Sept 26
Wed

§ 3.2 The Wronskian. Solutions of linear homogeneous equations.

Differential Operator Notation

* Open time interval I $\quad a < t < b$
* p, q two continuous functions of time on I
* Define the differential operator L by its action on any twice differentiable function φ:

$$L[\phi] = \varphi'' + p(t)\varphi' + q(t)\varphi$$

↳ $L[\phi]$ is a new function!

$$L[\phi](t) = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t)$$

Example:

* $I = (0, \infty)$ i.e. $0 < t < \infty$
* $p(t) = t$ $\quad q(t) = \ln(t)$

Take $\varphi(t) = \cos(2t)$

then $\varphi'(t) = -2\sin(2t)$, $\varphi''(t) = -4\cos(2t)$

$$L[\phi](t) = -4\cos(2t) + t\left(-2\sin(2t)\right) + \ln(t)\cos(2t)$$

Theorem 1 Existence and Uniqueness.

The initial value problem

$$\begin{cases} L[y] = y'' + p(t)\,y' + q(t)\,y = g(t) \\ y(t_0) = y_0, \quad y'(t_0) = y'_0 \end{cases}$$

where p, q, g are continuous on an interval I containing t_0,
has a unique solution in the whole interval.

* Existence of a solution!
* It's the only one!
* It's defined on the whole interval, as long as p, q, g continuous!

Example 1: $y'' - y = 0, \quad y(0) = 2 \quad y'(0) = -1$

has only one solution: $y(t) = \frac{1}{2} e^{t} + \frac{3}{2} e^{-t}$ in the whole interval $-\infty < t < \infty$.

Example 2: $\begin{cases} (t^2-1)\,y'' + y' + ty = 0 \\ y(0) = 0 \quad y'(0) = 1 \end{cases}$

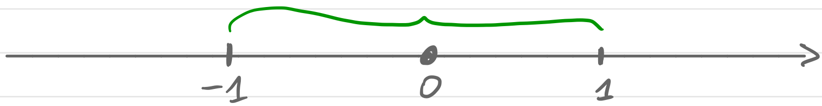
First, write the solution as in the theorem:

$$y'' + \underbrace{\frac{1}{t^2-1}}_{p(t)}\,y' + \underbrace{\frac{t}{t^2-1}}_{q(t)}\,y = \underbrace{0}_{g(t)}$$

Points of discontinuity are $t=-1$ and $t=+1$.

The longest interval containing the initial point $t=0$ in which coefficients are continuous is

$$\underline{-1 < t < 1}$$



Theorem 2   <u>Principle of superposition</u>

If y_1 and y_2 are two solutions of the D.E.

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

then any linear combination $c_1 y_1 + c_2 y_2$ is <u>also</u> a solution for any value of the constants c_1, c_2.

Proof: compute

$$L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2] = 0$$

Observe now how we determine the constants from initial conditions:

$$\begin{cases} c_1 y_1(t_0) + c_2 y_2(t_0) = y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0' \end{cases}$$

This is a linear system in the variables c_1, c_2.

Such a system has the determinant:

$$W(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}$$

Recall that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

⇝ If $W(t_0) \neq 0$, the linear system will have a unique solution (c_1, c_2).

Cramer's formula:

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{W(t_0)} \quad ; \quad c_1 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{W(t_0)}$$

Conversely, if $W(t_0) = 0$ then the initial conditions cannot always be satisfied by any choice c_1, c_2.

Definition: The determinant $W(t)$ is the WRONSKIAN DETERMINANT.

Explicitely: $W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$

Theorem 3 Suppose y_1, y_2 are two solutions of $L[y] = y'' + p(t)y' + q(t)y = 0$, with initial conditions $y(t_0) = y_0,\ y'(t_0) = y'_0$.

If their Wronskian is $\neq 0$ at t_0,

$$W[y_1, y_2] = y_1 y_2' - y_1' y_2$$

then there exists two constants C_1, C_2 such that

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

is a solution of the IVP.

Example $y'' + 3y' + 2y = 0$

Characteristic equation: $r^2 + 3r + 2 = 0$

Discriminant: $\Delta = 3^2 - 4 \cdot 2 = 1$

$$r_1 = \frac{-3-1}{2} = -2 \ ; \quad r_2 = \frac{-3+1}{2} = -1$$

This gives us two solutions:

$$y_1(t) = e^{-2t} \text{ and } y_2(t) = e^{-t}.$$

Let us compute their Wronskian:

$$W[y_1, y_2] = \begin{vmatrix} e^{-2t} & e^{-t} \\ -2e^{-2t} & -e^{-t} \end{vmatrix} = e^{-2t}(-e^{-t}) - e^{-t}(-2e^{-2t})$$
$$= e^{-3t}$$

Note that $W[y_1, y_2] = e^{-3t}$ is always nonzero:
y_1 and y_2 can be used to construct solutions of the differential equation with any differential conditions!

Theorem 4 Take two solutions y_1, y_2 of the equation,

$$\mathcal{L}[y] = y'' + p(t)y' + q(t) = 0.$$

The family of solutions with two parameters,

$$y : t \longmapsto C_1 y_1(t) + C_2 y_2(t)$$

includes every possible solution of the equation if and only if there is <u>one point</u> where the Wronskian is not zero:

$$W[y_1, y_2](t_0) = y_1 y_2' - y_1' y_2 \neq 0 \,.$$

Details: see the textbook.

Definition: We call a couple of solutions y_1, y_2 such that $W[y_1, y_2](t_0) \neq 0$ for some t_0 a FUNDAMENTAL SET OF SOLUTIONS.

Definition: The GENERAL SOLUTION of the equation $L[y]=0$ is an expression

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

with a fundamental set of solutions y_1, y_2.

Examples Take the equation $t^2 y'' + t y' - y = 0$.

* Two solutions for $t > 0$:

$$y_1(t) = t \quad \text{and} \quad y_1(t) = 1/t$$

* Wronskian:

$$W[y_1, y_2](t) = \begin{vmatrix} t & 1/t \\ 1 & -1/t^2 \end{vmatrix} = t\left(\frac{-1}{t^2}\right) - \frac{1}{t} = \frac{-2}{t}$$

* $W \neq 0$ for $t > 0$, so

y_1, y_2 form a fundamental set of solutions.

The general solution of this equation is

$$\boxed{y(t) = C_1 t + \frac{C_2}{t} \text{ for } t > 0.}$$

Theorem 5 Consider the D.E.

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

with coefficients $p(t)$, $q(t)$ continuous on some open interval I.

Choose t_0 in I, and let

* $y_1(t)$ be the solution which satisfies $y_1(t_0) = 1, \quad y_1'(t_0) = 0$
* $y_2(t)$ be the solution which satisfies $y_2(t_0) = 0, \quad y_2'(t_0) = 1$

Then y_1 and y_2 form a fundamental set of solutions

Proof: $$W(t_0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

Example $\quad y'' - y = 0 \quad$ and $\quad t_0 = 0.$

Solutions $t \mapsto e^t$, $t \mapsto e^{-t}$ are not the set in the theorem.

(they still form a fundamental set!)

Let us use the general solution

$$y(t) = c_1 e^t + c_2 e^{-t}.$$

* Conditions $y_1(0) = 1, \quad y_1'(0) = 0$

$$\begin{cases} C_1 + C_2 = 1 \\ C_1 - C_2 = 0 \end{cases} \quad \text{so} \quad C_1 = C_2 = 1/2$$

$$\boxed{y_1(t) = \frac{1}{2}\left(e^t + e^{-t}\right) = \cosh(t)}$$

* Conditions $y_2(0) = 0, \quad y_2'(0) = 1$

$$\begin{cases} C_1 + C_2 = 0 \\ C_1 - C_2 = 1 \end{cases} \quad \text{so} \quad C_1 = -C_2 = 1/2$$

$$\boxed{y_2(t) = \frac{1}{2}\left(e^t - e^{-t}\right) = \sinh(t)}$$

* Let us compute the Wronskian:

$$W[y_1, y_2] = \begin{vmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{vmatrix}$$

$$= \cosh^2(t) - \sinh^2(t) = 1$$

This is a fundamental set of solutions!

Theorem 6 Abel's theorem

Take the D.E. $L[y] = y'' + p(t)y' + q(t)y = 0.$

If p, q are continuous on the interval I, then the Wronskian of <u>any</u> two solutions y_1, y_2 is given by

$$W[y_1, y_2](t) = C \exp\left(-\int p(t)\right)$$

where C is a constant which depends only on y_1, y_2.

Then, W is either zero on I (if $C=0$) or never zero if $C \neq 0$!

Proof

$$\frac{d}{dt} W[y_1, y_2] = \frac{d}{dt}\left(y_1 y_2' - y_1' y_2 \right)$$

$$= \cancel{y_1' y_2'} + y_1 y_2'' - \cancel{y_1' y_2'} - y_1'' y_2$$

$$= y_1(t)\left[-p(t)y_2'(t) - \cancel{q(t)y_2(t)}\right]$$

$$- y_2(t)\left[-p(t)y_1'(t) - \cancel{q(t)y_1(t)}\right]$$

$$= -p(t)\left[y_1(t) y_2'(t) - y_2'(t) y_1(t)\right]$$

$$= -p(t)\, W[y_1, y_2].$$

→ W is the solution of a 1st order D.E:

$$W' + p(t)W = 0.$$

From Chapter 2, integrating factor method:

$$W(t) = C\exp\left(-\int p(t)\right) \therefore$$

Example 1 $t^2y'' + ty' - y = 0.$

By Abel's formula: $p(t) = 1/t$

so $W(t) = C\exp(-\ln t)$

$$\boxed{W(t) = \frac{C}{t}}$$

Above solutions: $C = -2$!

Example 2 Constant coefficients

$$ay'' + by' + cy = 0$$

Per Abel's theorem: $W(t) = C\exp\left(-\frac{b}{a}t\right)$

Check: we have two solutions for $b^2 > 4ac$:

$$r_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

The Wronskian of the two corresponding solutions is:

$$W[y_1, y_2](t) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix}$$

$$= r_2 e^{r_1 t} e^{r_2 t} - r_1 e^{r_1 t} e^{r_2 t}$$

$$= (r_2 - r_1)\, e^{(r_1 + r_2)t}$$

With the above values for $r_{1,2}$ this gives

$$W[y_1, y_2](t) = \frac{\sqrt{b^2 - 4ac}}{a}\, e^{-b/a\, t} \quad \therefore$$