Configurations with Subset Restrictions

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1 Introduction

An $r-$*set* is an $r-$element set. The problem we consider in this paper was suggested by a theorem of Bjorn Poonen [1] in connection with finite union-closed families of sets. Poonen showed that if a finite union-closed family of sets S contains three of the $3-$sets in some $4-$set, then one of the elements of that $4-$set is contained in at least half the members of the family S. This condition intrigued us, and we began to investigate collections of $3-$sets for which it fails. The alternative statement that no $4-$set contain more than two $3-$sets in the collection is reminiscent of conditions for Steiner systems. A *Steiner system* $S(r, s, t)$ is a collection C of $s-$subsets of a $t-$set T such that every $r-$set in T lies in exactly one $s-$set of C. The most familiar of these are the Steiner triple systems, or Steiner systems $S(2, 3, n)$, namely a collection C of $3-$sets in an $n-$set N in which every $2-$set in N lies in exactly one of the $3-$sets of C.

In contrast with Steiner systems, we consider collections C of $r-$sets in $\{1, \ldots, n\}$ such that every $s-$set in $\{1, \ldots, n\}$ contains at most t of the $r-$sets in C. That is, we remove the "exactly" condition, and the collection of smaller sets is restricted—not the collection of larger sets. Obviously, we could generalize this condition in many ways, but in this paper we treat the original condition.

Let $\mathcal{P}(n)$ be the power set of $\{1, \ldots, n\}$. An $[r, s, n, t]-$*configuration* is a subset C of the $r-$sets of $\mathcal{P}(n)$ such that each $s-$set in $\mathcal{P}(n)$ contains at most t of the $r-$sets in C. In this paper, we consider only $[3, 4, n, 2]-$configurations, and refer to them as $n-$*configurations*; by an $(n, k)-$*configuration* we mean an $n-$configuration containing exactly k $3-$sets. An $(n, k)-$configuration is *maximal* if it is not contained in any $(n, k + 1)-$configuration; finally, $L(n)$ is the largest integer k for which an $(n, k)-$configuration exists.

In this paper, we determine $L(n)$ for $4 \leq n \leq 9$; these values are given by the following table:

n	4	5	6	7	8	9
$L(n)$	2	5	10	15	22	32

In addition to some general results, we characterize all the maximal $n-$ configurations for $n = 4, 5$ and 6, as well as the $(n, L(n))-$ configurations for $n = 7, 8$ and 9. The approach we take in our proofs involves an analysis of the structure of certain graphs associated with the configurations.

2 Notation, the $W(i)$ graphs and the Main Bound Theorem

In this section, we give the definitions and general results which are used in carrying out the analysis for specific cases. To simplify notation, we will often denote a $3-$set $\{a, b, c\}$ merely as abc; a $2-$set $\{a, b\}$ as ab, and so on. Naturally, $abc = acb$, $ab = ba$, etc.

Definition 1. *Let S be a collection of $3-$sets in $\mathcal{P}(n)$. Let $N = N(S)$ be the number of members of S, let $W_S(i) = W(i) = \{(a, b)|\{a, b, i\} \in S\}$ and set $t_i(S) = t_i = |W(i)|$. The score of S is the vector $[t_1, t_2, \ldots, t_n]$, and if σ is any permutation of $\{1, 2, \ldots, n\}$, then $\sigma(S) = \{\{\sigma(a), \sigma(b), \sigma(c)\}|\{a, b, c\} \in S\}$. If S and T are collections of $3-$sets, and if $S = \sigma(T)$ for some permutation σ, then we say that S and T are permutation-isomorphic, or just isomorphic, and write $S \cong T$. The map σ is called an isomorphism (or an automorphism if $S = T$). The graph $G(S)$ of S is the graph with vertex set S in which two $3-$sets are adjacent provided they belong to the same $4-$set or, equivalently, if they contain a common $2-$set. The group of automorphisms of S is denoted by $H(S)$.*

The members of $W(i)$ can also be regarded as the edges of a graph on the vertex set $\{1, 2, \ldots, n\} - \{i\}$, and for simplicity we will use the name $W(i)$ for this graph. We recall a few definitions from graph theory: a triangle in $W(i)$ is a set of three edges of the form $(a, b), (a, c), (b, c)$; an independent set in $W(i)$ is a set of vertices no two of which are connected by an edge; and the independence number of $W(i)$ is the size of the largest independent set in $W(i)$.

Definition 2. *For an n-configuration S, the independence number of the graph $W_S(i)$ will be denoted by $I_S(i) = I(i)$.*

It is clear that $\cong$ is an equivalence relation and so we may speak of isomorphism classes of configurations. It is also clear that if two configurations have different scores, then they are not isomorphic; however, we shall see that there exist nonisomorphic configurations with the same scores. Evidently, two permutation-isomorphic collections of $3-$sets will have correspondingly isomorphic $W(i)$-graphs. Moreover, if S is a collection of $3-$sets, then S is an $n-$configuration if and only if the graph of S contains no triangles of the form abc, abd, acd.

Our first result includes a graph-theoretic characterization of $n-$configurations and a restriction on the score of an $n-$configuration.

Lemma 1. *a) S is an n-configuration if and only if for each $i = 1, 2, \dots, n$, the graph $W(i)$ contains no triangles. (b) $\sum_{i=1}^{n} t_i = 3N$.*

Proof. (a) Suppose that $W(i)$ contains a triangle, say ab, ac, bc. Then the $3-$sets abi, aci, bci are all members of S, and then the $4-$set $abci$ contains three $3-$sets from S, and S is not an n-configuration. On the other hand, suppose some $4-$set $xyzw$ contains three of the $3-$sets of S. Then, since any three of the $3-$sets of a $4-$set must have a common element, these three $3-$sets must have the form (say) xyz, xyw, xzw. Then $W(x)$ contains the triangle yz, yw, zw.

(b) Since t_i is just the number of members of S which contain i, then since every member of S is a $3-$set, the sum of the t_i is equal to $3|S| = 3N$. □

The next theorem gives a bound on the size of $L(n)$, which is sharp in some cases.

Theorem 1. *(**The Main Bound**) For $n \geq 5$, $L(n) \leq \dfrac{n}{n-3} \cdot L(n-1)$.*

Proof. Let $n \geq 5$. Evidently $L(n) \geq L(n-1) + 1$, since if S is an $(n-1)$-configuration, and we add on a $3-$set containing n, the result will be an n-configuration.

Thus, suppose that $L(n) = L(n-1) + k$ for some positive integer k, and let S be an n-configuration with $N(S) = L(n)$. For any $1 \leq i \leq n$, consider the configuration C consisting of the members of S which do not contain i. This is (isomorphic to) an $(n-1)$-configuration, and $N(C) = L(n) - t_i = L(n-1) + k - t_i$. Since $N(C) \leq L(n-1)$, it follows that $t_i \geq k$.

Thus, $3L(n) = \sum_{i=1}^{n} t_i \geq nk$, from which we get $3(L(n-1)+k) \geq nk$ and so $k \leq \dfrac{3L(n-1)}{n-3}$. Then $L(n) = L(n-1) + k \leq \dfrac{n}{n-3} \cdot L(n-1)$, as required. □

The basic idea of our constructions, is to extend an n-configuration to an $(n+1)$-configuration, by "adding on" more $3-$sets. We will reserve the use of the word "add" for the situation when a given configuration S , together with the new $3-$sets, is still a configuration. If S together with a new $3-$set abc is not a configuration, then we would say that we "cannot add" abc to S. Finally, we also use the word "add" in referring to the new edges which appear in the graphs $W(i)$ when new $3-$sets are added on to S, and to the corresponding vertices of the $W(i)$-graphs for S itself. This usage is best illustrated by an example.

Suppose $S = \{123, 234, 345, 451, 512\}$, so that $W_S(1) = \{23, 45, 52\}$. We draw the graph $W(1)$ as $3-2-5-4$, and observe that the independence number $I(1)$ is two, and the independent sets of size two are $\{3,5\}$, $\{3,4\}$, and $\{2,4\}$.

Since $T = \{123, 234, 345, 451, 512, 136, 126\}$ is not a configuration (the $4-$set 1236 has three $3-$sets from T), we would say that we cannot add 136 and 126 to S, and we cannot add the edges 36, 26 to $W_S(1)$.

We may add the $3-$sets $136, 156$ to S; when we do this, we obtain a configuration $T = \{123, 234, 345, 451, 512, 136, 156\}$. In T, we now have $W_T(1) = \{23, 45, 52, 36, 56\}$, and we say that we have added the edges 56 and 36 to $W_S(1)$. This last statement will usually be abbreviated to "add 5, 3 to $W(1)$", to avoid cumbersome notation; it is always understood that in using this phraseology, there is only one new vertex (in this case, 6), and the new edges all involve that new vertex.

It is clear that in adding new edges of the form $a6$ to $W_S(1)$, the vertices adjacent to the new vertex 6 must be an independent set in $W_S(1)$, since otherwise we would get a triangle. Furthermore, if we add 136 and 156 to S, we cannot also add 356 (for then 1356 would have three $3-$sets present). That is, in adding these new edges to $W(1)$, we eliminate certain possibilities from $W(3)$ and $W(5)$: we cannot add the edge 56 to $W(3)$ and we cannot add the edge 36 to $W(5)$. This reflects the fact that $W_T(6)$ already has edges 13 and 15, so cannot have an edge 35. These observations are summed up in the following lemma.

Lemma 2. *Let $n \geq 5$, and let S be an $(n-1)$-configuration. Suppose that T is an n-configuration containing S, such that every $3-$set in $T - S$ contains the element n. Put $W(i) = W_S(i)$, $t_i = |W_S(i)|$, $I(i) = I_S(i)$, and $u_i = |W_T(i)|$. For each $i = 1, 2, \ldots, n-1$ put $K(i) = \{a | \{a, n\} \in W_T(i)\}$. Then:*

(a) For each $i = 1, 2, \ldots, n-1$, $K(i)$ is an independent set in the graph $W(i)$.

(b) For each $i = 1, 2, \ldots, n-1$, $u_i \leq t_i + I(i)$.

(c) If $a, b \in K(i)$, $a \neq b$, then $W_T(a)$ does not contain the edge bn.

Proof. (a) If $K(i)$ were not an independent set in $W(i)$, there would be an edge connecting two members a, b of $K(i)$; then the graph $W_T(i)$ would contain the triangle an, bn, ab, a contradiction. The statement (b) follows immediately.

(c) If $a, b \in K(i)$, $a \neq b$, then T contains the 3−sets nai, nbi and therefore does not contain abn. That is, $W_T(n)$ contains the edges ai, bi, $W_T(n)$ does not have an edge ab; and $W_T(a)$ does not contain the edge bn. □

Throughout this paper, if one configuration is contained in another, then we will use t_i (resp., u_i) for the size of $W(i)$ in the smaller (resp., larger) configuration.

3 The Cases $n = 4$ and $n = 5$

In this section we characterize the maximal n−configurations for $n = 4, 5$, determining $L(4)$ and $L(5)$ in the process.

Theorem 2. *$L(4) = 2$ and there are unique $(4, 1)-$ and $(4, 2)-$configurations.*

Proof. By inspection, the only 4−configurations contain either one or two 3−sets, all $(4, 1)-$configurations are isomorphic to $\{123\}$ and all $(4, 2)-$configurations are isomorphic to $M_4 := \{123, 234\}$. Hence, $L(4) = 2$ and there are unique $(4, 1)-$ and $(4, 2)-$configurations. □

We need one more definition before proceeding to the case $n = 5$. If S is an $n-$configuration and $i \in \{1, \ldots, n\}$, then we define $S - i$ to be the collection of 3−sets in S not containing i. It is clear that $S - i$ is an $(n-1)-$configuration.

Theorem 3. *$L(5) = 5$, and there is a unique $(5, 5)-$configuration up to isomorphism, namely M_5=$\{123, 234, 345, 451, 512\}$. Furthermore, there are two isomorphism classes of maximal $(5, 4)-$configurations, namely*

$$54a := \{123, 124, 125, 345\} \text{ and } 54b := \{123, 124, 135, 145\}.$$

Proof. Let S be a 5−configuration. By Theorem 1, $L(5) \leq (5/2)L(4) = 5$. It is easy to check that M_5 is a 5−configuration, so $L(5) = 5$.

If $k = 5$, then each If $k = 5$, then each $t_i \geq 3$) (by Lemma 2) and $\sum_{i=1}^{5} t_i = 15$ (by Lemma 1) so the only possible score is $[3, 3, 3, 3, 3]$. If $k = 4$, then each $t_i \geq 2$ and $\sum_{i=1}^{5} = 12$, and the only possible scores are $[4, 2, 2, 2, 2]$ and $[3, 3, 2, 2, 2]$.

Since $\mathcal{P}(5)$ contains just ten 3-sets, it is easy to check cases directly. We find that if S contains $123, 124, 125$, then S cannot contain any other 3-set containing either 1 or 2, and so S must have $k \leq 4$, and if $k = 4$, S must be (up to isomorphism) the configuration 54a (which is therefore maximal). If S has $t_1 = 4$ and does not contain three 3-sets with a common 2-set, then $k = 4$ and S is the configuration 54b (up to isomorphism).

Suppose now that $k = |S| = 5$. Since each of the five 3-sets in S contains three 2-sets, there must be some repeated 2-sets, i.e., S must contain two three sets of the form abc, abd. (From above, S cannot contain three 3-sets with a common 2-set.) Thus suppose that $S - 5 = \{123, 234\}$. Since each $t_i = 3$, there must be at least two additional pairs of the form $15a, 15b$ and one of a, b must be 4, and the remaining 3-set must then have the form 45c. The only possibilities are $M_5 = \{123, 234, 345, 451, 512$ and $\{123, 234, 245, 145, 135\}$, and these are isomorphic by the transposition (23). Thus M_5 is (up to isomorphism) the only (5,5)-configuration.

As for their graphs, $G(M_5)$ is a 5-cycle, $G(54a)$ is a triangle together with an isolated point, and $G(54b)$ is a 4-cycle. This completes the proof. □

4 The case $n = 6$

In this section we show that $L(6) = 10$ and characterize the maximal $(6, k)-$configurations for $k = 8, 9$ and 10. Let $i, j \in \{1, \ldots, 6\}$. If S is a 6−configuration, then let u_i be the number of 3−sets in S containing i, and let t_i be the number of 3−sets in $S - j$.

Theorem 4. *Let* $M_6 = \{123, 234, 345, 451, 512, 163, 264, 365, 461, 562\}$*. Then* M_6 *is the unique* $(6, 10)-$*configuration,* $L(6) = 10$*, and* $G(M_6)$ *is the Peterson Graph PG.*

Proof. By Theorem 1, $L(6) \leq 6L(5)/3 = 30/3 = 10$. If S is a $(6, 10)-$ configuration, then $u_i \geq 5$ for all i: otherwise, $S - i$ would be a $(5, k)-$configuration for $k \geq 6$. Since $30 \leq 3L(6) \geq \sum_{i=1}^{6} u_i \geq 6 \cdot 5 = 30$, it follows that $L(6) = 10$ and that $u_i = 5$ for all i. Since $S - 6$ is a $(5, 5)-$configuration, we may assume that $S - 6 = \{123, 234, 345, 451, 512\}$. Now $S - 5$ is a $(5, 5)-$configuration containing 123 and 234, so it follows that

$$\text{either } S - 5 = \{123, 234, 346, 461, 612\} \text{ or } S - 5 = \{132, 324, 246, 461, 613\}.$$

In any case, S contains 461; in order to maintain $u_i = 5$ for all i, S also contains 256 and 356. If $S - 5 = \{123, 234, 346, 461, 612\}$, then S would contain 125, 126, and 256, which cannot happen. It follows that $S - 5 = \{132, 324, 246, 461, 613\}$, and so

$$S = M_6 := \{123, 234, 345, 451, 512, 163, 264, 365, 461, 562\}.$$

Hence, $L(6) = 10$, the $(6, 10)-$configuration M_6 is unique, and by inspection, $G(M_6)$ is the Peterson Graph PG. □

The following theorem tells the story about $(6, 8)-$ and $(6, 9)-$configurations.

Theorem 5. *There is a unique (nonmaximal)* $(6, 9)-$*configuration (which contains* M_5*), and there are seven* $(6, 8)-$*configurations, three of which contain* M_5 *and four of which do not.*

Proof. First, suppose that S is a $(6, k)-$configuration which contains M_5, with $k = 8$ or 9. If S contains only 3−sets in M_6, symmetry considerations reveal that there are three nonisomorphic possibilities, namely

$$\begin{aligned} M_6' &:= M_6 - 562 = \{123, 234, 345, 451, 512, 163, 264, 365, 461\}, \\ 68a &:= M_5 \cup \{163, 264, 365\}, \text{ with score } [4, 4, 5, 4, 4, 3], \text{ and} \\ 68b &:= M_5 \cup \{163, 264, 461\}, \text{ with score } [5, 4, 4, 5, 3, 3]. \end{aligned}$$

(Note that none of $H(M_6'), H(68a)$ and $H(68b)$ are transitive groups.) If S contains a 3−set not in M_6, then it can contain at most one other such 3−set. For example, if $126 \in S$, then 136, 156, 236 and 256 are excluded from S, and so S contains a subset of $\{146, 246, 346, 356, 456\}$. Moreover, at most one of 346, 356 and $456 \in S$, so S must contain a 3−set $ab6 \in M_6$. Without loss of generality, we may assume $163 \in S$. Again, symmetry considerations reveal that there are no $(6, 9)-$configurations and two nonisomorphic $(6, 8)-$configurations, namely

$$\begin{aligned} 68c &:= M_5 \cup \{163, 264, 465\}, \text{ with score } [4, 4, 4, 5, 4, 3], \text{ and} \\ 68d &:= M_5 \cup \{163, 364, 165\}, \text{ with score } [5, 3, 5, 4, 4, 3]. \end{aligned}$$

(Note that $H(68c)$ is trivial.) Hence there are four $(6, 8)-$configurations containing M_5, two maximal and two nonmaximal, and a unique (nonmaximal) $(6, 9)-$configuration (which contains M_5).

Now suppose that S is a $(6, 8)-$configuration not containing M_5; then $u_i \geq 4$ for all i, and since $24 = 3 \cdot 8 = \sum u_i \geq 6 \cdot 4 = 24$, it follows that $u_i = 4$ for all i. If

$S - 6 \cong 54a = \{125, 135, 145, 234\}$, then $u_6 = 4$ implies that S contains $1x6$ and $2y6$ for distinct $x, y \in \{3, 4, 5\}$. The remaining two triples must be 356 and 456, implying that S contains $135, 136$ and 356, which cannot happen. Hence $S - 6$ does not contain a copy of $54a$.

Let $\{a, b, c, d, e, f\} = \{1, \ldots, 6\}$. If S contains a copy of $54b$, we may assume that $S - f = \{abc, abd, ace, ade\}$, that a and e are not together in any pair of S, and that f appears with each of b, c, d, e twice. A brief exhaustive search reveals that there are two nonisomorphic ways to do this, namely

$$\begin{aligned} 68e &:= \{123, 234, 345, 145, 126, 156, 246, 356\}, \text{ with score } [4,4,4,4,4,4], \text{ and} \\ 68f &:= \{123, 234, 345, 135, 126, 156, 246, 456\}, \text{ with score } [4,4,4,4,4,4]. \end{aligned}$$

Although they have the same score, these two are not isomorphic: the graph of $68e$ has four vertices of degree 3 and four of degree 2, and the graph of $68f$ is isomorphic to Q_3, the familiar cube.

Finally, if S contains neither $54a$ nor $54b$, then without loss of generality we may assume $S - 6 = \{123, 234, 345, 451\}$. Since $u_1 = u_6 = 4$, two of the 3−sets $126, 136, 146, 156 \in S$, not both of $126, 136 \in S$, and not both $146, 156 \in S$. If $126, 146 \in S$, the only possibility is if

$$S = 68g := \{123, 234, 345, 451, 126, 146, 256, 356\}, \text{ with score } [4,4,4,4,4,4].$$

The graph of $68g$ has six vertices of degree 2 and two of degree 3. If $126, 156 \in S$, then 256 and 456 are excluded; this is impossible since $u_5 = 4$. Including 136 and 146 leads to either $68e$ or $68g$, and including 136 and 156 leads to either $68c$ or $68g$. Thus, there are seven $(6, 8)$−configurations, three of which contain M_5 and four of which do not. □

With that, we have finished the case $n = 6$.

5 The case $n = 7$

In this section we show that $L(7) = 15$, and characterize the $(7, 15)$-configurations and the $(7, 14)$-configurations.

Theorem 6. *$L(7) = 15$, and all $(7, 15)$-configurations are isomorphic to $M_7 = M_6 \cup \{173, 274, 375, 471, 572\}$.*

Proof. By Theorem 1, $L(7) \leq 17$; we show first that $L(7) \leq 15$.

Suppose that T is a $(7, 17)$-configuration. Then every $u_i \geq 7$, and also $\sum_{i=1}^{7} u_i = 3 \times 17 = 51$, and so we may assume that $u_7 = 7$. Then $S = T - 7$ is a $(6, 10)$-configuration , so in S, every $W(i)$ is a cycle on five vertices, and hence every $W(i)$ has independence number 2. It follows that $u_i \leq 5 + 2 = 7$ for $1 \leq i \leq 6$; but then $\sum_{i=1}^{7} u_i \leq 49$, a contradiction. So $L(7) \leq 16$.

Suppose that T is a $(7, 16)$-configuration. Then every $u_i \geq 6$, and also $\sum_{i=1}^{7} u_i = 3 \times 16 = 48$, and so we may assume that $u_7 = 6$. Then $S = T - 7$ is a $(6, 10)$-configuration; we assume that $S = M_6$. As above, every $u_i \leq 7$ for $1 \leq i \leq 6$, and it follows that the score can only be $[7, 7, 7, 7, 7, 7, 6]$. There are two triples in T containing both 6 and 7, say a67 and b67. Then ab6 cannot be in T, so ab must be one of the pairs $12, 15, 23, 34, 45$. Suppose that 167 and 267 are in T; then T cannot contain any of $137, 147, 247, 257, 127$. The remaining pairs containing 7 and not 6 are $157, 237, 347, 357, 457$, and of these we can have at most one of 347, 357, 457 (since 345 is in T). Then $u_7 \leq 5$, a contradiction. The argument is similar for the other possible values for the pair ab. Thus $L(7) \leq 15$.

Put $M_7 = M_6 \cup \{173, 274, 375, 471, 572\}$. It is easy to verify that M_7 is a $(7, 15)$-configuration, so $L(7) = 15$, and it remains only to show that M_7 is unique up to isomorphism.

Suppose that T is a $(7, 15)$-configuration. Then every $u_i \geq 5$, and also $\sum_{i=1}^{7} u_i = 3 \times 15 = 45$, so we can assume $u_7 \leq 6$. Then $T - 7$ is either a $(6, 10)$-configuration or a $(6, 9)$-configuration, and from this (considering the independence numbers of the $W(i)$-graphs for $T - 7$), we can say that every $u_i \leq 7$, and then the only two possible scores are (up to permutations) $[7, 7, 7, 6, 6, 6, 6]$ and $[7, 7, 7, 7, 7, 5, 5]$.

For the score $[7, 7, 7, 7, 7, 5, 5]$ we may assume without loss of generality that $u_7 = 5$. Since $H(M_6)$ is a transitive group, we may also suppose that $u_6 = 5$; then $T - 7$ and $T - 6$ are both isomorphic to M_6 and both $T - 7$ and $T - 6$ contain M_5. Then the graphs $W(6)$ in $T - 7$, and $W(7)$ in $T - 6$ are identical, and it follows that $T = M_7$.

We next show that the score $[7, 7, 7, 6, 6, 6, 6]$ is not possible. Suppose that the score is $[7, 7, 7, 6, 6, 6, 6]$ and that $u_7 = u_6 = 6$. Then $T - 7$ and $T - 6$ are both $(6, 9)$-configurations, and we may assume they both contain M_5; then T contains precisely ten triples containing either 6 or 7 (or both). Then $W(6)$ in

$T-7$, and $W(7)$ in $T-6$ are subsets (of cardinality four) of $W(6)$ in M_6, that is, of $\{13, 24, 35, 41, 52\}$; so they must have at least three common pairs. Suppose for example that T contains 137, 136, 247, 246, 357, 356. Then T cannot contain any of the triples 167, 367, 267, 467, 367, 567, which implies that T has no more than eight triples containing either 6 or 7, whence $|T| \leq 13$, a contradiction. A similar argument applies to any choice of three common pairs, and so the score $[7, 7, 7, 6, 6, 6, 6]$ is not possible. It follows that all $(7, 15)$-configurations are isomorphic to $M_7 := M_6 \cup \{173, 274, 375, 471, 572\}$. □

For $(7, 14)$-configurations, there are eight different isomorphism types, as the following theorem shows.

Theorem 7. *There are eight isomorphism types of* $(7, 14)-$*configurations. Two contain* M_6*; three contain* $M_6 - \{abc\}$ *but no copy of* M_6*; two contain* M_5 *but no* $(6, 9)$*-configuration or* $(6, 10)$*-configuration; and one contains no copy of* M_5*. The eight different types are as follows:*

$$\begin{aligned}
T_1 &= M_7 - \{572\} \text{ (score } [7, 6, 7, 7, 6, 5, 4])) \\
T_2 &= M_6 \cup \{712, 714, 726, 735\} \text{ (score } [7, 7, 6, 6, 6, 6, 4]) \\
T_3 &= (M_6 - \{562\}) \cup \{723, 725, 726, 745, 716\} \text{ (score } [6, 7, 6, 6, 6, 6, 5]) \\
T_4 &= M_6 - \{562\} \cup \{712, 714, 734, 756, 726\} \text{ (score } [7, 6, 6, 7, 5, 6, 5]) \\
T_5 &= M_6 - \{562\} \cup \{712, 716, 724, 745, 756\} \text{ (score } [7, 6, 5, 7, 6, 6, 5]) \\
T_6 &= 68g \cup \{713, 715, 724, 725, 736, 746\} \text{ (score } [6, 6, 6, 6, 6, 6, 6]) \\
T_7 &= 68c \cup \{762, 765, 761, 753, 723, 714\} \text{ (score } [6, 6, 6, 6, 6, 6, 6]) \\
T_8 &= 68d \cup \{167, 267, 467, 237, 257, 457\} \text{ (score } [6, 6, 6, 6, 6, 6, 6])
\end{aligned}$$

Proof. Suppose that T is a (7,14)-configuration. Then $\sum_{i=1}^{7} t_i = 42$, and every $t_i \geq 4$. If some $t_i = 4$, then T contains (a copy of) M_6, and the possible scores are $[7, 7, 6, 6, 6, 6, 4]$ and $[7, 7, 7, 6, 6, 5, 4]$. If every $t_i > 4$ and some $t_i = 5$, then T contains a $(6, 9)$-configuration and the possible scores are $[7, 6, 6, 6, 6, 6, 5]$ and $[7, 7, 6, 6, 6, 5, 5]$. If every $t_i > 5$, then it must be that every $t_i = 6$ (the score is $[6, 6, 6, 6, 6, 6, 6]$); then for $1 \leq i \leq 7$, $T - i$ is a $(6, 8)$-configuration.

Suppose that T has the score $[7, 7, 7, 6, 6, 5, 4]$, with $t_7 = 4$. Since the score for M_6 is $[5, 5, 5, 5, 5, 5]$, then for some $1 \leq i \leq 6$, T contains no triple having both i and 7. In view of the transitivity of $H(M_6)$, we may assume that $i = 6$; then $T-6$ is a $(6, 9)$-configuration containing M_5, and so $W(7)$ is a subset (of cardinality four) of $W(6)$, i.e. T results from removing one of the triples containing 7 from the configuration M_7. This is the first isomorphism type: $T_1 = M_7 - \{572\}$.

Suppose that T has the score $[7, 7, 6, 6, 6, 6, 4]$, with $t_7 = 4$, so that T contains M_6. In view of the transitivity of $H(M_6)$, we may assume that $t_1 = t_2 = 7$, and $t_i = 6$ for $3 \leq i \leq 6$. We adjoin four triples to M_6, say $71a, 71b, 7cd, 7ef$. Since the vertices a, b must be independent in $W(1, M_6)$, then ab can only be one of $24, 26, 56, 53, 43$, and since $t_i = 6$ for $3 \leq i \leq 6$, one of c, d, e, f must be equal to either a or b. We are requiring $t_2 = 7$, so we assume that $a = c = 2$. Then we have either $712, 714, 72d, 7ef$ or $712, 716, 72d, 7ef$. If we have $712, 714, 72d, 7ef$ then d cannot be $4, 3, 5, 1$ so $d = 6$, and then e, f can only be $3, 5$. Similarly, if we have $712, 716, 72d, 7ef$, we find that d must be 4 and e, f must be $3, 5$. The automorphism $(12)(35)$ transforms one of these into the other, so this is the only automorphism type with this score: $T_2 = M_6 \cup \{712, 714, 726, 735\}$.

If every $t_i > 4$ and some $t_i = 5$, then T contains a $(6, 9)$-configuration and the possible scores are (up to isomorphism) $[7, 6, 6, 6, 6, 6, 5]$ and $[7, 7, 6, 6, 6, 5, 5]$. Assume without loss of generality that $u_7 = 5$, and $T - 7 = M_6 - \{562\}$. The score for $T - 7$ is $[5, 4, 5, 5, 4, 4]$, and the corresponding independence numbers are $2, 3, 2, 2, 3, 3$, and so $t_i \leq 7$ for every $1 \leq i \leq 6$.

In $T - 7 = M_6 - \{562\}$, note that $2, 5, 6$ are interchangeable under the action of $H(M_6 - \{562\})$, and so are $1, 3, 4$. We will consider two cases: first, if some $t_i = 7$ for $i = 2, 5, 6$, and second, if $t_i \leq 6$ for $i = 2, 5, 6$.

Without loss of generality, suppose that $u_2 = 7$. There is only one independent set of size three in $W(2, T - 7)$, i.e. $5, 3, 6$; so we adjoin $725, 723, 726$. The only remaining possible triples with 7 are easily seen to be $754, 716$, and the score is $[6, 7, 6, 6, 6, 6, 5]$. If $t_5 = 7$ or $t_6 = 7$, we get an isomorphic configuration. This gives the third isomorphism class:

$T_3 = (M_6 - \{562\}) \cup \{723, 725, 726, 745, 716\}$ (score $[6, 7, 6, 6, 6, 6, 5]$).

Now suppose that $t_i \leq 6$ for $i = 2, 5, 6$. Then two of t_1, t_3, t_4 must be equal to 7, and we may suppose without loss of generality that $t_1 = 7$. The independent pairs in W(1) are: $24, 26, 34, 35, 56$; under the automorphism $(3, 4)(2, 5) \in H(M_6 - \{562\})$, 24 transforms into 35, and 26 into 56; so we need only consider the three possibilities $24, 26, 34$.

Case 1. Add 24 to $W(1)$: adjoin the triples $712, 714$. We have either $t_3 = 7$ or $t_4 = 7$. If $t_3 = 7$, we must adjoin $736, 734$, and then also $t_4 = 7$ and either $t_5 = 4$ or $t_6 = 4$, a contradiction. So $t_4 = 7$; we must adjoin $734, 756, 726$; the resulting configuration is $C_1 = M_6 - \{562\} \cup \{712, 714, 734, 756, 726\}$.

Case 2. Add 26 to $W(1)$: adjoin the triples $712, 716$. Then the only possible triple of the form $73a$ is either 734 or 735, so that $t_3 \leq 6$; then $t_4 = 7$ and we must have $724, 745$. The only remaining possibility is 756; the resulting configuration is $C_2 = M_6 - \{562\} \cup \{712, 716, 724, 745, 756\}$.

Case 3. Add 34 to $W(1)$: adjoin the triples $713, 714$. Either $t_3 = 7$ or $t_4 = 7$, and from the automorphism $(3,4)(2,5)$, it does not matter which one; assume that $t_3 = 7$. Then the only possibility is to adjoin $735, 726, 725$; the resulting configuration is $C_3 = M_6 - \{562\} \cup \{713, 714, 735, 726, 725\}$.

In C_2, there is no triple of the form $a37$; in C_1 and C_3, for every pair xy there is some triple of the form axy; thus C_2 is not isomorphic to either C_1 or C_3. The map $(1,3,4)(5,6,2)$ transforms C_1 into C_3, so they are isomorphic.

Suppose now that T has the score $[6,6,6,6,6,6,6]$ (i.e. T contains no $(6,9)$-configuration or $(6,10)$-configuration).

Suppose first that T does not contain an M_5. Then for every pair $i \neq j$, $1 \leq i, j \leq 7$, there must be precisely two members of T containing the pair ij. To see this: since $t_i = t_j = 6$, the number of members of T containing either i or j is $12 - k$ where $k = k(i,j)$ is the number of members of T containing the pair ij, so the number of members of T not containing either i or j is $k + 2$. If $k(i,j) = 3$, we would have an M_5, so $k(i,j) \leq 2$. Since $\sum k(i,j) = 3 \times 14 = 42$, and there are just 21 pairs in $\mathcal{P}(7)$, it must be that every $k(i,j) = 2$.

Evidently a $(6,8)$-configuration not containing an M_5 must be an extension of one of $68e$, $68f$, or $68g$, and we consider these in turn.

In the configuration $68e$ (respectively $68f$), the pair 25 (resp. the pairs $14, 25, 36$) is not contained in any triple, so in any extension to a 7-configuration, the pair 25 (resp. $14, 25, 36$) would appear at most once. So neither $68e$ nor $68f$ can be extended to a $(7,14)$-configuration not containing M_5.

In the configuration $68g$, there are precisely six pairs appearing once only $(13, 15, 24, 25, 36, 46)$ and all other pairs appear twice. So the only possible extension is to add on $713, 715, 724, 725, 736, 746$; it is easy to check that this gives a $(7,14)$-configuration not containing M_5.

Now suppose that T contains an M_5. Using the same methods as before, we find that any extension of $68a$ or $68b$ to a $(7,14)$-configuration, must contain some $(6,9)$-configuration. For example, to extend $68a$, requiring all $t_i = 6$, we begin by adjoining three triples of the form $76a$; these can only be either $761, 765, 762$ or

761, 765, 764; neither of these allows an extension without a $(6, 9)$-configuration.

Extending $68c$ and requiring all $t_i = 6$, we must begin by adjoining either (i) 762, 765, 761 or (ii) 762, 765, 763 since $68c$ has just three triples containing 6, and the independent sets of size 3 for $W(6)$ in $68c$ are 1, 2, 5 and 3, 2, 5. Proceeding as usual, we find that Case (i) leads to the $(7, 14)$-configuration $68c \cup \{762, 765, 761, 753, 723, 714\}$, and Case (ii) does not admit any extension not containing a $(6, 9)$-configuration.

Extending $68d$ and requiring all $t_i = 6$, we must begin by adjoining either (i) 762, 764, 761 or (ii) 762, 764, 765 (there are other independent sets in $W(6)$, but using the group $H(68d)$, we need only consider two of them); Case (i) leads to the $(7, 14)$-configuration $68d \cup \{167, 267, 467, 237, 257, 457\}$, and Case (ii) does not admit any extension not containing a $(6, 9)$-configuration. □

6 The Case $n = 8$

In this section, we prove that $L(8) = 22$ and characterize the $(8, 22)-$ configurations.

Theorem 8. *$L(8) = 22$, and there are exactly four isomorphism classes of $(8, 22)-$configurations. Three of them contain M_7, namely*

$$\begin{aligned} S_1 &:= M_7 \cup \{158, 168, 178, 238, 268, 278, 358\}, \text{score } [10, 10, 9, 7, 9, 7, 7, 7], \\ S_2 &:= M_7 \cup \{138, 158, 268, 348, 478, 578, 678\}, \text{score } [9, 8, 9, 9, 9, 7, 8, 7], \text{ and} \\ S_3 &:= M_7 \cup \{158, 168, 268, 348, 478, 578, 678\}, \text{score } [9, 8, 8, 9, 9, 8, 8, 7]. \end{aligned}$$

One of them does not, namely

$$\begin{aligned} S_4 := & \bigcup_{i=0}^{6} \{\{1+i, 2+i, 3+i\}, \{1+i, 2+i, 5+i\}\} \\ & \bigcup \{138, 148, 168, 248, 268, 278, 358, 578\}, \text{score } [9, 9, 8, 8, 8, 8, 8, 8]. \end{aligned}$$

where the addition in S_4 is mod 7, and we write 7 instead of 0.

Proof. By Theorem 1, we see that $L(8) \leq 8L(7)/5 = 24$. In any $(8, 24)-$configuration S, we have $u_i \geq 9$ for all i; otherwise, $S - i$ would contain a $(7, 16)-$configuration, which cannot happen. Hence, $S - 8 \cong M_7$; by renumbering if necessary, we may suppose that $t_7 = 5$. In M_7, $W(7)$ consists of the edges 13, 35, 52, 24, 41 and the isolated point 6. Since $I(W(7)) = 3$, it follows that

$u_7 \leq 8$, which contradicts the assumption that an $(8, 24)-$configuration exists. Hence $L(8) \leq 23$.

If S is an $(8, 23)-$configuration, then the previous argument shows that all $u_i \geq 8$, and since $69 = 3|S| = \sum u_i$, S must have a score of $[9, 9, 9, 9, 9, 8, 8, 8]$. Without loss of generality, let $u_8 = 8$, so that $S - 8 \cong M_7$. By relabeling if necessary, we may set $t_6 = t_7 = 5$, recalling that M_7 has score $[7, 7, 7, 7, 7, 5, 5]$. Now in M_7, $I(W(6)) = I(W(7)) = 3$, so that $u_6, u_7 \leq 8$. Since $u_i \geq 8$ we see that $u_6 = u_7 = 8$ and that 6 and 7 are each in three $3-$sets with 8. The only way this can happen is that $678 \in S$ (otherwise we violate the Pigeonhole Principle).

If, say, $178, 278 \in S$, then a study of the $W(i)$ graphs reveals that $128, 168, 138$ and 148 are excluded; the only other possible $3-$set containing both 1 and 8 is 158. But then $u_1 = 8$, contrary to the assumption that $u_1 = 9$. The four other possible pairs of $3-$sets $\{a78, b78\} \in S$, namely $\{278, 378\}, \{378, 478\}, \{478, 578\}$ and $\{578, 178\}$ lead similarly to contradictions. We conclude that $L(8) \leq 22$.

Let S be an $(8, 22)-$configuration, and first suppose that S contains a copy of M_7. Then some $u_i = 7$, and by renumbering if necessary, we may suppose that $u_8 = 7$ and the $W(i)$ graphs in M_7 are as follows:

$$\begin{aligned}
W(1) &= \{23, 25, 36, 37, 45, 46, 47\} \\
W(2) &= \{13, 15, 34, 46, 47, 56, 57\} \\
W(3) &= \{12, 16, 17, 24, 45, 56, 57\} \\
W(4) &= \{15, 16, 17, 23, 26, 27, 35\} \\
W(5) &= \{12, 14, 26, 27, 34, 36, 37\} \\
W(6) &= \{13, 24, 35, 41, 52\} \cup \{7\} \\
W(7) &= \{13, 24, 35, 41, 52\} \cup \{6\}
\end{aligned}$$

It is easy to check that $I(W(i)) = 3$ for all i, and so $u_i \leq 3 + t_i$ for all i; hence, $u_6 \leq 8$, $u_7 \leq 8$ and $u_i \leq 10$ for $1 \leq i \leq 5$. We will appeal to the symmetry of M_7—both $(1, 2, 3, 4, 5)$ and $(6, 7)$ are automorphisms of M_7—to streamline the argument.

First, suppose $u_1 = 10$ (the same argument works for $u_i = 10$ for $2 \leq i \leq 5$). Then S contains 168, 178 and either 128 or 158. Including 158 excludes all other $3-$sets $ab8$ except 238,248,268,278,348 and 358, of which four must be in S. The $W(i)$ graphs reveal that the only way to do this is to include 238, 268, 278 and 358, i.e. if $S = S_1$. A similar argument shows that if $128 \in S$, then $S \cong S_1$.

Next, suppose $u_1 = 9$ and $u_i \leq 9$ for all i (again, the same argument works for

$u_i = 9$ and $2 \le i \le 5$.) There are eight possible pairs of 3−sets $\{1a8, 1b8\} \in S$; by symmetry, the choices are $\{138, 148\}$, $\{138, 158\}$, $\{158, 168\}$, and $\{168, 178\}$.

Including 138 and 148 excludes all 3−sets except 248, 258, 268, 278, 358, 568, 578 and 678; a study of the $W(i)$ graphs reveals that at most four of these can be in S, which would imply that $u_8 \le 6$, contrary to assumption.

Including 138 and 158 excludes all 3−sets except 248, 268, 278, 348, 468, 478, 568, 578 and 678. Again, the $W(i)$ graphs reveal that there are only two ways to include five of them: the five are either $\{268, 348, 478, 578, 678\}$, in which case $S = S_2$, or $\{278, 348, 468, 568, 678\}$, in which case $S \cong S_2$.

Similarly, including 158 and 168 leads, by analyzing the $W(i)$ graphs, to eight possible configurations. Two of these are isomorphic to S_1, one to S_2, and five to S_3.

Finally, including 168 and 178 excludes all but ten triples, at most four of which could be in S, contrary to the assumption that $u_8 = 7$.

We conclude that if S is an $(8, 22)$−configuration which contains a copy of M_7, then S is in one of three isomorphism classes, namely those of S_1, S_2 and S_3 as defined in the statement of the theorem.

Now suppose that S is an $(8, 22)$−configuration which does not contain a copy of M_7. Then $\min(u_i) \ge 8$, and if $\min(u_i) \ge 9$, then $\sum u_i \ge 72 > 66 = 3 \cdot 22$, a contradiction. Hence $\min(u_i) = 8$, so we set $u_8 = 8$ without loss of generality, and see that $T = S - 8$ is one of the eight types of $(7, 14)$−configurations. A tedious but straightforward analysis of the $W(i)$ graphs, as done in the previous cases, reveals the following information. Up to permutation isomorphism: (a) T_1 and T_8 each have only one extension to an $(8, 22)$−configuration, which is isomorphic to S_4; (b) T_2, T_6 and T_7 have no extensions; (c) T_4, T_5 and T_6 each have extensions, but each such extension also contains a copy of T_8 and is isomorphic to S_4. □

7 The Case $n = 9$

In this section, we show that $L(9) = 32$, describe the six different isomorphism types for a $(9, 32)$− configuration, and sketch the main idea for the proof. The argument is similar to that used for $n = 8$, namely analysis of the graphs $W(i)$.

Theorem 9. $L(9) = 32$ *and there are six isomorphism types for a* $(9, 32)-$ *con-*

figuration. These are:

$$\begin{aligned}
R_1 &= S_1 \cup \{239, 259, 369, 379, 459, 469, 479, 589, 689, 789\},\\
R_2 &= S_2 \cup \{179, 189, 239, 289, 369, 459, 469, 489, 579, 679\},\\
R_3 &= S_2 \cup \{179, 189, 239, 279, 289, 369, 459, 469, 489, 679\},\\
R_4 &= S_3 \cup \{129, 169, 289, 379, 389, 459, 479, 569, 589, 679\},\\
R_5 &= S_3 \cup \{129, 169, 279, 289, 379, 389, 459, 569, 589, 679\},\\
R_6 &= S_3 \cup \{179, 189, 239, 279, 289, 369, 459, 469, 489, 679\}.
\end{aligned}$$

R_1 has a score consisting of three 12's and the rest 10's. The others have scores consisting of one 12, four 11's and four 10's.

Proof. By Theorem 1, $L(9) \leq 9L(8)/6 = 33$. Now, in a $(9, 33)-$ configuration we have $\sum u_i = 99$ with each $u_i \geq 11$, so that $u_i = 11$ for all i. Thus, a $(9, 33)-$configuration must contain one of the four $(8, 22)-$configurations S_i. Similarly, in a $(9, 32)-$configuration we have $\sum u_i = 96$ with each $u_i \geq 10$, so that $u_i = 10$ for some i. Thus, a $(9, 32)-$configuration must also contain one of the four S_i.

We will not give all the arguments in detail, since all of them are based on the following idea, which we call "the coloring procedure". (This amounts to successive applications of Lemma 2.) Suppose it is desired to extend an $(8, 22)$-configuration T to a $(9, k)$-configuration U. Let $t_i = t_i(T)$ and $u_i = t_i(U)$. Draw the graphs $W(i)$, and list the maximal independent sets in each $W(i)$. Choose a value of i, $1 \leq i \leq 8$, and a (non-empty) independent set $B = \{a, b, \dots\}$ of vertices in the graph $W(i)$, and let R be the set of vertices of $W(i)$ which are adjacent to some member of B. The vertices in B are colored blue, and the vertices in R are colored red. Then:

1. For each $j \in B$, the vertex i in the graph $W(j)$ is colored blue, and all vertices adjacent to i in $W(j)$, are colored red.

2. For each $j \in R$, the vertex i in the graph $W(j)$ is colored red.

3. For every pair of vertices $a, b \in B$, the vertex a in $W(b)$ is colored red, and the vertex b in $W(a)$ is colored red.

4. Iterate: For any $1 \leq x, y \leq 8$, if vertex x in $W(y)$ is colored red, then also vertex y in $W(x)$ is colored red. Continue until no new vertices can be colored red.

It is possible that some $W(k)$, $1 \leq k \leq 8$, has a non-empty set of independent vertices which have not yet been colored. Then we may color these vertices blue, and repeat the procedure described above. Continuing in this way, at some point we will have all the vertices in the $W(i)$ colored either red or blue, and then there can be no further extension to a configuration. The triples in the configuration which we get, are those of the form $ab9$ where vertex a is colored blue in $W(b)$; if there is a total of $2k$ blue vertices in the graphs $W(i)$, then the configuration we get has k new triples adjoined to T.

We will examine each of the $(8,22)-$configurations and show that none of them can be extended to a $(9,33)-$configuration; since $(9,32)-$configurations exist (e.g., R_1), it will follow that $L(9) = 32$. If U is a $(9,33)-$configuration then $u_9 = 11$ and $U - 9$ is isomorphic to one of the S_i.

For S_1, the vertices 4, 6, 7 and either 3 or 5 form a maximal independent set in $W(8)$. Since $t_8 = 7$, it follows that U must contain 489 and 689. But since $t_4 = 7$, examining $W(4)$ reveals that 469 must be in U. This is a contradiction, so S_1 cannot be extended to a $(9,33)-$configuration.

For S_2, $t_8 = 7$ but $W(8)$ does not contain an independent set of size 4, so $u_8 \leq 10$, which means that S_2 cannot be extended to a $(9,33)-$configuration.

For S_3, $t_8 = 7$ and $W(8)$ does contain an independent set of size 4—namely, $\{1,2,3,7\}$—so that U must contain 189, 289, 389, and 789. Their presence in U eliminates all other $3-$sets containing 9 except for 149, 249, 259, 359, 369, 459, 469 and 569. But examining $W(5)$ shows that at most two of 259, 359, 459 and 569 can be in U, which implies $u_9 \leq 10$. But then U cannot have 33 $3-$sets. Thus, S_3 cannot be extended to a $(9,33)-$configuration.

Finally, for S_4, $t_8 = 8$ and $W(8)$ contains six independent sets xyz of size 3, namely 125, 346, 347, 367, 456 and 457. Examining each of these in turn, as above, reveals that including all of the $3-$sets $x89, y89, z89$ and assuming that $u_i = 11$ for all i leads to a contradiction. That is, S_4 cannot be extended to a $(9,33)-$configuration.

We will go through some of the details for finding the $(9,32)-$configurations which extend S_3, which is both representative and the most interesting of the cases. The other cases are similar in approach but differ in the outcome. As stated in the theorem, S_1 admits one extension, S_2 admits two, S_3 admits three and S_4 admits none.

The $(8,22)$-configuration S_3 is equal to $M_7 \cup \{158, 168, 268, 348, 478, 578, 678\}$. Its score is $[t_1, t_2, \dots] = [9,8,8,9,9,8,8,7]$ and its automorphism group is trivial; for, an automorphism must fix 8, and none of the non-trivial automorphisms of M_7 extend to $W(8)$.

We list first the maximal independent sets for the graphs $W(i)$ of S_3 together with the bounds on the u_i given by $t_i + k_i$ (recall that k_i is the size of the largest independent set). We shall see that in fact several of these cannot be attained, i.e. these bounds are not all exact.

$$\begin{array}{lcl} \text{For } W(1) & : & 384, 627, 657, 428, 287;\ u_1 \le 12 \\ \text{For } W(2) & : & 418, 518, 617, 637, 358, 178, 378;\ u_2 \le 11 \\ \text{For } W(3) & : & 185, 467, 582, 2678;\ u_3 \le 12 \\ \text{For } W(4) & : & 736, 756, 258, 128, 568;\ u_4 \le 12 \\ \text{For } W(5) & : & 238, 167, 467, 248, 468;\ u_5 \le 12 \\ \text{For } W(6) & : & 127, 157, 237, 348, 458, 347, 457;\ u_6 \le 11 \\ \text{For } W(7) & : & 128, 238, 346, 456, 156, 126, 236;\ u_7 \le 11 \\ \text{For } W(8) & : & 1237, 456, 356, 124, 245, 235;\ u_8 \le 11 \end{array}$$

Suppose that U is a $(9, 32)$-configuration containing S_3. As previously noted, every $u_i \ge 10$, and $u_8 = 12$ is impossible (as is $u_3 = 12$ if $u_8 = 11$), so we may assume that $u_3 \le 11$ and $u_8 = 10$. We consider the independent sets of size 3 in $W(8)$; from the coloring procedure we find that just two of them, namely $124, 235$, will allow extensions to a $(9, 32)$-configuration, and each of these leads to three possibilities.

Using 235 for $W(8)$ — i.e. we adjoin triples $982, 983, 985$ to S_3 initially — determines the set of triples $X = \{982, 983, 985, 945, 965, 937, 967, 916, 912\}$, and leaves a choice of one of 947, 924, or 927. Using 124 for $W(8)$ — we adjoin triples $981, 982, 984$ to S_3 initially — determines the set of triples $Y = \{981, 982, 984, 945, 946, 967, 963, 923, 917\}$, and leaves a choice of one of 957, 925, or 927. These result in the following six possibilities:

$$\begin{aligned} U_1 &= S_3 \cup X \cup \{947\} \\ U_2 &= S_3 \cup X \cup \{942\} \\ U_3 &= S_3 \cup X \cup \{927\} \\ V_1 &= S_3 \cup Y \cup \{957\} \\ V_2 &= S_3 \cup Y \cup \{952\} \\ V_3 &= S_3 \cup Y \cup \{927\} \end{aligned}$$

Now $(1, 5, 4, 3, 2)(6, 7)(8, 9)$ is an isomorphism from U_2 to R_2, $(1, 2, 8, 4, 5, 6, 7)(3, 9)$ is an isomorphism from V_1 to U_3, and $(1, 2)(3, 5)(6, 7)(8, 9)$ is an isomorphism from

V_2 to R_3. It remains to show that $R_4 := U_1$, $R_5 := U_3$, and $R_6 := V_3$ are not isomorphic to each other or to any of R_1, R_2, R_3.

In each configuration, we color the vertices of the graphs $W(i)$ as follows: i is red if $u_i = 11$, i is green if $u_i = 10$, and i is yellow if $u_i = 12$. The degree sequence of $W(i)$ is $(n_0, n_1, n_2, \dots)$, where n_j is the number of vertices of degree j in $W(i)$. An isomorphism from configuration C to configuration D must preserve degree sequences, and if $W(i, C)$ is assigned to $W(m, D)$, then $W(i, C)$ and $W(m, D)$ must have not only the same degree sequence, but the same color pattern.

In V_3 there are four $W(i)$ with degree sequence $(0, 0, 2, 6)$, and none of the other configurations has more than two $W(i)$ with this degree sequence.

While U_1 and R_2 have the same set of degree-sequences, they differ in color patterns: In U_1, $W(2)$ and $W(3)$ have degree sequence $(0, 0, 4, 4)$, and the four vertices of degree 3 are colored red, red, red, yellow. In R_2, every $W(i)$ with degree sequence $(0, 0, 4, 4)$ has the four vertices of degree 3 colored red, red, green, yellow.

While U_3 and R_3 have the same set of degree-sequences, they differ in color patterns: In U_3, $W(2)$ and $W(7)$ are the only ones with degree sequence $(0, 0, 2, 6)$ and the two vertices of degree 2 are colored green, green in $W(2)$ and red, red in $W(7)$. In $R(3)$, $W(2)$ and $W(7)$ are the only ones with degree sequence $(0, 0, 2, 6)$, but in each one the two vertices of degree 2 are colored red, green.

Since R_1, R_2, R_3 all have different degree sequences, this accounts for all possibilities. Similarly, one verifies that R_4, R_5 and R_6 are the only other possible isomorphism types for a $(9, 32)-$configuration. This completes the proof of Theorem 9.

□

8 Questions for Further Study

In this paper, we have determined $L(n)$ for $n \leq 4 \leq 9$; the work has raised several questions which point the way to further research:

- Is the bound on $L(n)$ given by Theorem 1 sharp—i.e., is

$$L(n) = \frac{n}{n-3} \cdot L(n-1)$$

  for infinitely many values of n?

- Find a good estimate for the number of isomorphism types of $[r, s, n, t]-$configurations of a given size — in particular, for $[3, 4, n, 2]-$configurations of size $L(n)$.

- Is there a way to compute $L(n)$ without checking the extendibility of a large number of $[3, 4, n-1, 2]-$configurations?

- Are there special values of k for which there are systematic ways of constructing $[r, s, n, t]-$configurations of size k — e.g., the circular construction of M_5?

- What can be said about the maximum size of an $[r, s, n, t]-$configuration for general r, s, n and t?

References

[1] Bjorn Poonen, Union-Closed Families, *J. Combinatorial Theory Series A* **59** (1992), 253–268.