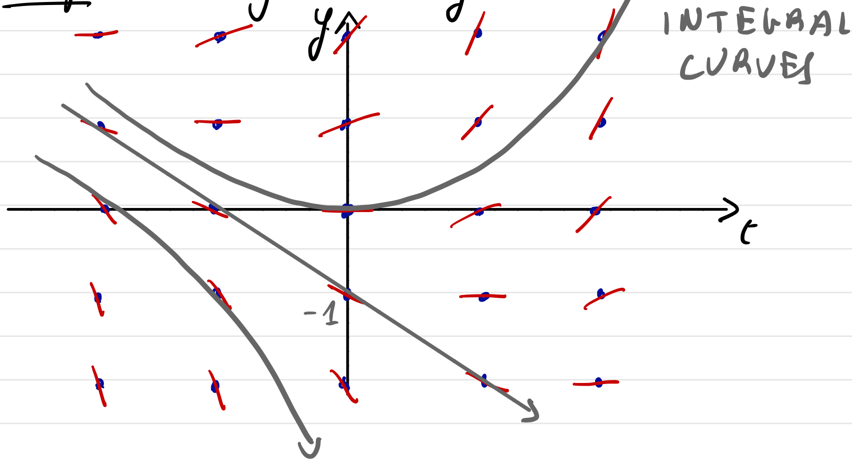
Week 2: August 27–31

(cont. study of 1^{st} order ODE $y' = f(t,y)$)

Case 2 General function $y' = f(t,y)$

Example: $y' = t + y$



Conclusion: Direction fields are a visual aid to gain intuition about 1^{st} order ODEs.

III) Solutions of a differential equation.

Definition: A solution of the n-th order ODE:

$$\frac{d^n y}{dt^n} = f(t, y, y', \ldots, y^{(n-1)})$$

is a function φ such that $\varphi', \ldots, \varphi^{(n)}$ exist and

$$\frac{d^n \varphi}{dt^n} = \varphi^{(n)}(t) = f(t, y, \ldots, y^{(n-1)}) \text{ at all times.} \quad (1)$$

GENERAL SOLUTION: An expression that contains ALL possible solutions of a D.E.

INTEGRAL CURVES: The geometrical representation of the general solution is (typically) an infinite family of curves called integral curves.

PARTICULAR SOLUTION: a single solution of a D.E. typically characterized by an additional condition such as an initial condition (IVP).

Examples: see next chapter (now)!

CHAPTER 2 FIRST ORDER ODEs

§2.1 Integrating factors

* In the general IVP case: $\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$

There is no generic way of arriving at an explicit expression for the solution.

(Although one should know that a solution always exists under very relaxed conditions: Cauchy-Lipschitz existence theorem)

We saw last week that equations of the form $y' = ay + b$ for a, b constant have explicit solutions.

Trick : we know how to solve equations of type

$$\frac{dy}{dt} = g(t) \longrightarrow \text{direct integration!}$$

This turns out to work in general for any ODE of type

$$y' + p(t)\, y = g(t),$$

where coefficients are not constants.

$\rightarrow$ Method of integrating factors.

Example 1 $\quad y' + y = 2e^{-3t}$.

Step 1 : find an appropriate function $m(t)$ such that by multiplying the whole equation by $m(t)$:

$$m(t)\frac{dy}{dt} + m(t)\,y(t) = 2\,m(t)\,e^{-3t}$$

we may rewrite the equation as

$$\frac{d}{dt}\Big(m(t)y(t)\Big) = 2\,m(t)\,e^{-3t}$$

$\hookrightarrow$ this can be solved directly by integrating both sides!

By the product rule, this will happen if

$$m \cdot \cancel{\frac{dy}{dt}} + \frac{dm}{dt} \cdot y = m \cancel{\frac{dy}{dt}} + m \cdot y$$

or $$\frac{dm}{dt} = m$$

For example, choose $m(t) = e^t$

(Rule: $m(t) > 0$)

Now we rewrite the initial problem as

$$\frac{d}{dt}\left(e^t y(t)\right) = e^t(y' + y) = e^t\left(2e^{-3t}\right)$$

or $$\frac{d}{dt}\left(e^t y(t)\right) = 2e^{-2t}.$$

Step 2

Integrate both sides:

Constant of integration.

$$e^t y(t) = \int 2e^{-2t} = -e^{-2t} + C$$

Divide by $m(t) = e^t$:

$$y(t) = Ce^{-t} - e^{-3t}.$$

This is the general solution!

Now: general case, $y' + p(t)y = g(t)$.
Let us find the right integrating factor $m(t)$:

$$\underbrace{m(t)\frac{dy}{dt} + m(t)p(t)y}\ = m(t)g(t)$$

Want $\dfrac{d}{dt}\big(m(t)y\big) = m(t)\dfrac{dy}{dt} + m'(t)y(t)$

↳ need to adjust $m(t)$ such that

$$m'(t) = p(t)\,m(t)$$

Assuming $m>0$, rewrite as $\dfrac{m'(t)}{m(t)} = p(t)$.

Chain rule: $\dfrac{m'(t)}{m(t)} = \dfrac{d}{dt}\ln m(t) = p(t)$ so

integrating both sides, $\ln m(t) = \int p(t)$

or $m(t) = \exp\left(\int p(t)\right)$.

METHOD OF INTEGRATING FACTORS

① Write the ODE in standard form,
$$y' + p(t)\,y = g(t)$$

② Compute the integrating factor,
$$m(t) = \exp\left(\int p(t)\right)$$

③ Compute general solution:
$$y(t) = \frac{1}{m(t)}\left(\int m(t)g(t) + C\right)$$

Note: any primitive of $p(t)$ works for $m(t)$. Do not worry about constants of integration!

Example. Solve the IVP

$$\begin{cases} (t^2+1)\dfrac{dy}{dt} + 2t\,y = 1 \\ y(0) = 0 \end{cases}$$

① Standard form: divide by (t^2+1)

$$y' + \underbrace{\frac{2t}{t^2+1}}_{p(t)}\, y = \underbrace{\frac{1}{t^2+1}}_{g(t)}$$

② Integrating factor:

$$m(t) = \exp\left(\int p(t)\right)$$

Here, $p(t) = \dfrac{2t}{t^2+1}$ $\dfrac{u'}{u}$ so $\int p(t) = \ln(t^2+1)$

$$m(t) = \exp\left(\ln(t^2+1)\right) = t^2+1$$

③ General solution: two ways.

- Direct use of formula:

$$y(t) = \frac{1}{m(t)}\left(\int m(t)\,g(t) + c\right)$$

Here, $m(t)g(t) = 1 \longrightarrow \int m(t)g(t) = t$

General solution: $y(t) = \dfrac{t+C}{t^2+1}$ C arbitrary.

- If you do not remember this formula:

remember to multiply the ODE by $m(t)$ and transform the LHS into $d/dt\,[m(t)y(t)]$.

$$\underbrace{(t^2+1)\frac{dy}{dt} + 2ty}_{} = 1$$

$$\frac{d}{dt}\Big((t^2+1)\,y(t)\Big) = 1$$

so $$(t^2+1)\,y(t) = t + C$$

$$\boxed{y(t) = \frac{t+C}{t^2+1}}$$ General solution of the ODE.

4 Particular solution from initial condition.

We want $y(0) = \dfrac{C+0}{1+0} = C = 0$

so $y(t) = \dfrac{t}{t^2+1}$ is the particular solution of the IVP.

Note: we could have seen immediately that the LHS is a derivative of $d/dt\big((t^2+1)y\big)$. But the method guides us.

Example 2 Solve the IVP

$$\begin{cases} y' + \dfrac{t}{3} y = e^{-t^2/6} \cos(t) \\ y(0) = 1 \end{cases}$$

1 standard form: OK.

$p(t) = t/3,\quad g(t) = e^{-t^2/6} \cos(t)$

2 integrating factor: $m(t) = \exp\left(\int p(t)\right)$

Here, $p(t) = t/3 \Rightarrow m(t) = \exp\left(t^2/6\right)$

3 general solution:

$$\frac{d}{dt}\left(e^{t^2/6} y(t)\right) = \cancel{e^{t^2/6}} \cdot \cancel{e^{-t^2/6}} \cos(t)$$

integrating: $e^{t^2/6} y(t) = \sin(t) + C$

finally: $y(t) = e^{-t^2/6} \sin(t) + C e^{-t^2/6}$

4 particular solution:

$$y(0) = e^{-0} \cancel{\sin(0)} + C e^{-0} = C = 1$$

so $\boxed{y(t) = e^{-t^2/6}\left(1 + \sin(t)\right)}$

solution of the IVP.

Conclusion

* General-purpose method for linear, 1^{st} order ODE of type

$$y' + p(t)\,y = g(t).$$

* Not always possible to write the solution! We may not be able to compute the integrals.

* Solution always write as

$$y(t) = \underbrace{y_p(t)}_{\text{1 particular solution}} + \underbrace{y_h(t)}_{C/m(t)}$$

"homogeneous solution"

Let $h(t) = 1/m(t) = \exp\left(-\int p(t)\right)$

$$\frac{dh}{dt} = -p(t)\exp\left(-\int p(t)\right) = -p(t)\,h(t)$$

(chain rule)

OR, $\boxed{\dfrac{dh}{dt} + p(t)\,h = 0}$

homogeneous equation ($g \equiv 0$)
NO RHS.

§2.2 Separable equations

What can we do when the previous method does not apply?

Observation: we were able to solve directly for the integrating factor,

$$\frac{dm}{dt} = p(t)\, m,$$

because we "separated variables" between left/right:

$$\frac{1}{m}\frac{dm}{dt} = p(t)$$

inverse Chain rule

$$\frac{d}{dt}\ln(m) = p(t)$$

integrate both sides

$$\ln(m) = \int p(t)$$

This idea can be useful in other cases, when some other function of m than $1/m$ appears "on the left"

$\Rightarrow$ We identify particular cases of 1^{st} order ODEs which can be transformed to be of the form

$$M(t) + N(y)\frac{dy}{dt} = 0$$

that is if $\quad f(t,y) = -\dfrac{M(t)}{N(y)}.$

In this case, the equation can be solved directly by integration.

Note: switch to indep variable x instead of t.

Example 1: $$\frac{dy}{dx} = \frac{x}{1+y^2}$$

① Rewrite by separating variables:

$$\underbrace{x}_{\text{only } x} - \underbrace{(1+y^2)}_{\text{only } y}\frac{dy}{dx} = 0$$

Integrate: $$\frac{d}{dx}\left[\frac{x^2}{2} + \left(y(x) + \frac{y^3(x)}{3}\right)\right] = 0$$

or $$\boxed{y + \frac{1}{3}y^3 = C - \frac{x^2}{2}}$$

↳ IMPLICIT FORM of the solutions.
NOT $y(x) \stackrel{?}{=} ?$

↳ Also called an equation for the integral curves of the equation.

Note: usually impossible to solve for y!
But if it is — do it!

Examples of equations: which are separables?

(a) $y' = \dfrac{\sin(y)}{\sin(t)}$ (b) $y^2 y' = \dfrac{2}{t}$

(c) $\dfrac{dy}{dt} = e^{y-t}$ (d) $\dfrac{dy}{dt} = \sin(ty)$

Domain of validity

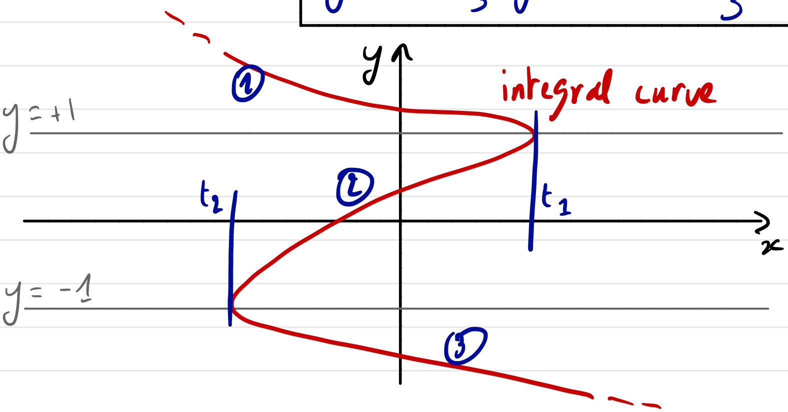
Example (textbook): $\dfrac{dy}{dx} = \dfrac{x^2}{1-y^2}$

- It is separable: $x^2 - (1-y^2)\dfrac{dy}{dx} = 0$

- Integral curves:

$$\frac{d}{dx}\left[\frac{x^3}{3} - \left(y(x) - \frac{1}{3}y^3(x)\right)\right] = 0$$

OR $$\boxed{y(x) - \frac{1}{3}y^3(x) = \frac{x^3}{3} + C}$$



→ Such a curve is not the graphical representation of a solution (multiple values around zero!)

In fact the curve corresponds to 3 distinct solutions for each value of C:

* one for $y \geq 1$, ① valid for $t \leq t_1$
* one for $-1 \leq y \leq 1$, ② valid for $t_2 \leq t \leq t_1$
* one for $y \leq -1$, ③ valid for $t \geq t_2$

Note how each solution is only valid on a certain interval of time, beyond which it cannot be extended.

This is the domain of validity of the solution.

Friday August 31

§2.2 (cont.) **SEPARABLE EQUATIONS**

Consider the equation $\frac{dy}{dx} = -\frac{x}{y+1}$.

We can solve it by separation of variables:

$$\int y+1 \, dy = \int -x \, dx$$

integrate $\times 2$

$$\frac{1}{2}y^2 + y = -\frac{1}{2}x^2 + C$$

C an arbitrary constant.

The "integral curves" or graphical representation of solutions are 'level sets' of the resulting expression,

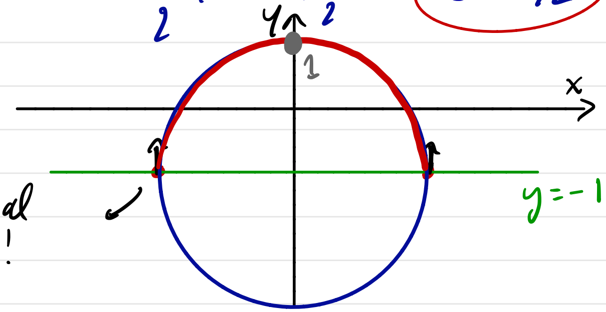
$$F(x,y) = \frac{1}{2}(x^2+y^2)+y = \underset{\text{constant.}}{C}$$

Here, level sets are circles!

$$x^2 + (y+1)^2 = 2C+1$$

Particular solution: when given an initial condition, $y(0) = 1$ i.e. $y = 1$ for $x = 0$ we can find the value of C:

$$\frac{1}{2}1^2 + 1^2 + \frac{1}{2}0^2 = \boxed{C = 3/2}$$



vertical tangents!

Looking at the equation we observe that $\left|\frac{dy}{dx}\right| \to \infty$ when $y + 1 \to 0$ i.e. $\boxed{y = -1}$

This corresponds to $\frac{1}{2}x^2 + \frac{1}{2}(-1)^2 - 1 = 3/2$

Solve for x: $\boxed{x = \pm 2}$

Thus solution exists for $\boxed{-2 \leq x \leq 2}$ domain of validity

Conclusion: even if we only have the implicit form of the solution (no $y(x) = \ldots$), lots to say.

1. Separate variables
2. Integrate both sides. Write implicit form.
3. If possible, find explicit solution.

OR

Use graphical tools; determine interval of validity.

Recipe : look for values where

* $y(x) \longrightarrow \infty$: Example, $y'(x) = y^2$

$$\frac{y'(x)}{y^2(x)} = 1 \Rightarrow \frac{-1}{y(x)} = x + C \Rightarrow y(x) = \frac{-1}{x+C}$$

Solution exists for $x < -C$ or $x > +C$

* $y'(x) \longrightarrow \infty$: Example, $y'(x) = k/y$

$$y y'(x) = k \Rightarrow \frac{1}{2} y^2(x) = kx + C \Rightarrow y(x) = \sqrt{2(kx+C)}$$

Solution exists for $x > -C/k$

Last example : consider IVP $\begin{cases} dy/dx = \dfrac{xy}{1-y^2} \\ y(0) = 2 \end{cases}$

① Separation of variables :

$$\left(\frac{1-y^2}{y}\right) \frac{dy}{dx} = x$$

⚠ $y=0$ forbidden!

② Integrate :

$$\left(\frac{1}{y} - y\right) \frac{dy}{dx} = x \Rightarrow \frac{d}{dx}\left(\ln|y| - \frac{y^2}{2} - \frac{x^2}{2}\right) = 0$$

$$\boxed{\ln|y| - y^2/2 - x^2/2 = C}$$ C arbitrary.

(3) Particular solution: $y = 2$ for $x = 0$

$$\ln 2 - 2 = C \approx -1.3$$

Interval of definition: vertical tangents $y' \to \infty$ for $y = \pm 1$, here

$$\ln|\pm 1| - \frac{(\pm 1)^2}{2} - \frac{x^2}{2} = \ln 2 - 2$$

so $x = \pm\sqrt{3 - 2\ln 2} \approx \pm 1.27$

↳ Since $-1.27 < 0 < 1.27$ this gives the domain

$$-\sqrt{3 - 2\ln 2} < x < \sqrt{3 - 2\ln 2}$$

Consider Other case: $y(1) = 0$

$$\rightsquigarrow \quad \cancel{\ln|0|}^{\to \infty} - \frac{0^2}{2} - \frac{1^2}{2} = C$$

⇒ The expression above does not include all possible solutions!

Forbidden case: $\underline{y = 0}$

Check: $\frac{dy}{dx} = 0 = \frac{x \cdot 0}{1 - 0} = 0$ ✓