# Chapter 7 Systems of First-Order Linear Equations

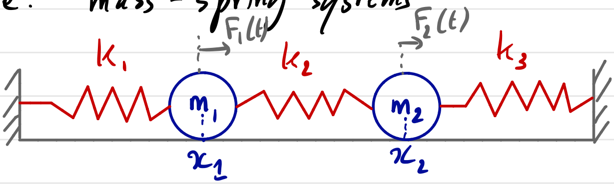
## §7.1 Introduction.

Real systems usually involve several interconnected components and several variables.

Typically we may consider

- One independent variable $t$
- Several dependent variables $x_1(t), x_2(t), \dots$

Example: mass-spring systems




Balance of forces:

$$m_1 \frac{d^2 x_1}{dt^2} = \underbrace{-k_1 x_1}_{\text{First spring}} + \underbrace{k_2 (x_2 - x_1)}_{\text{Second spring}} + F_1(t)$$

$$m_2 \frac{d^2 x_2}{dt^2} = \underbrace{-k_2 (x_2 - x_1)}_{\text{Second spring}} \underbrace{- k_3 x_2}_{\text{Third spring}} + \underbrace{F_2(t)}_{\text{External forces.}}$$

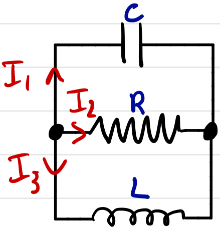
This is a second order system, but it can be transformed into a first-order one by introducing velocity variables:

$$v_1 = dx_1/dt \quad \text{and} \quad v_2 = dx_2/dt$$

We write then:

$$\begin{cases} \dfrac{dx_1}{dt} = v_1 \\ m_1 \dfrac{dv_1}{dt} = -(k_1+k_2)x_1 + k_2 x_2 + F_1(t) \\ \dfrac{dx_2}{dt} = v_2 \\ m_2 \dfrac{dx_2}{dt} = +k_2 x_1 - (k_2+k_3)x_2 + F_2(t) \end{cases}$$

Another example: LRC circuit in parallel.




Law: $I_1 + I_2 + I_3 = 0$

$$V = \frac{Q}{C} = RI_2 = L\frac{dI_3}{dt}$$

with $\dfrac{dQ}{dt} = I_1$

so $$C\frac{dV}{dt} = -I_2 - I_3 = -\frac{V}{R} - I_3$$

$$\begin{cases} \dfrac{dV}{dt} = -\dfrac{1}{RC}V - I_3 \\ \dfrac{dI_3}{dt} = \dfrac{1}{L}V \end{cases}$$

Any 2nd order equation: for example,

$$m u'' + \gamma u' + ku = F(t)$$

$v = u'$

$$\begin{cases} u' = v \\ m v' = F(t) - \gamma v - ku \end{cases}$$

Any n-th order ODE:

$$y^{(n)} = F(t, y, y', \dots, y^{(n-1)})$$

Introduce auxiliary variables $x_j = y^{(j-1)}(t)$

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \vdots \\ x_n' = F(t, x_1, \dots, x_n) \end{cases}$$

A solution of a system with n unknowns consists of

- interval $I$
- n functions, differentiable on $I$, such that $x_1 = \varphi_1(t), \dots, x_n = \varphi_n(t)$ satisfy the system of differentiable equations.

Initial conditions are n conditions $x_1(t_0) = x_1^0, \dots, x_n(t_0) = x_n^0$ where $t_0 \in I$.

An initial value problem (IVP) is a system of n equations with initial conditions:

$$\begin{cases} x_1' = F_1(t, x_1, \dots, x_n), & x_1(t_0) = x_1^0 \\ \vdots \\ x_n' = F_n(t, x_1, \dots, x_n), & x_n(t_0) = x_n^0 \end{cases}$$

Existence of solutions for a general $1^{st}$ order system:

If $F_1, \dots, F_n$ and $\frac{\partial F_i}{\partial x_j}$ with $1 \leq i, j \leq n$ are continuous in a region of space delimited by $a < t < b$, $a_j \leq x_j \leq b_j$ including $t_0, x_1^0, \dots, x_n^0$ then the IVP has a unique solution in some smaller interval $|t - t_0| < h$, $a \leq t_0 \pm h \leq b$.

**Linear systems** have the general form

$$x_1' = a_{11}(t)\,x_1 + \dots + a_{1n}(t)\,x_n + f_1(t)$$
$$\vdots$$
$$x_n' = a_{n1}(t)\,x_1 + \dots + a_{nn}(t)\,x_n + f_n(t)$$

**Vector notation:**

$$\underbrace{\frac{d}{dt}\vec{x}(t)}_{\vec{x}(t):\ n\times 1 \text{ solution vector of differentiable functions on interval } I} = \underbrace{A(t)}_{A(t):\ n\times n \text{ matrix of continuous functions on interval } I}\vec{x}(t) + \underbrace{\vec{f}(t)}_{\vec{f}(t):\ \text{forcing } n\times 1 \text{ vector of continuous functions.}}$$

The system is **homogeneous** if $\vec{f} = \vec{0}$.
Otherwise, non-homogeneous.

An IVP can be written in this compact notation:

$$\vec{x}'(t) = A(t)\,\vec{x}(t) + \vec{f}(t)$$
$$\vec{x}(0) = \vec{x}_0 = \begin{bmatrix} x_1^0 \\ \vdots \\ x_n^0 \end{bmatrix}$$

Previously introduced principles apply:

*** Superposition principle:**

Take $\neq$ forcings $\quad \vec{f}_k(t) \longrightarrow$ solutions $\vec{x}_k(t)$
then combination $\quad c_1\vec{f}_1(t) + \dots + c_N\vec{f}_N(t) \longrightarrow c_1\vec{x}_1 + \dots + c_N\vec{x}_N.$

* **Non-homogeneous principle:**

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

General solution non-homogeneous system ($\vec{x}(t)$); General solution homogeneous system ($\vec{x}_h(t)$); Particular solution ($\vec{x}_p(t)$)

## § 7.2 Matrices. Elementary linear algebra

**A matrix** is a table of numbers, with **m rows, n columns.**

↳ $m \times n$ matrix (also: order $m,n$)

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

**A vector** is a matrix for which either

$m = 1$: row vector

or $n = 1$: column vector

**Operations on matrices:**

① **Equality:**

* same order
* All corresponding entries are equal

② **Addition:** If two matrices $A, B$ have same order,

$A+B$ is the matrix obtained by adding the corresponding entries of $A$ and $B$:

$$\underline{[A+B]_{i,j}} = A_{ij} + B_{ij}$$

Denotes entry on i-th row, j-th column.

③ **Substraction:** If two matrices $A, B$ have same order,

$A-B$ is the matrix obtained by substracting the corresponding entries of $A$ and $B$:

$$[A-B]_{i,j} = A_{ij} - B_{ij}$$

④ **Multiplication by a number** (scalar)

- Any matrix can be multiplied by any real or complex number.
- $cA$ is the matrix obtained by multiplying all entries of matrix $A$ by the number $c$.

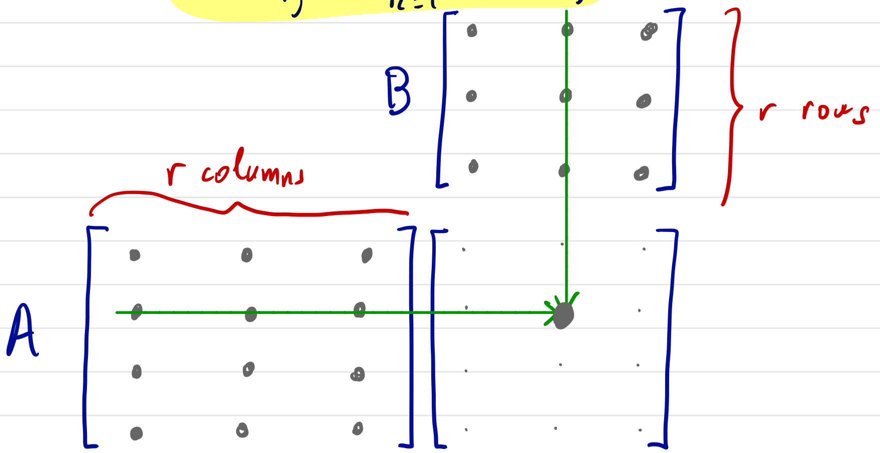
$$[cA]_{ij} = c\,A_{ij}.$$

## 5 Multiplication of matrices $A$, $B$

- Possible only if the # of columns of $A$ equals the # of rows of $B$.

- If $A$ has order $m \times r$ and $B$ order $r \times n$, the product $C = AB$ is a $m \times n$ matrix where the $(i,j)$-th entry is the scalar product of the $i$-th row of $A$ and $j$-th column of $B$:

$$[AB]_{ij} = \sum_{k=1}^{r} a_{ik} b_{kj}$$




$C = AB$

Example $\quad A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} -4 & 1 & 0 \\ 2 & -4 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

## 6 NO DIVISION

### Properties.

Addition: $\begin{cases} A+B = B+A & \text{(commutativity)} \\ A+(B+C) = (A+B)+C & \text{(associativity)} \end{cases}$

Multiplication: $\begin{cases} AB \neq BA & \text{(non-commutativity)} \\ (AB)C = A(BC) & \text{(associativity)} \end{cases}$

$\begin{cases} A(B+C) = AB + AC \\ (A+B)C = AC + BC \end{cases}$ (distributivity)

$$(c+d)A = cA + dA$$

## 7 TRANSPOSE

$$\underset{m,n \text{ matrix}}{A} \longrightarrow \underset{n,m \text{ matrix}}{A^T} \quad \text{transpose of } A$$

The transpose operation interchanges rows and columns of $A$:

$$A = (a_{ij}) \longrightarrow A^T = (a_{ji})$$

## 8 Adjoint:

for complex matrices, the adjoint is the conjugate of the transposed matrix:

$$\underset{m,n \text{ matrix}}{A} \longrightarrow \underset{n,m \text{ matrix}}{A^*} \quad \text{transpose of } A$$

$$A = (a_{ij})_{m,n} \longrightarrow A^* = (\overline{a_{ji}})_{n,m}$$

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## ZOO OF SPECIAL MATRICES

* **Square matrices:** $m = n$, same number of rows and columns. We say such a square matrix has order $n$.

The **diagonal** of a square matrix is the collection (vector) of entries $A_{j,j}$ for $1 \leq j \leq n$.

$$A = \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix}$$

* **(Column) vectors:** $1 \times n$ matrices.
  Notation: either arrows, as in $\vec{x}$
  or bold (in type)

* **Zero matrix:** all entries are zero.

* **Diagonal matrix:** a SQUARE matrix where only diagonal entries are different from zero:

ex. $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

* Identity matrix of order $n$, $I_n$ :

square matrix of order $n$, where diagonal entries are $1$ and all other are $0$.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Special property: this matrix acts as the identity for the multiplication operation,

$$AI_n = I_m A = A \text{ for } A \text{ } m \times n \text{ matrix.}$$

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**INVERSE** If $A$ is a square matrix of size $n$, there MAY exist a unique matrix $A^{-1}$ such that

$$AA^{-1} = A^{-1}A = I_n.$$

(Note: $(A^{-1})^{-1} = A$ )

**Example:** $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$

Take the matrix $B = \begin{bmatrix} 1 & -3/2 \\ 0 & 1/2 \end{bmatrix}$ then

$$AB = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -3/2 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -3/2 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So $B = A^{-1}$.

# DETERMINANTS in a nutshell.

The determinant of the matrix $A$ is a number denoted $\det A$.

**Main property:**
An inverse $A^{-1}$ exists if and only if $\det A \neq 0$.

**Definition:** By induction.

* if $n=1$: $A = [a_{11}]$ $\det A = a_{11}$
* if $n>1$: Assume the determinant is defined for matrices of order $1, \dots, n-1$.
The following procedure defines a determinant for order $n$ matrices.

For $1 \leq i, j \leq n$
① Define a smaller square matrix, the minor $M_{ij}$ by deleting row $i$ and column $j$
② Define the cofactor
$$c_{ij} = (-1)^{i+j} \det(M_{ij})$$

Then we choose either:
③ Development along the $i$-th row: define

(for any $j$)
$$\det A = \sum_{i=1}^{n} a_{ij} c_{ij} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij})$$

③ bis Development along the $j$-th column: define

(for any $i$)
$$\det A = \sum_{j=1}^{n} a_{ij} c_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij})$$

**Properties:**

$$\det(AB) = (\det A)(\det B)$$
$$\det(I_n) = 1$$
$$\det(A^{-1}) = (\det A)^{-1}$$

**Link with matrix inverse:** if $\det A \neq 0$ then

$$A^{-1} = \underbrace{(\det A)^{-1}}_{\text{inverse of determinant}} \underbrace{C^T}_{\text{transpose of the matrix of cofactors}} \quad \text{i.e.} \quad [A^{-1}]_{ij} = \frac{c_{ji}}{\det A}$$

**Note:** this formula is NOT FOR PRACTICAL COMPUTATIONS

In practice, the inverse is computed through GAUSSIAN ELIMINATION.

**Notation:** vertical lines

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ or } \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{ then } \det A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

**Examples.**

* 2×2 determinants: $A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Development along...

$$\left.\begin{array}{ll}
* \text{ row } 1: & (-1)^{1+1} a\,|d| + (-1)^{1+2} b\,|c| = ad - bc \\
* \text{ row } 2: & (-1)^{2+1} c\,|b| + (-1)^{2+2} d\,|a| = -cb + da \\
* \text{ column } 1: & (-1)^{1+1} a\,|d| + (-1)^{2+1} c\,|b| = ad - cb \\
* \text{ column } 2: & (-1)^{1+2} b\,|c| + (-1)^{2+2} d\,|a| = -bc + da
\end{array}\right\} = \det A$$

$$\begin{vmatrix} 1 & -1 & -1 \\ 3 & -1 & 2 \\ 2 & 2 & 3 \end{vmatrix} \underset{\text{row }1}{=} +1 \times \begin{vmatrix} -1 & 2 \\ 2 & 3 \end{vmatrix} - (-1) \times \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} + (-1) \times \begin{vmatrix} 3 & -1 \\ 2 & 2 \end{vmatrix}$$

$$= (-3-4) + (9-4) - (6+2) = \underline{-10}$$

Hint: if possible, develop along row or column with the most zeros.

## § 7.3 Linear systems; linear independence, Diagonalization

A linear system of algebraic equations is a problem: solve for $x_1, \dots, x_n$ such that

$$\begin{cases} a_{11} x_1 + \dots + a_{1n} x_n = b_1 \\ \vdots \qquad\qquad \vdots \qquad \vdots \\ a_{n1} x_1 + \dots + a_{nn} x_n = b_n \end{cases}$$

or

$$A \vec{x} = \vec{b},$$

where $A$ - $n \times n$ square matrix, $\vec{x}, \vec{b}$ – $n{\times}1$ column vectors.

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If $\det A \neq 0$ then such a system has solution:

$$\vec{x} = A^{-1} \vec{b}.$$

If $\det A = 0$ then either there is no solution, or infinitely many solutions.

In practice, one does not compute $A^{-1}$, but we use techniques such as Gaussian elimination.

① Form the augmented matrix:

$$(A \mid b) = \left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & b_n \end{array}\right)$$

② Perform elementary operations on rows such as to obtain the identity matrix on the left:

* Row exchange
* Multiplication by number $\neq 0$
* Add a multiple of a row to another

## Example

$$\begin{cases} x_1 - 2x_2 + 3x_3 = -3 \\ -x_1 + x_2 - 2x_3 = 1 \\ 2x_1 - x_2 - x_3 = 4 \end{cases} \longleftrightarrow \begin{array}{c} \left(\begin{array}{ccc|c} 1 & -2 & 3 & -3 \\ -1 & 1 & -2 & 1 \\ 2 & -1 & -1 & 4 \end{array}\right) \end{array} \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

First, eliminate all numbers under the diagonal, put 1 on it:

"pivot"

$$\left(\begin{array}{ccc|c} \boxed{1} & -2 & 3 & -3 \\ 0 & -1 & 1 & -2 \\ 0 & 3 & -7 & 10 \end{array}\right) \begin{array}{l} \\ \leftarrow (2)+(1) \\ \leftarrow (3)-2(1) \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & -3 \\ 0 & \boxed{1} & -1 & 2 \\ 0 & 0 & -4 & 4 \end{array}\right) \begin{array}{l} \\ \leftarrow -(2) \\ \leftarrow (3)+3(2) \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & -3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & \boxed{1} & -1 \end{array}\right) \begin{array}{l} \\ \\ \leftarrow -(3) \end{array}$$

Row-Reduced Echelon Form "RREF"
+ All 1's on the diagonal:
There will be solutions.
Continue

Next, eliminate all numbers above the diagonal:

$$\left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \boxed{1} & -1 \end{array}\right) \begin{array}{l} \leftarrow (2)-3(3) \\ \leftarrow (2)+(3) \\ \\ \end{array}$$

$$\underbrace{\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 1 \end{array}\right.}_{\text{Identity matrix}} \underbrace{\left|\begin{array}{c} 2 \\ 1 \\ -1 \end{array}\right)}_{\text{Solution}} \begin{array}{l} \leftarrow (1)+2(1) \\ \\ \\ \end{array}$$

Solution $\to$ $\vec{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

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## Linear dependence and independence of vectors

Given $k$ vectors $\vec{x}_1, \dots, \vec{x}_k$ of size $n$, they are said

* **Linearly dependent** if there exists numbers $c_1, \dots, c_k$ which are not all zeros such that
$$c_1 \vec{x}_1 + \dots + c_k \vec{x}_k = \vec{0}.$$

* **Linearly independent** if the above relation implies
$$c_1 = \dots = c_k = 0.$$

**How to compute such a property?**

1. If there are $n$ vectors $\vec{x}_1, \dots, \vec{x}_n$ of size $n$, rewrite relation as

$$\begin{cases} x_{11} c_1 + \dots + x_{1n} c_1 = 0 \\ \vdots \qquad \qquad \vdots \qquad \vdots \\ x_{n1} c_1 + \dots + x_{nn} c_n = 0 \end{cases}$$ where $x_{ij} = [\vec{x}_j]_i$

that is we form the matrix $X$ such that vectors $\vec{x}_1, \ldots, \vec{x}_n$ are the columns of $X$.

Then the relation above reads $X\vec{c} = \vec{0}$ so

* dependent $\Leftrightarrow$ $\det X = \begin{vmatrix} \vec{x}_1 & \cdots & \vec{x}_n \end{vmatrix} = 0$
* independent $\Leftrightarrow$ $\det X = \begin{vmatrix} \vec{x}_1 & \cdots & \vec{x}_n \end{vmatrix} \neq 0$.

② We can also compute the RREF of $X$, for any $k$:

$$\text{RREF}\left[\begin{array}{ccc|c} \vec{x}_1 & \cdots & \vec{x}_k & \vec{0} \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} ? & ? & ? & ? & \vec{0} \\ 0 & ? & ? & ? & \\ \vdots & & \ddots & ? & \\ 0 & \cdots & 0 & ? & \end{array}\right]$$

If there is AT LEAST ONE "non-pivot" column (zero on the diagonal) then $\vec{x}_1, \ldots, \vec{x}_k$ are linearly dependent.

**Example**

Are the vectors $\begin{pmatrix} 3 \\ 2 \\ -1 \\ 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} -2 \\ -2 \\ 3 \\ -1 \end{pmatrix}$ linearly independent?

$$\text{RREF}\left[\begin{array}{ccc|c} 3 & 1 & -2 & 0 \\ 2 & 0 & -2 & 0 \\ -1 & 2 & 3 & 0 \\ 4 & 3 & -1 & 0 \end{array}\right] \xrightarrow{\text{calc}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \boxed{0} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{non-pivot}$$

So the vectors are linearly dependent. 3rd variable free:
Choose $c_3 = 1 \rightarrow c_1 = 1, \; c_2 = -1$

Thus

$$1 \times \begin{pmatrix} 3 \\ 2 \\ -1 \\ 4 \end{pmatrix} + (-1) \times \begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix} + 1 \times \begin{pmatrix} -2 \\ -2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

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## Eigenvalues, eigenvectors

Idea: some vectors are left in the same direction by the action of the matrix,

$$\underbrace{A}_{\text{matrix}} \vec{v} = \underbrace{\lambda}_{\text{eigenvalue}} \vec{v}$$

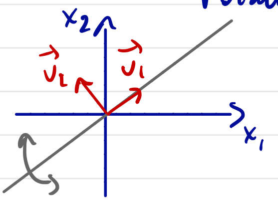
$\vec{v}$ → eigenvector: $\vec{v} \neq \vec{0}$

## Geometric interpretation: 2D example

$A \longrightarrow$ transformation of space

rotation, dilation, symmetry

ex:




$$A \vec{v}_1 = \vec{v}_1$$
$$A \vec{v}_2 = -\vec{v}_2$$

special directions: eigenvectors.

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To find such vectors, we write the equation,

$$(A - \lambda I_n)\vec{v} = \vec{0}$$

This equation has non-zero solutions if and only if

$$\det(A - \lambda I_n) = 0$$

This is the characteristic equation of the matrix

# Practical computation: algorithm

1. Form the characteristic equation
   (a polynomial equation of order $n$)
2. Find the roots $\lambda_1, \dots, \lambda_k$
3. Use RREF to find solutions of linear pbs,
   $$(A - \lambda_j I_n)\vec{v}_j = 0$$

**Example** Let $A = \begin{bmatrix} 3 & 1 \\ 5 & 3 \end{bmatrix}$

## 1. Characteristic equation

$$\det(A - \lambda I_2) = \begin{vmatrix} 3-\lambda & 1 \\ 5 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 5$$
$$= \lambda^2 - 6\lambda + 4$$

## 2. Roots

$$\Delta = 36 - 16 = 20$$

$$\lambda_{1,2} = \frac{6 \pm \sqrt{20}}{2} = 3 \pm \sqrt{5}$$

$\boxed{\text{Eigenvalues of } A}$

## 3. Eigenvectors

* $\lambda = 3 + \sqrt{5}$ : solve

$$\left[\begin{array}{cc|c} -\sqrt{5} & 1 & 0 \\ 5 & -\sqrt{5} & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & -1/\sqrt{5} & 0 \\ 0 & \boxed{0} & 0 \end{array}\right]$$

Extended matrix

Free variable

Choose $v_2 = 1 \to v_1 = 1/\sqrt{5}$

$$\vec{v} = \begin{pmatrix} 1 \\ 1/\sqrt{5} \end{pmatrix}$$ Eigenvector for $\lambda = 3+\sqrt{5}$

* $\lambda = 3-\sqrt{5}$ : solve

$$\left[\begin{array}{cc|c} +\sqrt{5} & 1 & 0 \\ 5 & +\sqrt{5} & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 1/\sqrt{5} & 0 \\ 0 & \boxed{0} & 0 \end{array}\right]$$

Extended matrix

Free variable

Choose $v_2 = 1 \to v_1 = -1/\sqrt{5}$

$$\vec{v} = \begin{pmatrix} 1 \\ -1/\sqrt{5} \end{pmatrix}$$ Eigenvector for $\lambda = 3-\sqrt{5}$

Conclusion

The two eigenpairs of the matrix are

$$(\lambda_1, \vec{v}_1) = \left(3+\sqrt{5}, \begin{bmatrix} 1 \\ 1/\sqrt{5} \end{bmatrix}\right) \text{ and } (\lambda_2, \vec{v}_2) = \left(3-\sqrt{5}, \begin{bmatrix} 1 \\ -1/\sqrt{5} \end{bmatrix}\right)$$

Notes

- Not all $n \times n$ matrices have $n$ distinct eigenvalues.
- Eigenvectors are always defined up to multiplication by a (nonzero) constant.
- Characteristic equation: polynomial of degree $n$.

Multiplicity of an eigenvalue $\to$ Repeated root

Algebraic multiplicity: # of times $\lambda$ is a root of poly.

$\geq$ Geometric multiplicity: # of linearly independent eigenvectors for $\lambda$.

**Special case:** when a matrix is SYMMETRIC that is $A = A^T$ or $a_{ij} = a_{ji} \ \forall i,j$

* All eigenvalues are real
* There are always n linearly independent eigenvectors.

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## Diagonalization

If $P = [\vec{v}_1, \dots, \vec{v}_n]$ is the matrix with columns formed by "n linearly independent eigenvectors of $A$":

$$A = P D P^{-1}$$

where $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ is a diagonal matrix formed with the eigenvalues.

**Example 1:**

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$$

① Characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{vmatrix}$$

$3^{rd}$ column

$$= (-1)^{1+3} \times 1 \times \begin{vmatrix} 6 & -1-\lambda \\ -1 & -2 \end{vmatrix} + (-1)^{2+3} \times 0 \times \begin{vmatrix} 1-\lambda & 2 \\ -1 & -2 \end{vmatrix} + (-1)^{3+3} \times (-1-\lambda) \begin{vmatrix} 1-\lambda & 2 \\ 6 & -1-\lambda \end{vmatrix}$$

$$= \left(-12 - (1+\lambda)\right) - (1+\lambda)\left( (1-\lambda)(-1-\lambda) - 12 \right)$$

$$(\cdots) = \lambda(-\lambda^2 - \lambda + 12)$$
$$= -\lambda(\lambda-3)(\lambda+4)$$

② Compute eigenvalues (roots):

$$\lambda_1 = -4 \qquad \lambda_2 = 0 \qquad \lambda_3 = 3$$

③ Compute eigenvectors:

$\lambda_1 = -4$ Solve eq $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -4\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 5 & 2 & 1 & 0 \\ 6 & 3 & 0 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & \boxed{0} & 0 \end{array}\right]$$

z free variable

Choose $z = 1 \to \vec{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

$\lambda_2 = 0$ Solve eq $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 6 & -1 & 0 & 0 \\ -1 & -2 & -1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 1/13 & 0 \\ 0 & 1 & 6/13 & 0 \\ 0 & 0 & \boxed{0} & 0 \end{array}\right]$$

z free variable

Choose $z = 13 \to \vec{v}_2 = \begin{bmatrix} -1 \\ -6 \\ 13 \end{bmatrix}$

$\lambda_3 = 3$ Compute $\vec{v}_3 = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$

④ Assemble $P$ and $D$:

$$P = \begin{bmatrix} -1 & 1 & 2 \\ 2 & 6 & 3 \\ 1 & -13 & -2 \end{bmatrix} \qquad D = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Check:

$$AP = PD = \begin{bmatrix} 4 & 0 & 6 \\ -8 & 0 & 9 \\ -4 & 0 & -6 \end{bmatrix}$$

**Example 2** 2×2 matrix: $A = \begin{bmatrix} 3 & 1 \\ 5 & 3 \end{bmatrix}$

Eigenvalues / eigenvectors: computed above,

$$\lambda_1 = 3 + \sqrt{5}, \ \vec{v}_1 = \begin{bmatrix} 1/\sqrt{5} \\ 1 \end{bmatrix}; \quad \lambda_2 = 3 - \sqrt{5}, \ \vec{v}_2 = \begin{bmatrix} -1/\sqrt{5} \\ 1 \end{bmatrix}$$

Then we assemble

$$P = \begin{bmatrix} 1/\sqrt{5} & -1/\sqrt{5} \\ 1 & 1 \end{bmatrix} \longrightarrow \quad C = \begin{bmatrix} 1 & -1 \\ 1/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

$$D = \begin{bmatrix} 3+\sqrt{5} & 0 \\ 0 & 3-\sqrt{5} \end{bmatrix} \qquad P^{-1} = \frac{\sqrt{5}}{2} \begin{bmatrix} 1 & 1/\sqrt{5} \\ -1 & 1/\sqrt{5} \end{bmatrix}$$

so
$$PDP^{-1} = \begin{bmatrix} 1/\sqrt{5} & -1/\sqrt{5} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3+\sqrt{5} & 0 \\ 0 & 3-\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{5}/2 & 1/2 \\ -\sqrt{5}/2 & 1/2 \end{bmatrix}$$

$$(\dots) = \begin{bmatrix} 3/\sqrt{5}+1 & 1-3/\sqrt{5} \\ 3+\sqrt{5} & 3-\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{5}/2 & 1/2 \\ -\sqrt{5}/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 3/2 + \sqrt{5}/2 - \sqrt{5}/2 + 3/2 & 3/2\sqrt{5} + 1/2 + 1/2 - 3/2\sqrt{5} \\ 3\sqrt{5}/2 + 5/2 - 3\sqrt{5}/2 + 5/2 & 3/2 + \sqrt{5}/2 + 3/2 - \sqrt{5}/2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \\ 5 & 3 \end{bmatrix} = A \quad \therefore$$

**Example 3** (Complex eigenvalues)

$$A = \begin{bmatrix} 2 & 1 \\ -2 & 4 \end{bmatrix}$$

① $\det(A - \lambda I) = (2-\lambda)(4-\lambda) + 2$
$= \lambda^2 - 6\lambda + 10$

② Eigenvalues: $\Delta = 36 - 40 = -4 < 0$

$$\lambda_1 = \frac{6+2i}{2} = 3+i \;;\quad \lambda_2 = \frac{6-2i}{2} = 3-i$$

③ Eigenvectors: $\boxed{\lambda_1 = 3+i}$

$$A - \lambda_1 I_2 \Rightarrow \left[\begin{array}{cc|c} -1-i & 1 & 0 \\ -2 & 1-i & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & \frac{-1-i}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_1 = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$

$\boxed{\lambda_2 = 3-i}$

$$A - \lambda_1 I_2 \Rightarrow \left[\begin{array}{cc|c} -1+i & 1 & 0 \\ -2 & 1+i & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & \frac{-1+i}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_2 = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$$

# §7.4 Basic theory of 1st order linear systems

Return to ODE systems!

$$\vec{x}'(t) = P(t)\,\vec{x}(t) + \vec{g}(t)$$

$\vec{x}(t)$ vector of size $n$, $P(t)$ $n\times n$ square matrix and $\vec{g}(t)$ vector of size $n$.

<u>Basic assumptions</u>: $P(t)$, $\vec{g}(t)$ are continuous on some interval $a < t < b$ meaning that each of the components $p_{ij}(t)$, $g_i(t)$ is a continuous function on the interval $a < t < b$.

## I) Homogeneous case: $\vec{g}(t) = \vec{0}$

<u>Notation</u>: $\vec{x}^{(1)}, \ldots, \vec{x}^{(k)}, \ldots$ are specific solutions of the system.

$x_{ij}(t) = x_i^{(j)}(t)$ is the $i$th component of the $j$-th solution (book notation)

**Theorem (Principle of Superposition)**

If $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ are solutions of the system $\vec{x}' = P(t)\vec{x}$, then the linear combination $c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)}$ is also a solution for any constants $c_1$ and $c_2$.

In fact, all linear combinations of solutions are solutions:

$$\vec{x}(t) = c_1 \vec{x}^{(1)}(t) + \dots + c_k \vec{x}^{(k)}(t)$$

---

**Example** $P(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$

We can check that the following are two solutions:

$$\vec{x}^{(1)}(t) = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} \quad \text{and} \quad \vec{x}^{(2)}(t) = \begin{bmatrix} t e^{-t} \\ (1-t)e^{-t} \end{bmatrix}$$

Then, any linear combination

$$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) = \begin{bmatrix} c_1 e^{-t} + c_2 t e^{-t} \\ -c_1 e^{-t} + c_2 (1-t) e^{-t} \end{bmatrix}$$

is a solution.

**Question:** are all solutions of this form?

↳ Can we solve for all initial conditions?

Suppose $n$ solutions are known: we need to solve

$$c_1 \vec{x}^{(1)}(t_0) + \dots + c_n \vec{x}^{(n)}(t_0) = \vec{x}_0$$

or $X \vec{c} = \vec{x}_0$

where $X = \begin{bmatrix} \vec{x}^{(1)}(t_0) & \dots & \vec{x}^{(n)}(t_0) \end{bmatrix} = \begin{bmatrix} x_{11}(t_0) & \dots & x_{1n}(t_0) \\ \vdots & & \\ x_{n1}(t_0) & \dots & x_{nn}(t_0) \end{bmatrix}$

⇝ Unique solution if $\det X \neq 0$ !

↳ We say that the solutions $\vec{x}^{(1)}, \dots \vec{x}^{(n)}$ are linearly independent on the interval $a < t < b$ if the WRONSKIAN

$$W[\vec{x}^{(1)}, \dots, x^{(n)}] = \det(X)$$

is non-zero on the whole interval.

If this is the case:

**Theorem** If $W[\vec{x}^{(1)}, \dots, \vec{x}^{(n)}] \neq 0$ for $a < t < b$,

- $\vec{x}^{(1)}, \dots, \vec{x}^{(n)}$ form a fundamental set of solutions,
- $\vec{x}(t) = c_1 \vec{x}^{(1)}(t) + \dots + c_n \vec{x}^{(n)}(t)$ is the general solution.

**Abel's theorem** If $\vec{x}^{(1)}, \dots, \vec{x}^{(n)}$ are solutions of

$$\vec{x}' = P\vec{x}$$

on the interval $a < t < b$, then

$$W[\vec{x}_1, \dots, \vec{x}_n] = \det X(t) \text{ is}$$

$$W(t) = C \exp\left( \int p_{11}(t) + \dots + p_{nn}(t)\, dt \right)$$

(Abel's formula)

so
* either 0 at all points ($C = 0$)
* or 0 nowhere ($C \neq 0$).

**Note:** A fundamental set of solutions always exists.

Idea: choose $\vec{x}^{(k)}(t)$ as solution of $\vec{x}' = P\vec{x}$ with initial condition

$$\vec{x}(0) = [0 \dots 0, \underset{\substack{\uparrow \\ k\text{-th position}}}{1}, 0 \dots 0]^T$$

then

$$X(0) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} = I_n \quad \text{and} \quad \det X(0) = 1.$$

**Note 2** Connexion between Wronskians.

$$y'' + p(t)y' + q(t)y = 0 \longleftrightarrow \begin{cases} u_1' = u_2 \\ u_2' = -q(t)u_1 - p(t)u_2 \end{cases} \quad \begin{pmatrix} u_1 = y \\ u_2 = y' \end{pmatrix}$$

or $\vec{u}' = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix} \vec{u}$ so $W(t) = C\exp\left(-\int p(t)\right)$ $\therefore$

## II) Non-homogeneous systems $\vec{x}' = P(t)\vec{x} + \vec{g}(t)$

The general solution has the form

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$ (non-homogeneous principle)

$\vec{x}_h(t)$: general solution of homogeneous system

$\vec{x}_p(t)$: particular solution of original system (undetermined coeff...)

$$\vec{x}_h(t) = c_1 \vec{x}^{(1)}(t) + \dots + c_n \vec{x}^{(n)}(t)$$