib$, a, b real.

| a: real part of z $\qquad$ $Re(z) = a$
| b: imaginary part of z $\qquad$ $Im(z) = b$

Equality: $a + ib = a' + ib'$ if $a = a'$ AND $b = b'$.

Addition: $(a+ib) + (a'+ib') = (a+a') + i\,(b+b')$

Conjugate: $\overline{a+ib} = a - ib$

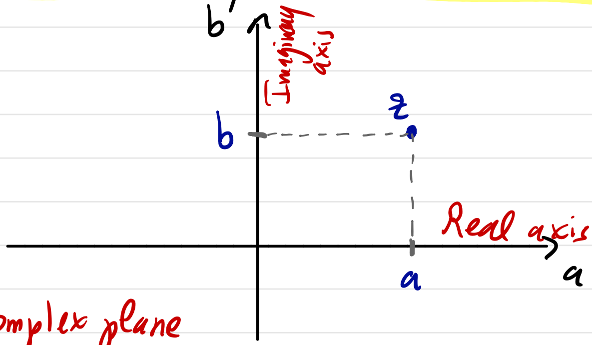
↳ Notation: $\bar{z} = Conj(z)$

Changes the sign of the imaginary part of z.

Multiplication:

$$(a+ib)(c+id) = ac + ibc + iad + i^2bd$$
$$= (ac-bd) + i(bc+ad)$$

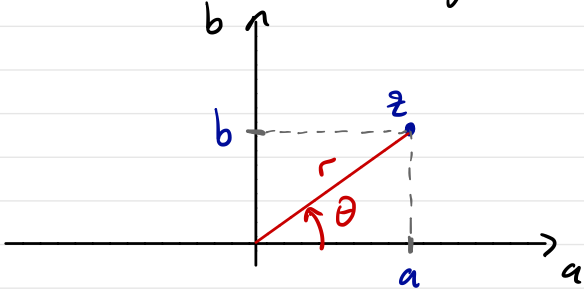
Interpretation: points in the plane



Complex plane

"GEOMETRICAL INTERPRETATION"

$\rightsquigarrow$ Polar form – Euler's formula:



Radial coordinates:

* Modulus: $r = |z| = \sqrt{a^2+b^2}$
* Argument: $\theta = \arg(z)$ such that $\cos\theta = a/r$, $\sin\theta = b/r$

Notes:

- $z\bar{z} = r^2$
- θ defined up to a multiple of 2π.

so
$$z = r\cos\theta + i(r\sin\theta) = \underbrace{r(\cos\theta + i\sin\theta)}_{\text{trigonometric form}}$$

Euler's formula:
$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Proof: Taylor series,
$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \dots$$
$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right) + i\left(x - \frac{x^3}{3!} + \dots\right)$$

Finally, **polar form:**
$$\boxed{z = re^{i\theta}}$$

Complex exponential
$$e^z = e^a e^{ib} = e^a(\cos a + i\cos b)$$

Usual rules: $e^{z+z'} = e^z e^{z'}$

(Use this to write the usual trig formulae! Sum of angles, doubling of angles...)

⚠ Log not so easy – due to $\arg(z)$ being defined only up to 2π!

Example: solutions of polynomial equations.

1. $x^2 = -1 \longrightarrow x = \pm i$

Apply usual formulae: $x^2 + 1 = 0$

$$\Delta = 0^2 - 4.1.1 = -4$$

so $x = \dfrac{-0 \pm 2i}{2} = \pm i$ $\therefore$

2. $x^2 + 2x + 2 = 0$

Usual formula: $\Delta = 2^2 - 2 \cdot 4 = -4$

$$x = \frac{-2 \pm 2i}{2} = -1 \pm i$$

Return to differential equations.

Complex roots of the characteristic equation

Recall the problem: $ay'' + by' + cy = 0$.

a, b, c real; $a \neq 0$.

Roots of the characteristic equation: take $y(t) = e^{rt}$

$\rightarrow \quad ar^2 + br + c = 0$

Results from §3.1: $\Delta = b^2 - 4ac > 0$, real roots.
§3.2: General solution writes

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Here : case $\Delta = b^2 - 4ac < 0$.

Then we have complex roots,

$$r_1 = \frac{-b - i\sqrt{-\Delta}}{2a} \qquad r_2 = \frac{-b + i\sqrt{-\Delta}}{2a}$$

Then, we have complex solutions,

$$y_1(t) = e^{r_1 t} \quad ; \quad y_2(t) = e^{r_2 t}$$

Question: how to extract real-valued solutions?

Observation: for the ODE $y'' + p(t)y' + q(t)y = 0$ with continuous, real valued $p, q(t)$.

If $u(t) + i v(t)$ is a complex-valued solution, then both $u(t)$ and $v(t)$ are real solutions.

Check:

$$L[u + iv] = \overbrace{\left(u'' + p(t)u' + q(t)u\right)}^{L[u],\ \text{real}} + i\underbrace{\left(v'' + p(t)v' + q(t)v\right)}_{L[v]\ \text{real}} = 0$$

so $\quad L[u] = 0 \quad$ and $\quad L[v] = 0$

(complex numbers equality !)

Let's return to our solutions:

observe $\quad r_1 = \lambda + i\mu, \qquad r_2 = \lambda - i\mu$

where $\quad \lambda = \dfrac{-b}{2a}$ and $\mu = \dfrac{\sqrt{4ac - b^2}}{2a}$

$$y_1(t) = e^{(\lambda + i\mu)t} = e^{\lambda t} + e^{i\mu t}$$

Euler's formula !

$$= e^{\lambda t}\left(\cos(\mu t) + i \sin(\mu t)\right)$$

$$\rightarrow \quad y_1(t) = \underbrace{e^{\lambda t}\cos(\mu t)}_{\text{Real part}} + \underbrace{i\, e^{\lambda t}\sin(\mu t)}_{\text{Imaginary part}}$$

Then we get two real solutions

$$\begin{cases} u(t) = e^{\lambda t}\cos(\mu t), \\ v(t) = e^{\lambda t}\sin(\mu t). \end{cases}$$

Is this a set of fundamental solutions ?

$$\begin{vmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{vmatrix} = \begin{vmatrix} e^{\lambda t}\cos(\mu t) & e^{\lambda t}\sin(\mu t) \\ \lambda e^{\lambda t}\cos(\mu t) - \mu e^{\lambda t}\sin(\mu t) & \lambda e^{\lambda t}\sin(\mu t) + \mu e^{\lambda t}\cos(\mu t) \end{vmatrix}$$

6 $$= e^{2\lambda t}\left(\cos(\mu t)\left(\cancel{\lambda e^{\lambda t}\sin(\mu t)} + \mu e^{\lambda t}\cos(\mu t)\right) - \sin(\mu t)\left(\cancel{\lambda e^{\lambda t}\cos(\mu t)} - \mu e^{\lambda t}\sin(\mu t)\right)\right)$$

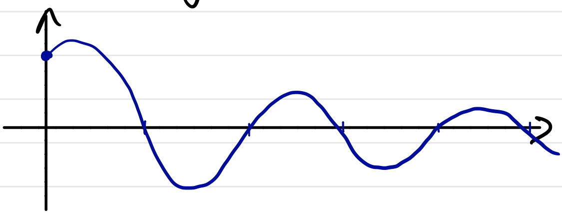
$$\ldots = \mu \left(\cos^2(\mu t) + \sin^2(\mu t)\right) e^{-2\lambda t}$$

So... $$W[u,v] = \mu\, e^{-2\lambda t} = \frac{\sqrt{4ac-b^2}}{2a}\, e^{-b/a\, t}$$

Check: Abel's theorem!

Observation: Oscillatory solutions!

Damping if $\lambda = -\frac{b}{2a} < 0$.



Example 1

$$\begin{cases} y'' + y = 0 \\ y(0) = 1 ; \quad y'(0) = 0 \end{cases}$$

① Seek the solution as $y(t) = e^{rt}$

↳ $(r^2+1)\, e^{rt} = 0$

The roots of the characteristic equation: $r^2 + 1 = 0$

$$r_1 = +i \qquad r_2 = -i$$

Identify real and imaginary parts:

$$r_1 = \underbrace{0}_{\text{Real, } \lambda} + \underbrace{1}_{\text{Imaginary part, } \mu} \cdot i$$

2 So two solutions:

$$\begin{cases} u(t) = e^{0.t}\cos(1.t) = \cos(t) \\ v(t) = e^{0.t}\sin(1.t) = \sin(t) \end{cases}$$

These form a fundamental set of solutions, so the general solution is

$$\boxed{y(t) = C_1\cos(t) + C_2\sin(t).}$$

3 Use initial conditions: $\begin{cases} y(0) = C_1 = 1 \\ y'(0) = C_2 = 0 \end{cases}$

The particular solution is $\boxed{y(t) = \cos(t)}$

Note: purely oscillatory solution. no decay or amplification.

Example 2

$$\begin{cases} y'' + y' + y = 0 \\ y(0) = 0, \quad y'(0) = 1 \end{cases}$$

1 Characteristic equation: $r^2 + r + 1 = 0$

Discriminant: $\Delta = 1^2 - 4.1.1 = -3 < 0$

Complex roots: $r_1 = \dfrac{-1 + i\sqrt{3}}{2}$, $r_2 = \dfrac{-1 - i\sqrt{3}}{2}$

Real part: $\lambda = -1/2$
Imaginary part: $\mu = \sqrt{3}/2$

② General solution:

$$y(t) = c_1 e^{-t/2} \cos\left(\sqrt{3}/_2\, t\right) + c_2 e^{-t/2} \sin\left(\sqrt{3}/_2\, t\right)$$

③ Particular solution:

Compute first the derivative:

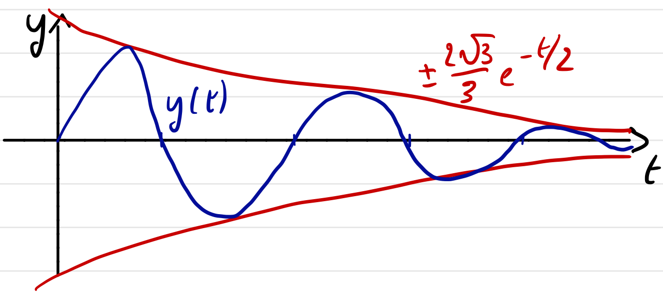
$$y'(t) = -\frac{1}{2} c_1 e^{-t/2} \cos\left(\sqrt{3}/_2\, t\right) - \frac{\sqrt{3}}{2} c_1 e^{-t/2} \sin\left(\sqrt{3}/_2\, t\right)$$
$$- \frac{1}{2} c_2 e^{-t/2} \sin\left(\sqrt{3}/_2\, t\right) + \frac{\sqrt{3}}{2} c_2 e^{-t/2} \cos\left(\frac{\sqrt{3}}{2}\, t\right)$$

So, $\begin{cases} y(0) = c_1 = 0 \\ y'(0) = -\frac{1}{2} c_1 + \frac{\sqrt{3}}{2} c_2 = 1 \end{cases}$

and $c_1 = 0$; $c_2 = \frac{2\sqrt{3}}{3}$

Finally: $\boxed{y(t) = \frac{2\sqrt{3}}{3} e^{-t/2} \sin\left(\frac{\sqrt{3}}{2}\, t\right)}$

Note: damped solution, $y(t) \longrightarrow 0$



Example 3

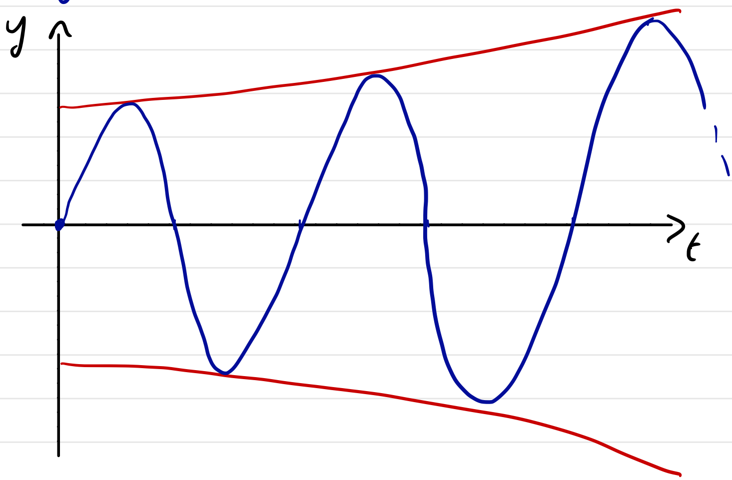
$$\begin{cases} y'' - y' + y = 0 \\ y(0) = 0, \quad y'(0) = 1 \end{cases}$$

Same discriminant: $\Delta = (-1)^2 - 4 \cdot 1 \cdot 1 = -3 < 0$

Roots $r_1 = \dfrac{+1 + i\sqrt{3}}{2}, \quad r_2 = \dfrac{+1 - i\sqrt{3}}{2}$

Same computation, sign change:

$$\boxed{y(t) = \frac{2\sqrt{3}}{3} e^{+t/2} \sin\left(\frac{\sqrt{3}}{2} t\right)}$$



Summary Coefficient b corresponds to friction.

If $b^2 < 4ac$: "damping too low" leads to oscillatory solutions.

Three possible cases:

- $b/a > 0$: damped osillations.
- $b = 0$: undamped osillations, constant amplitude.
- $b/a < 0$: exponentially increasing osillations.