Monday Sept 10

§2.5 Autonomous equations. Population dynamics.

Definition: a 1^{st} order ODE of the form

$$\frac{dy}{dt} = f(y)$$

is **AUTONOMOUS.**

→ The independent variable does not appear explicitely.

A good example is population dynamics $\begin{cases} \text{medicine} \\ \text{ecology} \\ \text{economics} \end{cases}$

In general, solution by separation of variables:

$$\int \frac{dy}{f(y)} = \int dt$$

It is not the subject here (simple integration pb).

Idea: Use geometric intuition to find quickly properties of solutions qualitatively.

↳ Typically, stability, basin(s) of attraction for equilibria.

Example 1: Malthus model, 1798 (economist)

$$\frac{dy}{dr} = ry \qquad \text{(proportional to population)}$$

$$\rightarrow \boxed{y(t) = y_0 e^{rt}} \quad \text{where } y(0) = y_0.$$

If $r > 0$, this model predicts exponential growth.

In practice, limits (lack of resources / space) ultimately reduce the growth rate – think petri dish.

Then exponential growth ends.

Example 2 – Logistic growth

Idea: growth rate actually depends on population

$$\frac{dy}{dt} = h(y)\, y,$$

where $h(y) \approx r$ for small y,
decreasing with $y \nearrow$
ultimately $h < 0$ for y large.

Simplest replacement: affine function,

$$h(y) = r\,(1 - y/K)$$

Then

$$\frac{dy}{dt} = r\,(1 - y/K)\, y$$

Logistic equation

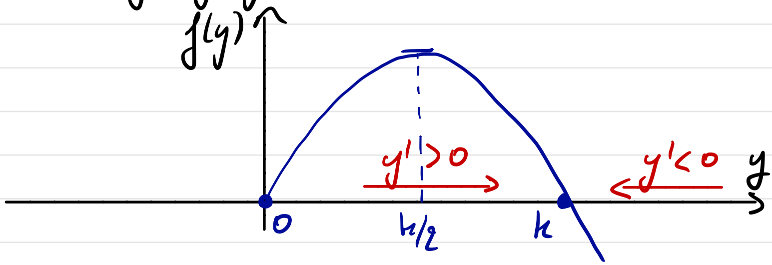
* r: intrinsic growth rate,
* K: equilibrium population (see later).

Analysis: Equilibrium solutions, $f(y)=0$

$$r\left(1-\frac{y}{k}\right)y = 0$$

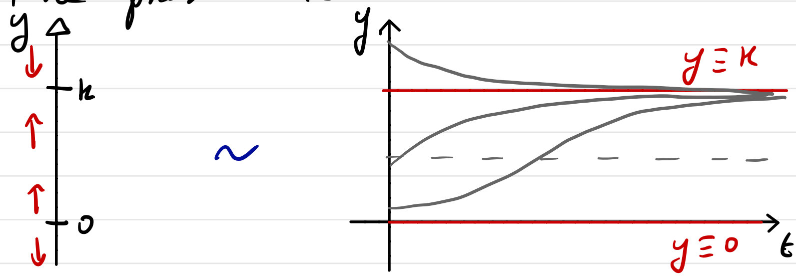
* $y(t) \equiv 0$ * $y(t) = k$

Zeros of $f(y)$ are also called critical points.



Behavior of $y(t)$ (increasing / decreasing) depends on the sign of $f(y)$.

→ The phase line:



⇝ $y \equiv K$: stable equilibrium

⇝ $y \equiv 0$: unstable equilibrium

Note: * solutions do not intersect

⇝ If a solution starts $0 < y < K$, it stays in that interval (and $y \to K$ as $t \to \infty$.)

⇝ If $y > K$ at any time, it also stays in that interval and $y \to K$ as $t \to \infty$.

* K is called the saturation level or environmental carrying capacity.

* **Concavity:** (curves up / down)

This depends on $\dfrac{d^2y}{dt^2} = \dfrac{d}{dt} f(y) = y' f'(y) = \underline{f(y) f'(y)}$

Remember: concave up $\smile$ $y'' > 0$
concave down $\frown$ $y'' < 0$

Inflection points when y'' changes sign : $f'(y) = 0$.

Here (graph of f) :
$$\begin{cases} y'' > 0 & \text{for } 0 < y < K/2 \\ y'' < 0 & \text{for } K/2 < y < K \\ y'' > 0 & \text{for } y > K \end{cases}$$

Note the inflection point at $y = K/2$.

Finally, solution! Separation of variables

$$\int \frac{dy}{y(1-y/k)} = \int r\,dt$$

$\downarrow$ partial fraction expansion

$$\frac{1}{y(1-y/k)} = \frac{1-y/k + y/k}{y(1-y/k)}$$

$$= \frac{1}{y} + \frac{1}{k-y}$$

Suppose $0 < y < k$ then

$$\ln(y) - \ln(k-y) = rt + C$$

$$\frac{y}{k-y} = e^{C} e^{rt}$$ Implicit solution

Initial condition: $$e^{C} = \frac{y_0}{k-y_0}$$

Solving for $y(t)$:

$$y = \frac{y_0}{k-y_0} e^{rt} (k-y)$$

$$\left(1 + \frac{y_0}{k-y_0} e^{rt}\right) y(t) = \frac{k y_0}{k-y_0} e^{rt}$$

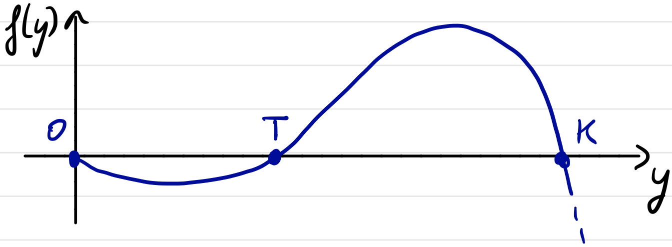
Thus:

$$y(t) = \frac{y_0 k}{y_0 + (k - y_0)e^{-rt}}$$

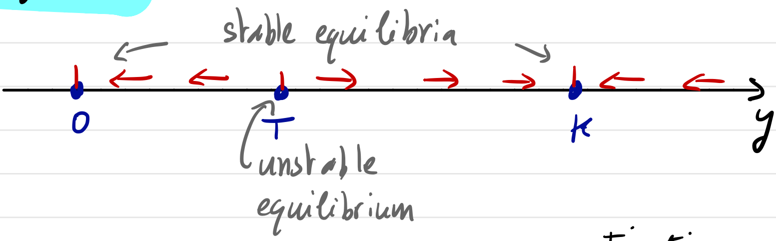
Example 3: Logistic growth with a threshold

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{k}\right) y$$

where $r > 0$ and $0 < T < k$



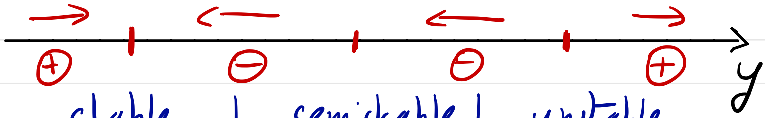
Phase line:



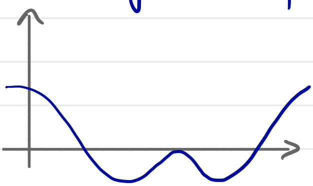
Example: hunted species
→ extinction
→ threshold for conservation
→ healthy population

Summary

Phase line: graphical representation helpful to determine equilibria and stability



stable	semistable	unstable
$f'<0$	$f'=0$, $f''\neq 0$	$f'>0$



§ 2.4 Differences between linear and non-linear diff. eq. (1st order)

Linear	Non-linear
$y' + p(t)\,y = g(t)$	$y' = f(t,y)$
(Specific form)	(Everything else)
Ex: $y' + \sin(t)\,y = \cos(t)$	Ex: $y' = t^2 + y^2$

TL:DR Linear usually much easier.
More results and methods.

Linear case: General solution formula

$$\begin{cases} y(t) = \dfrac{1}{m(t)}\left(\int m(t)\, g(t)\, dt + C \right) \\ \text{where } m(t) = \exp\left(\int p(t) \right) \end{cases}$$

[integrating factors !]

↳ All possible solutions are <u>explicit</u> as long as $p(t)$ and $g(t)$ is continuous in some interval.

Non linear case: At best, implicit solution of the form

$$F(t,y) = C$$

"First integral" of the differential equation.

(Separable equations; Exact equations)

But, no general method, no explicit solution.

Only reassurance:

Picard-Lindelöf existence and uniqueness theorem.

↳ "There exists a solution". (Friday)

Main difficulty: Interval of definition

* Linear eq: OK, as long as $p(t)$ and $g(t)$ are continuous.
* Nonlinear case: only guaranteed "some" interval.

Example: $y' = y^2$; $y(0) = y_0$

Solution: $\frac{y'}{y^2} = 1$ or $\frac{d}{dt}\left(\frac{-1}{y} - t\right) = 0$

so $\frac{-1}{y(t)} = t + C$ and $y(t) = \frac{-1}{t+C}$

Then $C = -\frac{1}{y_0}$ and $y(t) = \frac{y_0}{1 - y_0 t}$

The solution blows up at $t = 1/y_0$

↳ Interval of existence:

If $y_0 > 0$: $-\infty < t < 1/y_0$

If $y_0 < 0$: $1/y_0 < t < +\infty$

Note: If $y_0 = 0$: then $-\infty < t < \infty$, $y(t) = 0$!

Example 2: $$y' = \sqrt{y}$$

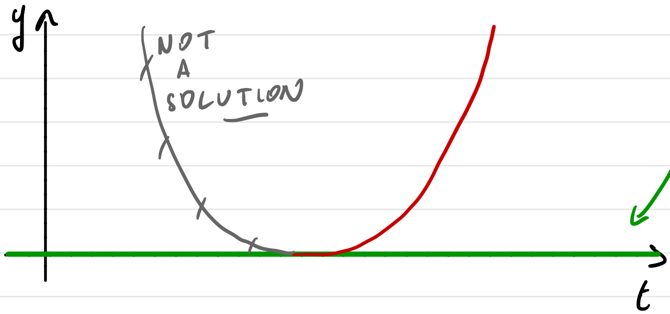
Solution: separation of variables.

$$\int \frac{dy}{\sqrt{y}} = \int dt \qquad \text{OR} \qquad y = 0$$

$$2\sqrt{y} = t + C \qquad \text{OR} \qquad y = 0$$

for $\underline{t + C \geq 0}$

$$\boxed{\begin{array}{ll} y(t) = \left(\frac{t+C}{2}\right)^2 \quad \text{OR} & y(t) = 0 \\ \text{for } t \geq -C & \text{for all } t \end{array}}$$



In fact, we can build new solutions by putting two pieces together:

$$y(t) = \begin{cases} 0 & \text{for } t \leq -C \\ \left(\frac{t+C}{2}\right)^2 & \text{for } t > -C \end{cases}$$

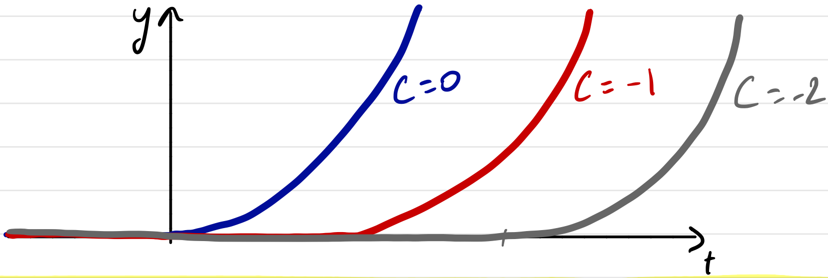
Indeed, y' is continuous: $y'(-c) = \begin{cases} 0 & (\text{left}) \\ 0 & (\text{right}) \end{cases}$

so such $y(t)$ satisfies the definition of a solution.

In particular, this means that the IVP

$$\begin{cases} y' = \sqrt{y} \\ y(0) = 0 \end{cases}$$

has many solutions:



Non-linear IVPs do not necessarily have unique solutions!

↳ This is annoying. Can we identify cases where it does not happen?

What is special about 0 here? Note

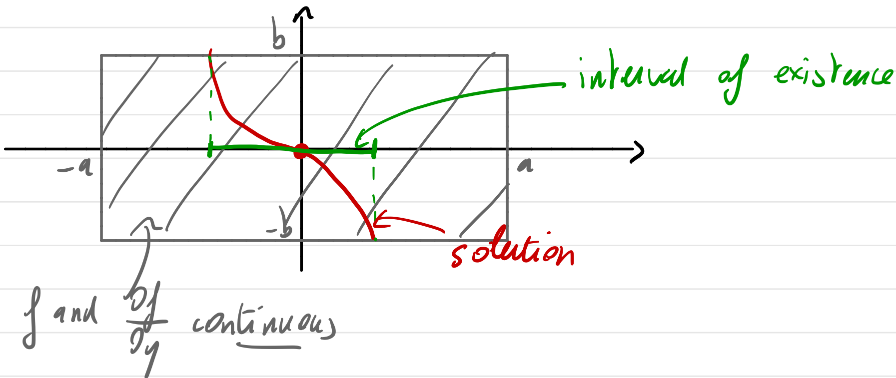
$f(t,y) = \sqrt{y}$ continuous for $y \geq 0$

but $\frac{\partial f}{\partial y}$ is not continuous for $y = 0$.

§2.8 Existence and Uniqueness Theorem

This is a fundamental result for 1st order IVPs.

> **Picard-Lindelöf Theorem**
> (or Cauchy-Lipschitz)
>
> Consider a function $f(t,y)$ such that f and $\frac{\partial f}{\partial y}$ are both continuous in a rectangle: $|t| < a, \quad |y| < b,$
>
> then there exists a time interval $|t| \leq h \leq a$ in which there exists a unique solution to the IVP $y' = f(t,y)$ and $y(0) = 0$.



An idea about the proof.

Or, How do we show that a solution exists without knowing anything about it?

Main idea: look at the eq in a $\neq$ way:

$$\begin{cases} y' = f(t, y) \\ y(0) = 0 \end{cases} \longrightarrow y(t) = \int_0^t f(t, y(t))$$

unknown!

Instead of a D.E we have now an integral eq!

The function $y(t)$ is then identified as a fixed point of the integral operation.

Picard Iteration Method

Start from $\varphi_0(t) = 0$

Then $\varphi_1(t) = \int_0^t f(s, \varphi_0(s))\,ds$

Then $\varphi_2(t) = \int_0^t f(s, \varphi_1(s))\,ds$

$\vdots$

$$\varphi_{n+1}(t) = \int_0^t f(s, \varphi_n(s))\,ds$$

This generates a sequence of function. Taking the limit we hope to write

$$\varphi_\infty(t) = \lim_{n\to\infty} \varphi_n(t) \longrightarrow \varphi_\infty(t) = \int_0^t f(s, \varphi(s))\,ds$$

For this to work, we need to make sure that things do not break down along the way:

* Do the values of φ_n get out of the rectangle at some point?
* Convergence? Properties of the limit
  ↳ continuity, derivatives...
* Uniqueness?

Example in action: $y' = y + 1$

↳ $y(t) = \int_0^t y(s)\,ds + t$

Picard iteration:

$$\varphi_0(t) = 0$$
$$\varphi_1(t) = \int_0^t 0\,ds + t = t$$
$$\varphi_2(t) = \int_0^t s\,ds + t = t + \frac{t^2}{2}$$
$$\varphi_3(t) = \int_0^t s + \frac{s^2}{2}\,ds + t = t + \frac{t^2}{2} + \frac{t^3}{6}$$

Extrapolating, we recognize the Taylor expansion of

$$\varphi_\infty(t) = e^t - 1 = t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \dots$$

⇝ at each step, add very small correction.

Main idea: fixed point iterations

Idea: if F is a contraction:

$$\text{dist}(F(x), F(y)) \leq k \, \text{dist}(x, y)$$

then

$$\begin{aligned} \text{dist}(\varphi_{n+2}, \varphi_{n+1}) &\leq k \, \text{dist}(\varphi_{n+1}, \varphi_n) \\ &\leq k^2 \, \text{dist}(\varphi_n, \varphi_{n-1}) \\ &\dots \\ &\leq k^{n+1} \, \text{dist}(\varphi_1, \varphi_0) \end{aligned}$$

If $k < 1$ $\longrightarrow$ convergence!