Mon 17 2018

§ 2.6 Exact differential equations

Idea: separable equations are a very particular case, but the idea of implicit solutions carries over...

For a function $F(x,y)$ one forms a differential equation

$$\frac{d}{dx} F(x, y(x)) = 0 \longrightarrow F(x,y) = C$$

↳ Couple of notions from Calc III to get there...

* Partial derivatives: take a simple function

$$F(x,y) = 2x^2y^3$$

Around a point (a,b) we can change either x or y. First we may hold y fixed and allow x to vary.

$$y = b \qquad\qquad g(x) := F(x,b) = 2x^2b^3$$

This is now a function of a single variable! Rate of change → derivative

$$g'(a) = 4ab^3$$

Call g' the partial derivative of F w.r.t. x.

This gives us $\frac{\partial F}{\partial x}(a,b) = F_x(a,b) = 4ab^3$

Similarly, we can hold $x = a$ fixed and change y:
set $h(y) = 2a^2y^3$

$$\frac{\partial F}{\partial y}(a,b) = F_y(a,b) = h'(y) = 6a^2b^2$$

* Chain rule: we have the situation here where we want to compute $\frac{d}{dx} F(x, y(x))$

More generally, extend chain rule from Calc I:

If $z(t) = F(x(t), y(t))$ then:

$$\frac{dz}{dt} = \frac{\partial F}{\partial x}(x(t), y(t))\frac{dx}{dt}(t) + \frac{\partial F}{\partial y}(x(t), y(t))\frac{dy}{dt}(t)$$

Apply here to our example, where $x(t) = t$

$$\underbrace{\frac{\partial F}{\partial x}(x,y)}_{M(x,y)} + \underbrace{\frac{\partial F}{\partial y}(x,y)}_{N(x,y)}\frac{dy}{dx} = 0$$

Any D.E. of the form $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ where there exists a function $F(x,y)$ such that $M = \frac{\partial F}{\partial x}$, $N = \frac{\partial F}{\partial y}$ is called exact.

Implicit solutions of such an equation are given as

$$F(x,y) = C, \quad C \text{ arbitrary.}$$

QUESTIONS

- When does such an F exist?
- How to find it based on M and N?

ANSWERS:

1. Assume that in some rectangle of (x,y) plane, M, N, M_y, N_x exist and are continuous.

Then the two statements are equivalent:

(1) there exist F such that in this rectangle,
$$M = \frac{\partial F}{\partial x}, \quad N = \frac{\partial F}{\partial y}$$

(2) $\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$ in the rectangle.

$\Downarrow$

This is a test for whether an equation is exact.

2. Idea: integrate one variable after the other.

Example:
$$\begin{cases} y^2/x + \left(2y\ln(x) + \cos(y)\right)\dfrac{dy}{dx} = 0 \\ y(1) = 1 \end{cases}$$

1. Test for exact D.E.

$$\frac{\partial M}{\partial y} = \frac{2y}{x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{2y}{x} \quad \checkmark$$

② Compute the "generating function" F

* First, write $F(x,y) = Q(x,y) + h(y)$

where Q is any function such that

here, we start with x $\rightarrow$ $\dfrac{\partial Q}{\partial x} = \dfrac{\partial F}{\partial x} = M(x,y) = y^2/x$.

↳ choose $\boxed{Q(x,y) = y^2 \ln x}$

* Next, use 2nd piece of information:

$$\frac{\partial F}{\partial y} = \frac{\partial Q}{\partial y} + h'(y) = N(x,y)$$

↳ $h'(y) = [\cancel{2y \ln(x)} + \cos(y)] - \cancel{2y \ln(x)}$

so we can take $\boxed{h(y) = \sin(y)}$

* Combine Q and h:

$$F(x,y) = y^2 \ln x + \sin(y)$$

* Now we can rewrite the exact D.E. in the form

$$\frac{d}{dx} F(x, y(x)) = 0$$

and the implicit solutions: $y^2 \ln(x) + \sin(y) = C$

4) Find constants, domain of validity

$y = 1$ for $x = 1$: $C = \sin(1) \approx 0.84$

⟹ General recipe for finding $F(x,y)$

(Textbook, p72).

* Choose to integrate first in x or y

$$\frac{\partial F}{\partial x} = M(x,y) \Longrightarrow F(x,y) = \underbrace{\int M\,dx}_{Q(x,y)} + h(y)$$

unknown!

OR

$$\frac{\partial F}{\partial y} = N(x,y) \Longrightarrow F(x,y) = \underbrace{\int N\,dy}_{P(x,y)} + g(x)$$

↳ Compute $Q = \int M\,dx$ OR $P = \int N\,dy$

* Next, do the other one

$$h'(y) = \underbrace{N(x,y) - \frac{\partial Q}{\partial y}(x,y)}$$

OR

IF EXACT EQN, THIS EXPRESSION WILL NOT DEPEND ON x/y.

$$g'(x) = \overbrace{M(x,y) - \frac{\partial P}{\partial y}(x,y)}$$

* Finally, combine: $F(x,y) = Q(x,y) + h(y) = P(x,y) + g(x)$

Wed 19, September

Summary: 1st order ODEs.

(1) Linear equations: $y' + p(t)y = g(t)$

SOLUTION METHOD: INTEGRATING FACTORS.

$$y(t) = \frac{1}{m(t)}\left(\int m(t)g(t)\,dt + C\right)$$

$$m(t) = \exp\left(\int p(t)\,dt\right)$$

(2) Separable equations: $\frac{dy}{dx} = g(x)f(y)$

SOLUTION: SEPARATE VARIABLES, INTEGRATE

$$\int \frac{dy}{f(y)} = \int g(x)\,dx$$

(3) Exact equations: $M(x,y) + N(x,y)y' = 0$

→ Check: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ ✓

SOLUTION: compute

$g(x,y) = \int M(x,y)\,dx$ and $h(y)$ such that

$$h'(y) = N(x,y) - \frac{\partial g}{\partial y}$$

↳ Solutions have the form

$$F(x,y) = g(x,y) + h(y) = C \quad \text{(constant)}$$

Models of interest:

(1) Mixing problems:

$$\frac{dQ}{dt} = \underbrace{r_{in} \cdot c_{in}}_{\text{rate IN}} - \underbrace{r_{out} \cdot \frac{Q(t)}{V(t)}}_{\text{rate OUT}}$$

↳ linear: integrating factors.

(2) Population models, autonomous equations

$$\frac{dy}{dt} = f(y) \qquad \left(= r\,(1 - y/K)\,y \right)$$

Phase line analysis; equilibria, stability.
Separation of variables.

(3) Heating / Cooling problems.

$$\frac{dT}{dt} = k\,(\underbrace{M(t)}_{\text{external temp.}} - \underbrace{T(t)}_{\text{object temp.}})$$

Chapter 3 Second Order Diff Eq

§3.1 Homogeneous Differential Equations Constant coefficients

In general, a 2^{nd} order D.E. has the form

$$y'' = f(t, y, y') \qquad (1)$$

where f is a given function. We will focus on cases where f takes the specific shape:

$$f(t, y, y') = g(t) - p(t)y' - q(t)y$$

i.e. f is a linear function of y and y'.

Then we can rewrite the linear 2^{nd} order ODE:

$$\underbrace{y'' + p(t)y' + q(t)y}_{\text{linear combination of } y, y', y''} = g(t). \qquad (2)$$

or equivalently,

$$P(t)y'' + Q(t)y' + R(t)y = G(t) \qquad \text{for } P(t) \neq 0.$$

Any eqn (1) which cannot be transformed as (2) is called non-linear. Not much to say as they are hard to solve analytically.

An initial value problem is here a D.E. of the form (1) or (2) with TWO initial conditions,

$$y(0) = y_0, \qquad y'(0) = y'_0.$$

two given numbers.

Why 2 conditions? Think most simple case,

$$y'' = 0 \implies y'(t) = A \implies y(t) = At + B$$

To fix each of the 2 constants introduced by integrating twice, need 2 conditions.

A linear 2nd order ODE is homogeneous if the right-hand side | forcing term, $g(t)$ or $G(t)$ is zero:

$$P(t)\,y'' + Q(t)\,y' + R(t)\,y = 0.$$

In the simplest case, the coefficients $P, Q, R(t)$ are simply constants:

$$ay'' + by' + cy = 0, \qquad a \neq 0.$$

This is a specific but important example which governs many physical/engineering situations close to equilibrium.

A first example

$$\begin{cases} y'' - y = 0 \\ y(0) = 2, \quad y'(0) = -1 \end{cases}$$

"Intuition" of solutions: e^{+t}, e^{-t} and multiples

Observation: sums of solutions are solutions!

Form linear combination of these elementary solutions:

$$y(t) = c_1 e^{t} + c_2 e^{-t}$$

Two arbitrary constants!

Initial conditions:

$$y(0) = c_1 e^{0} + c_2 e^{-0} = c_1 + c_2 = 2$$

$$y'(0) = c_1 e^{0} - c_2 e^{-0} = c_1 - c_2 = -1$$

Solve system of 2 equations for c_1, c_2:

$c_1 = 1/2$, $\quad c_2 = 3/2 \quad \rightarrow \quad y(t) = \frac{1}{2} e^{t} + \frac{3}{2} e^{-t}$

General technique

$$ay'' + by' + cy = 0$$

- Find two solutions $y_1(t)$, $y_2(t)$ which are different (not multiple of each other)
- Write general solution as a linear combination:
  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Use the 2 constants to fit the 2 initial conditions.

Check: if y_1, y_2 are two solutions then

$$a(C_1y_1+C_2y_2)''+b(C_1y_1+C_2y_2)'+c(C_1y_1+C_2y_2)$$
$$= C_1\underbrace{(ay_1''+by_1'+cy_1)}_{=0}+C_2\underbrace{(ay_2''+by_2'+cy_2)}_{=0}$$
$$= 0$$

How do we find the base solutions?

Idea: seek exponential solutions of the form:

$$y(t) = e^{rt} \qquad r \text{ unknown yet.}$$

Since $y'(t) = re^{rt}$, $y''(t) = r^2e^{rt}$,

$$ay''+by'+cy=0 \longleftrightarrow (ar^2+br+c)e^{rt}=0$$

Now $e^{rt}\neq 0$: satisfied if r solution of the **CHARACTERISTIC EQUATION**

$$\underbrace{ar^2+br+c}_{\text{Characteristic polynomial (order 2)}} = 0$$

Possible cases: Discriminant, $\Delta = b^2-4ac$

- $\Delta>0$: Two real roots
- $\Delta=0$: One real, repeated root
- $\Delta<0$: Two complex conjugate roots.

First, consider the case where $\Delta > 0$.
Other cases: § 3.4 and § 3.3.

Then we have two roots of the characteristic polynomial:

$$\boxed{r_1 = \frac{-b+\sqrt{\Delta}}{2a} \qquad r_2 = \frac{-b+\sqrt{\Delta}}{2a}.}$$

Then, $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are two different solutions of the equation and

The general solution has the form

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

with c_1, c_2 arbitrary constants.

Next, fit the initial conditions:

$$\begin{cases} y(t_0) = y_0 \\ y'(t_0) = y'_0 \end{cases}$$

This leads to two equations:

$$\begin{cases} c_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} = y_0 \\ c_1 r_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} = y'_0 \end{cases}$$

↳ system of linear equations: 2 unknowns, 2 equations.

Example 1

$$\begin{cases} y'' + y' - 2y = 0 \\ y(0) = 1, \quad y'(0) = 0 \end{cases}$$

Step 1 Solve the characteristic equation
⇝ solutions under the form $y(t) = e^{rt}$

$$\underline{r^2 + r - 2 = 0}$$

Discriminant: $\Delta = 1^2 - 4\cdot(-2) = 9$

The roots of the characteristic equation are

$$r_1 = \frac{-1-\sqrt{9}}{2} = -2 \quad \text{and} \quad r_2 = \frac{-1+\sqrt{9}}{2} = +1$$

Step 2 General solution:

$$\boxed{y(t) = C_1 e^{-2t} + C_2 e^t, \quad C_1 \text{ and } C_2 \text{ arbitrary}}$$

Step 3 Initial conditions, particular solution:

$$\begin{cases} C_1 + C_2 = 1 \\ -2C_1 + C_2 = 0 \end{cases} \quad \text{so} \quad \begin{cases} C_1 + 2C_1 = 1 \\ C_2 = 2C_1 \end{cases}$$

and $c_1 = 1/3$ $\quad c_2 = 2/3$

$$\boxed{y(t) = \frac{1}{3} e^{-2t} + \frac{2}{3} e^t}$$

Example 2

$$\begin{cases} y'' + 12y' + 35y = 0 \\ y(0) = 3, \quad y'(0) = -17 \end{cases}$$

Step 1 $\quad r^2 + 12r + 35 = 0$

Discriminant: $\quad \Delta = 12^2 - 4 \cdot 1 \cdot 35 = 4$

The roots of the characteristic polynomial are

$$r_1 = \frac{-12 - \sqrt{4}}{2} = -7 \qquad r_2 = \frac{-12 + \sqrt{4}}{2} = 5$$

Step 2 The general solution of the D.E. is

$$y(t) = C_1 e^{-7t} + C_2 e^{-5t}$$

Step 3 Particular solution:

$$\begin{cases} y(0) = C_1 + C_2 = 3 \\ y'(0) = -7C_1 - 5C_2 = -17 \end{cases} \quad \text{so} \quad \begin{cases} C_2 = 3 - C_1 \\ -7C_1 - 5(3 - C_1) = -17 \end{cases}$$

then, $\quad -2C_1 = -17 + 15 = -2 \quad$ so $\quad C_1 = 1$ and $C_2 = 2$

The solution to this IVP is

$$y(t) = e^{-7t} + 2e^{-5t}.$$

General behavior at infinity: depends on sign of r_1 and r_2

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \qquad r_1 < r_2$$

As $t \to \infty$ ...

r_2 \ r_1	$r_1 < 0$	$r_1 = 0$	$r_1 > 0$
$r_2 < 0$	$y \to 0$		
$r_2 = 0$	$y \to C_2$		
$r_2 > 0$	$y \to \pm\infty$ depends on sign of C_2; if $C_2 = 0$, $y \to 0$	$y \to \pm\infty$ depends on sign of C_2; if $C_2 = 0$, $y \to C_1$	$y \to \pm\infty$ depends on sign of C_2; if $C_2 = 0$, on sign of C_1; if $C_1 = 0$, $y = 0$!

Note: as long as $r_1 \neq r_2$, system for $C_{1,2}$ can always be solved!

$$\begin{cases} C_1 e^{r_1 t_0} + C_2 e^{r_2 t_0} = y_0 \\ C_1 r_1 e^{r_1 t_0} + C_2 r_2 e^{r_2 t_0} = y_0' \end{cases} \Rightarrow \begin{cases} C_1 = \dfrac{y_0' - y_0 r_2}{r_1 - r_2} e^{-r_1 t_0} \\ \\ C_2 = \dfrac{y_0 r_1 - y_0'}{r_1 - r_2} e^{-r_2 t_0} \end{cases}$$

by substitution!

There is always a solution in this form! But, is it the only possibility? Answer in the next section!