I) Motivation (Ch 1, textbook)

↳ Why are we here?

Study of objects that evolve with time

↳ described by variations of continuous quantities. (ultimately, discrete: atoms!)

Mathematical description: model.

Process of modeling: series of steps

- → What are the most important processes?
- → What important variables should I use to describe the system? (ex: position, temperature, concentration)
- → How do I describe their evolution?

Note: models can typically be improved by observing, analyzing, correcting errors.

Note 2: time not only possibility → ex: space. but very intuitive → many examples along the way!

Essential ingredient: differential equations.

Idea: often, known relation between the rate of change of one variable and the variables themselves. Example:

$x(t)$, $\qquad \dfrac{dx}{dt} = x^2 + t^2$

position as unknown function of time

some relation between $x(t)$, t and dx/dt.

Definition: A differential equation (D.E.) is an equation which contains derivatives of one (or more) dependent variables with respect to one (or more) independent variables

$\rightsquigarrow$ If only one independent variable:
$\to$ only ordinary derivatives
$\to$ ORDINARY diff. eq.
UNKNOWN(S) IS A FUNCTION OF 1 VARIABLE

$\rightsquigarrow$ If more than one independent variable:
$\to$ partial derivatives
$\to$ PARTIAL diff. eq.
UNKNOWN(S) IS FUNCTION OF SEVERAL VARIABLES

↳ MATH 647.

Definition (2)

The ORDER of a diff. eq. is the highest order of the derivative(s) appearing in it.

Notations : $y(t) \to \frac{dy}{dt}, \frac{d^2y}{dt^2}, \ldots$

OR $y', y'', \ldots$

Example 1 CREDIT

Model evolution of a credit card.

Step 1 : discover a D.E. modeling the problem at hand.

- $A(t)$ amount of money on the credit card after t years.

- Evolution of $A(t)$:

  * compound interest: 25% APY.

  * Minimum payment each month: \$35, or \$420 / year.

Together, these give the rate of change:

$$\frac{dA}{dt} = 0.25A - 420$$

Step 2: solve the model equation. Here, we know the solution:

$$A(t) = 1680 + Ce^{0.25t}$$

Check: $\frac{dA}{dt} = 0.25\,C e^{0.25t}$

$$0.25\,A(t) - 420 = \cancel{0.25 \times 1680 - 420} + 0.25\,C e^{0.25t}$$

$\therefore$

Note: we will learn soon how to compute such solutions!

C is an arbitrary constant. To determine the solution, need additional information: balance at time 0.

$\to$ $A(0) = \$2000 = 1680 + Ce^{0}$

$\Rightarrow$ $C = 2000 - 1680 = 320$

Then $A(t) = 1680 + 320\,e^{0.25t}$

Question: When is the loan paid off $(A(t)=0)$?

Answer: never. $A(t)$ is increasing.

After 10 years, you owe \$5578

Wednesday, August 24 Chapter 1

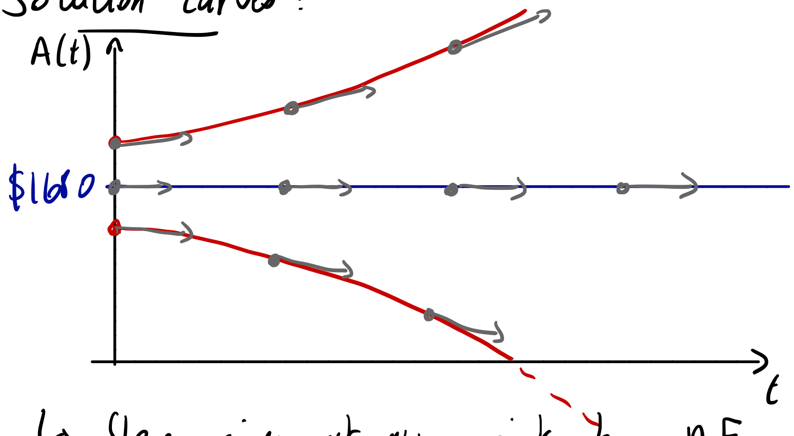
II) 1) Direction field: graphical analysis.

(Recall) from previous lecture: CC balance model

$$\frac{dA}{dt} = 0.25A - 420$$

$$\Rightarrow A(t) = 1680 + Ce^{0.25t}$$

Solution curves:



↳ Slope given at any point by D.E.

Remark: a problem of the form

$$\begin{cases} \text{Diff. eq:} & y' = f(t,y) \\ \text{Initial condition:} & y(t_0) = y_0 \end{cases}$$

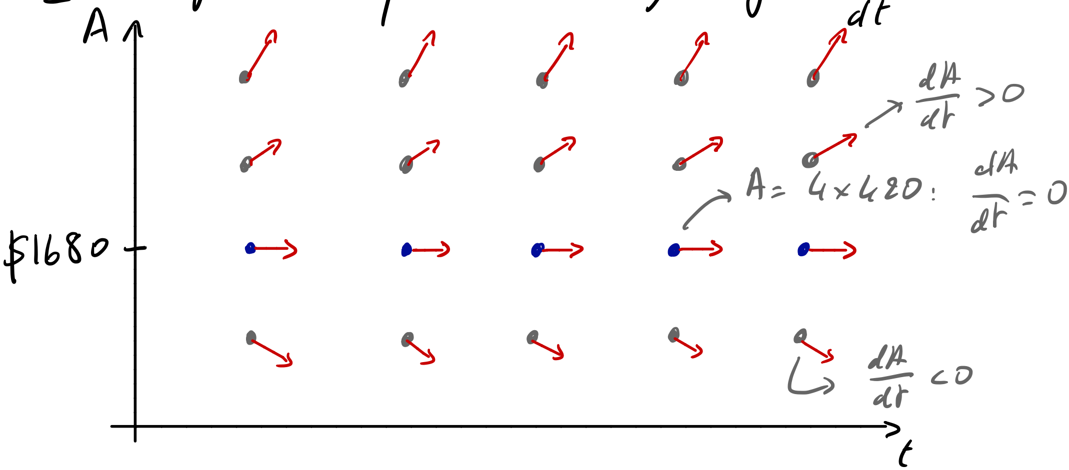
known quantities

is called an initial value problem (IVP)

Q When is the balance increasing / stable / decreasing?

$$\frac{dA}{dt} = 0.25A - 420$$

A Depends only on the sign of $\frac{dA}{dt}$!



- If $A < 1680$, it will decrease (until it reaches zero and I stop paying)
- If $A > 1680$, it will increase towards infinity
- If $A = 1680$, it will stay there.

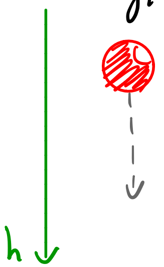
Direction field

General concept: $\dfrac{dy}{dt} = f(t, y)$

We can visualize the behavior of solutions $y(t)$ by drawing short segments with the slope $f(t, y)$ at a grid of points in the (t, y) plane.

Example 2 : Free Fall

Legend : Newton observes an apple falling from the tree.



Step 1 : interesting quantities:

$h(t)$: height of the apple.

$v(t)$: velocity of the apple.

Newton's 2nd law:

rate of change of velocity (acceleration) proportional to the net effect of all forces applied to the object.

$$m\,a(t) = m \frac{dv}{dt} = \sum F$$

The constant of proportionality is the mass (m).

What forces are applied to the object ?

* gravity : pushes down
* air resistance : pushes up

$\Rightarrow$ $F(t) = mg - \gamma v$

* m : mass of object
* g : acceleration due to gravity. $g \approx 9.8\ m/s^2$ near sea level
* γ : drag coefficient (depends on <u>shape</u>)

Together: $$m\frac{dv}{dt} = mg - \gamma v$$

Step I.5 | Can we say smth before solving?

Example: $m = 10kg$, $\gamma = 2\ kg/s$.

then $$\frac{dv}{dt} = 9.8 - 0.2v$$

Q When is the velocity stable / increasing / decreasing?

A Sign of dv/dt !

In particular, $\frac{dv}{dt} = 0$ if $9.8 - 0.2v = 0$ or

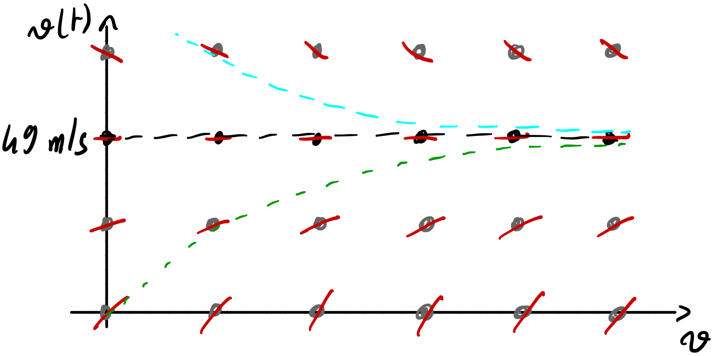
$$v(t) = 49\ m/s.$$

<u>Note</u>: this is a constant solution of the O.D.E!

called **equilibrium solution**

or, here, <u>terminal velocity</u>.

Direction field



- If $v < 49$ m/s, it will increase towards this value.
- If $v > 49$ m/s, it will decrease towards this value.

$\Rightarrow$ In both cases, as $t \to \infty$ the velocity approaches 49 m/s.

<u>Step II</u>) Solution.

write the equation in the form

$$\frac{dv}{dt} = 0.2\,(49 - v)$$

then $$\frac{dv/dt}{v-49} = -0.2$$

Recall chain rule:

$$\frac{d}{dx} \ln|x| = \frac{1}{x}$$

$$\frac{d}{dt} \ln|v(t)-49| = v'(t) \times \frac{1}{v-49}$$

Then $$\frac{d}{dt}\left[\ln|v(t)-49| \right] = -0.2$$

By integrating both sides, we find

$$\ln|v(t)-49| = C - 0.2 \times t$$

$$|v(t)-49| = \exp(C-0.2t)$$

$$v(t)-49 = \pm e^{C} e^{-0.2t}$$

$$v(t) = 49 + c\, e^{-0.2t}$$

where $c = \pm e^{C}$ is an arbitrary constant.

Note: method valid for all 1^{st} order D.E. of the form

$$y' = ay + b$$

$$\Rightarrow \quad y(t) = -\frac{b}{a} + c\, e^{at}.$$

Friday August 26

Recap: modeling

Examples of (ingredients for) simple 1st order models.

Key: proportionality between rate of change and some function of the quantity.

① $\frac{dy}{dt}$ proportional to y: $\frac{dy}{dt} = ky$

② $\frac{dy}{dt}$ " " $1/y$: $\frac{dy}{dt} = \frac{k}{y}$

③ Logistic growth: $\frac{dN}{dt} = k N (N_{tot} - N)$

↳ useful for disease modeling

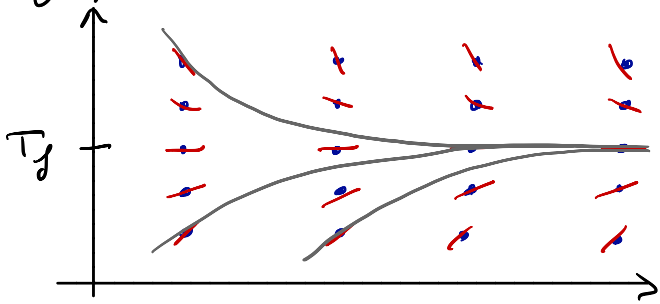
Rate of increase of # of infected people

proportional to product of # infected by # healthy

④ Newton's law of cooling or heating:

Rate of change in an object temperature proportional to difference between its temperature and surrounding temperature: $\frac{dT}{dt} = k(T_f - T)$

Graphical analysis: direction field!



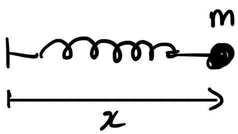
↳ solutions all tangent to direction field!

Higher order models

Ex 1: Free fall with variable $h(t)$, height

$$m\frac{d^2h}{dt^2} = mg - \gamma\frac{dh}{dt}$$

Ex 2: Hooke's law and the harmonic oscillator:



Displacement of a mass attached to a spring:

* Restoring force: $F_{spring} = -k\,(x - x_0)$

stiffness constant   length at rest

* Newton's 2nd law (neglect friction):

$$m\frac{d^2x}{dt^2} = -k\,(x - x_0)$$

This is a 2nd order ODE. Note that it's equivalent to a 1st order system:

Let $v = \frac{dx}{dt}$ then $\begin{cases} \frac{dx}{dt} = v \\ m\frac{dv}{dt} = -k(x-x_0) \end{cases}$

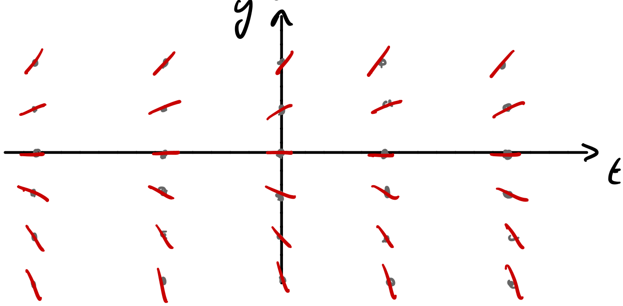
III) Direction fields; graphical solutions

Take a 1st order D.E: $\frac{dy}{dt} = f(t,y)$

A solution is a function $t \mapsto \varphi(t)$ such that $\varphi'(t) = f(t, \varphi(t))$ for all t.

The slope or direction field consists of small line segments on a grid, with slope $m = f(t,y)$

Case: $f(t,y) = 2y$



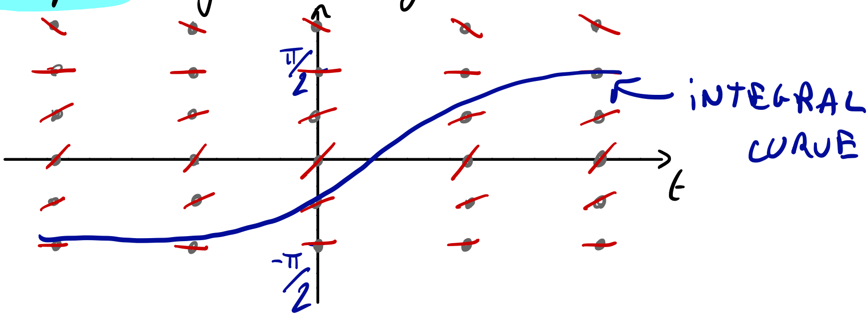
A solution is always tangential to the slope field.

↳ Graphical method to find or approximate an INTEGRAL CURVE representing a SOLUTION.

Case 1: f does not depend on t:

$$y' = f(y)$$

Example: $y' = \cos(y)$



Notion of equilibrium: solutions of $f(y) = 0$

$\Rightarrow$ $y(t) = $ Cste : equilibrium solution.

(Note: solutions reach max/min at equilibrium points)

Notion of stability: An equilibrium solution is **stable** if solutions "near it" tend towards it.

It is **unstable** if they tend away from it.

In the example above: $\begin{cases} +\pi \text{ is stable} \\ -\pi \text{ is unstable} \end{cases}$

Another example: $y' = y^2$

↳ Use computer. Matlab script.