Wed. Sept 4

§2.3 Modeling with First Order ODEs

Outlook:
- Recap some steps in formulating a mathematical model with ODEs!
- Propose and study some "useful" examples.

Physical / Engineering situation:

Step 1: Construction of the model

* Common sense
* Known laws
* Experience
* Concentrate on essential features.

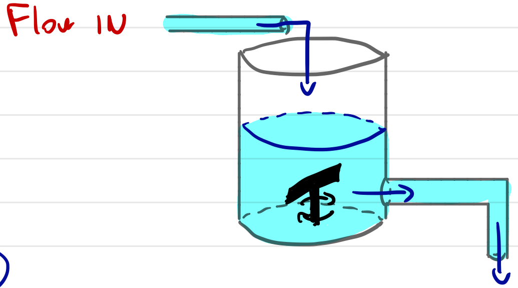
Step 2: Analysis

* Solutions and its properties, ideally
* Numerical approximation

Step 3: Comparison with reality

* Check (common sense)
* Data

Example 1: Salt tank mixing problems



$V(t)$: volume of mixture: water & salt

$Q(t)$: amount of salt in the tank.

Text problem :

"A tank contains, at time $t_0 = 0$, a quantity Q_0 lb of salt dissolved in 100 gallons of water.

Water containing 1/4 lb of salt per gallon is entering the tank at a rate of r gall/min,

the mixture is well-stirred at all times,

and drains out of the tank at a rate of 2r gal/min."

Questions:

1. How much mixture is in the tank at a time t, before it empties?

2. Set up an IVP for the amount of salt according to the description. Solve this IVP.

3. Assume r = 3 gal/min and $Q_0 = 0$.
   What is $Q(t = 10\text{ mn})$?

1 Volume in the tank:

$$\frac{dV}{dt} = \text{FLOW IN} - \text{FLOW OUT}$$
$$= r - 2r = -r$$

↳ $V(t) = -rt + C$ and $V(0) = 100$

$$\boxed{V(t) = 100 - rt}$$

∴

Note the tank empties at $t = 100/r$, after which expressions do not make sense.

2 IVP for salt amount:

$$\frac{dQ}{dt} = \underset{\text{lb/min}}{\text{RATE IN}} - \underset{\text{lb/min}}{\text{RATE OUT}}$$

$$= \underbrace{\frac{1}{4}}_{\substack{\text{concentration} \\ \text{going in} \\ \text{lb/gal}}} \times \underbrace{r}_{\substack{\text{flow rate} \\ \text{IN} \\ \text{gal/min}}} - \underbrace{\frac{Q(t)}{V(t)}}_{\substack{\text{concentration} \\ \text{in the tank} \\ \text{lb/gal}}} \times \underbrace{2r}_{\substack{\text{flow rate} \\ \text{OUT} \\ \text{gal/min}}}$$

$$\begin{cases} \dfrac{dQ}{dt} = \dfrac{r}{4} - \dfrac{2r}{100 - rt} Q(t) \\ Q(0) = Q_0 \end{cases}$$

Solution: integrating factor

* Standard form: $\frac{dQ}{dt} + \underbrace{\frac{2r}{100-rt}}_{p(t)} Q(t) = \underbrace{\frac{r}{4}}_{q(t)}$

* Integrating factor:

$$m(t) = \exp\left(\int p(t)\right)$$

Here, $$p(t) = \frac{2r}{100-rt}$$

To integrate, use $$\int \frac{u'}{u} = \ln(u(t))$$

Here $u(t) = 100 - rt$ so $u'(t) = -r$

Then, $$\int p(t) = -2\int \frac{-r}{100-rt}$$

$$= -2\ln(100-rt)$$

$$m(t) = \exp\left(-2\ln(100-rt)\right)$$

$$= (100-rt)^{-2}$$

* Solution:

$$Q(t) = \frac{1}{m(t)}\left(\int m(t)\,q(t) + C\right)$$

Now $$m(t)\,q(t) = \frac{r/4}{(100-rt)^2}$$

To integrate, use $$\int \frac{-u'}{u^2} = \frac{1}{u(t)}.$$

Here, $u(t) = 100 - rt$, $u'(t) = -r$

so now, $\int p(t)q(t) = \frac{1}{4}\int \frac{-(-r)}{(100-rt)^2}$

$$= \frac{1}{4(100-rt)} + C$$

$$Q(t) = \frac{100-rt}{4} + C\,(100-rt)^2$$

* Value of the constant:

$$Q(0) = Q_0 = 25 + 100^2\, C$$

$$C = 10^{-4}\,(Q_0 - 25)$$

so... $$Q(t) = \frac{1}{4}(100-rt) + \left(\frac{100-rt}{100}\right)^2 (Q_0 - 25)$$

(3) Numerical value: $t = 10$ mn, $r = 3$ gal/min; $Q_0 = 0$

First, check: $t = 10$ mn $< \frac{100}{r} = 33...$ min.

$$Q(10\text{ mn}) = \frac{1}{4}(100-30) + \left(\frac{100-30}{100}\right)^2 \times (0-25)$$

$Q(10\text{ mn}) = 5.25$ lb $\therefore$

Friday Sept. 7 — § 2.3 and 2.5

Note: 3 possible cases for salt tank problems.

- Flow IN > flow OUT

  ↳ amount of mixture increases until overflow.

- Flow OUT > flow IN

  ↳ amount of mixture decrease until empty.

- Flow IN = flow OUT.

  ↳ amount of mixture is constant.

Easiest case: ODE has constant coefficients

$$\frac{dQ}{dt} = c_{in} r_{in} - \frac{r_{out}}{V_0} Q(t)$$

See the textbook: example 1, § 2.3

Example 2: Compound interest.

Retirement account: regular deposits at fixed rate.

* Balance: $S(t)$, t years after start.

* IVP: $\dfrac{dS}{dt} = \underbrace{rS}_{\text{interest / year}} + \underbrace{k}_{\text{deposit per year}}$

* Solution: integrating factors.

(1) $\frac{dS}{dt} - rS = k$ (2) $m(t) = \exp(-rt)$

(3) $\frac{d}{dt}\left(e^{-rt}S(t)\right) = k e^{-rt}$

(4) $S(t) = e^{rt}\left(-\frac{k}{r}e^{-rt} + c\right)$

(5) $S(t) = \underbrace{S_0 e^{rt}}_{\text{initial investment + interest accruing}} + \underbrace{\frac{k}{r}\left(e^{rt} - 1\right)}_{\text{regular deposit or withdrawals.}}$

Note: * On the stock market, "interest rate" is not constant / predictible. RANDOM.

* See textbook for "discrete" model, more precise. (after example 2)

Example 3 Temperature models

Newton's law of cooling:

Rate of change of temperature proportional to difference of temperature between the object and its surroundings.

Math model:

$$\frac{dT}{dt} = k\left(T_s - T(t)\right)$$

where $T(t)$: temperature of the object,

$T_s(t)$: temperature of its surroundings, usually constant;

k: cooling constant/rate.

Solution: If T_s constant, use separation of variables:

$$\int \frac{dT}{T - T_s} = -\int k$$

thus $\ln|T - T_s| = -kt + C$

$$T - T_s = \underbrace{\pm e^{C}}_{c,\ \text{arbitrary constant}} e^{-kt}$$

Initial temperature $T(0) = T_0$:

$$T(t) = T_s + (T_0 - T_s)e^{-kt}$$

Since $k > 0$ always (thermodynamics!) we observe $T(t) \longrightarrow T_s$ (the equilibrium) independently of the original temperature (stability).