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The Ancient World's Magical Genius Thinks BIG

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For some historical figures, legend blends into fact almost from the beginning. So it was with antiquity's greatest scientific mind. They say that he discovered laws of physics while lying in a bathtub, set sailing vessels on fire with mirrors, and drew a ship out of the water with his bare hands. Maybe he did—and maybe he didn't. What is certain is that he described the mathematics of balance and the lever, for which moving the earth with a stick would be a "thought experiment." What is certain is that he described a way to deal with large numbers, and as an example he counted the number of grains of sand in the universe—two different universes, in fact. What is certain is that he described a herd of cattle that satisfied nine "simple" conditions, and this herd turned out to be very, very large. In short, he thought big. Some of the threads of his work lead all the way to the present. In this story, we will learn something of his life and just exactly how big he thought. All you need to know is his name: Archimedes.

The Life and Death of Archimedes

Facts about the life of Archimedes are sketchy, but he certainly did live in interesting times. He was born in the city of Syracuse (Siracusa) on the island of Sicily ca. 287 BCE, the son of Phidias the Astronomer. This was a decade or so after Ptolemy I Soter of Egypt established the Museum, a center of learning at Alexandria, and its Library. Euclid's arrival to set up a school of mathematics, and the subsequent appearance of Euclid's *Elements*, changed mathematics forever.

As a youth, Archimedes probably traveled to Alexandria to study, very likely with Euclid's successors. He then settled in Syracuse, where he did most of his work. He may have been related to Hieron, the ruler of that city. We know about his many remarkable achievements through Arabic, Greek, and Latin copies of his letters and other works. Unfortunately, none of his original writings survive.

His life would have drawn to an uneventful close, were it not for the fact that in 214 BCE, Syracuse, previously neutral in the conflict between Rome and Carthage, sided with Carthage. Led by the general Marcellus, Rome besieged Syracuse for three years. Archimedes directed the Syracusan resistance to the siege, about which many Archimedean legends arose, including the business about setting ships on fire with mirrors. The siege was broken and the city defeated in 212 BCE.

In his *Lives of the Noble Grecians and Romans*, Plutarch gives three different accounts of Archimedes' death. The most striking one has Marcellus breaking the siege and sacking the city. During the subsequent chaos, a Roman soldier encounters Archimedes sitting in the street, studying a problem in geometry. The soldier orders him to move. When he does not move, the soldier kills him. Other accounts vary, but they mostly agree on one point: Archimedes met his death during the siege of Syracuse.

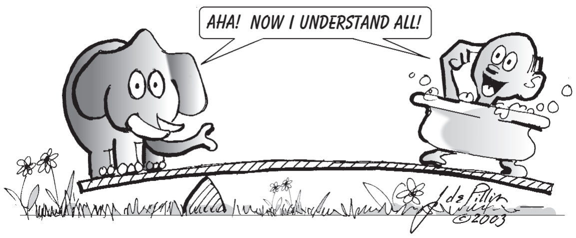
Thus ended the life of antiquity's finest mind; the Western World would not see his equal in intellect until the Renaissance. Alfred North Whitehead had this to say about the death of Archimedes:

> The death of Archimedes at the hands of a Roman soldier is symbolic of a world change of the first magnitude. The Romans were a great race, but they were cursed with the sterility which waits upon practicality. They were not dreamers enough to arrive at new points of view, which could give more fundamental control over the forces of nature. No Roman lost his life because he was absorbed in the contemplation of a mathematical diagram.

The Works of Archimedes

Archimedes' fame rests, in the final balance, on a handful of works preserved through translators over the last two millenia:

- *On the Equilibrium of Planes*, a two-part work, analyzes problems in mechanics, including the Principle of the Lever. Archimedes' knowledge of, and ability to apply, the basic principles of statics and hydrostatics were profound. In particular, he is supposed to have said that if he could find another world, and go to it, he could move



this one. As Pappus quotes Archimedes: "Give me a lever long enough and a place to stand, and I will move the world." We'll take Archimedes at his word and look at the ramifications of this statement later.

- *On the Quadrature of the Parabola* determines, with proof, the area bounded by a parabola and a line.
- *On the Method of Mechanical Theorems* is a letter to his friend Eratosthenes, explaining how he applies principles of statics to problems in geometry ("How I Did It"). The *Method* was lost for centuries, and how it was preserved, found, and brought to light is a great story in itself.
- *On the Sphere and the Cylinder* determines the 3:2 ratio of the volumes of a sphere to its circumscribing cylinder; the figure appears on his gravestone. The Roman statesman Cicero (*Tusculan Disputations*, Book I, Section II(5)) describes the rediscovery and restoration of this gravestone; the gravesite was overgrown and, like much of ancient mathematics at the time of the Romans, neglected and all but forgotten. In his book *Calculus Gems*, George Simmons notes that "The Romans were so uninterested in mathematics that Cicero's act of respect in cleaning up Archimedes' grave was perhaps the most memorable contribution of any Roman to the history of mathematics." Cicero ruefully admitted as much, writing that "Among them [the Greeks] geometry was held in highest honor; nothing was more glorious than mathematics. But we [the Romans] have limited the usefulness of this art to measuring and calculating."
- *On Spirals* determines, with proof, the area swept out by a spiral.
- *On Conoids and Spheroids* determines, with proof, various volumes bounded by cones, spheres, and related surfaces.
- *On Floating Bodies*, another two-part work, analyzes hydrostatical problems in mechanics, including the Principles of Buoyancy: A floating body displaces its weight, a submerged body displaces its volume.
- *On the Measurement of a Circle* contains Archimedes' celebrated proof that the area of a circle is equal to half the product of its radius with its circumference, as well as the determination that $3\ 1/7 < \pi < 3\ 10/71$.
- *The Sandreckoner*: establishes a system for expressing very large numbers—about more of which we discuss later.
- *The Cattle Problem*: a problem involving those very large numbers; we'll talk about this curious herd of cattle later.

Many of Archimedes' proofs use a technique known as Eudoxus' Method of Exhaustion in a way that is very close to calculus. However, calculus—due jointly to Newton and Leibniz—does not appear for another 1900 years.

How Far Away Did Archimedes Stand?

What if we take Archimedes at his word when he says, "Give me a lever long enough and a place to stand, and I will move the world." Let's do just that, ignoring physical limitations such as the inability to breathe in space, or to travel vast distances in the universe, or to manufacture a lever of the required length and stiffness.

What if he can do this? Where will he rest his lever, and how long would it be? We can answer this by appealing to his own Principle of the Lever. First, we define the *moment* of an object about a point to be equal to its mass times its *directed* distance to the point. Archimedes' Principle of the Lever is that a mechanical system is balanced about a point Q exactly when the sum of the moments about Q is zero. For example, 80-pound Ed and 60-pound Sal want to balance each other on a seesaw; the sum of their moments will be $80 \cdot d_E + 60 \cdot d_S$, where d_E and d_S are their respective directed distances to the balance point. A short calculation reveals that $d_S = -4/3\ d_E$. Thus, Sal should sit 4/3 as far from the balance point as Ed, and on the opposite side. (In practice, they figure this out by trial and error.)

And now, let's move the earth.

A convenient fulcrum, or balance point, would be the moon. Let m_E, d_E, m_A, and d_A be the earth's mass and distance and Archimedes' mass and distance, respectively. We know Archimedes and the earth will be on opposite sides of the moon, so we let d_A be positive; the moment equation becomes $m_E d_E = m_A d_A$.

In round figures, $m_E \sim 1.32 \times 10^{25}$ pounds, and d_E is roughly 240,000 miles from the earth to the moon. We don't have any idea how much Archimedes weighed, so let's guess $m_A = 150$ pounds. Working it out, we find that

$$d_A = m_E d_E / m_A \sim 2.1 \times 10^{28} \text{ miles, or } 3.6 \times 10^{15} \text{ light-years,}$$

or about 3.4×10^{31} meters.

How far is this, really? Only about 100,000 times the radius of our known universe!

Question 1. How far away would Archimedes have to stand in order to move the moon, using the Earth as a fulcrum? (You will have to look up the mass of the moon.)

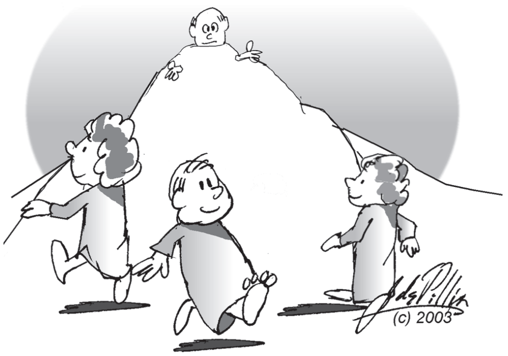
Question 2. If we placed the moon and the Earth on a 240,000 mile seesaw, where would the balance point be?

And now, let's fill the universe with sand.

How Many Grains of Sand Fill the Universe?

In *The Sandreckoner*, Archimedes describes a way to express very large numbers by means of myriads, periods, and octads. As an illustrative example, he calculates the number of grains of sand it would take to fill the universe. Here are Archimedes' assumptions:

- The volume of the universe is roughly the cube of its diameter.
- It takes about 10^4 (one myriad) grains of sand to equal one poppy seed in volume.
- A row of 40 poppy seeds in a row is about one finger-breadth.
- About 10^4 finger-breadths equal one *stadion*, an ancient unit of measure. The *stadion* was a unit of length anywhere from 500 to 700 feet. Since 10,000 of my finger-breadths come to about 625 feet, this is a fair assumption.
- The diameter of the earth is less than 10^6 *stadia*. This is generous, given that the earth is not quite 8000 miles in diameter, or about 75,000 *stadia*.
- Archimedes' universe was geocentric, with the earth at the center and the "sphere" of the sun and the fixed stars at its outer limits. He assumed that the diameter of this universe is less than a myriad earth diameters. This works out to be at most $10^4 \times 10^6 = 10^{10}$ *stadia*, or $10^{10} \times 10^4 = 10^{14}$ finger-breadths, or $10^{14} \times 40 \sim 4 \times 10^{15}$ poppy seed lengths.



Finally, with these assumptions Archimedes calculates the number of grains of sand to fill the universe to be about 10^4 grains per seed $\times (4 \times 10^{15})^3$, or about 10^{51} grains. But wait—it gets better.

In *The Sandreckoner*, Archimedes fills another much larger universe with sand. Namely, the universe of Aristarchus of Samos, who lived a century before Archimedes and placed the sun at the center of the universe.

Using (but not endorsing) Aristarchus' heliocentric Universe, Archimedes states that

$$\frac{\text{diameter of earth}}{\text{earth–sun distance}} = \frac{\text{earth–sun distance}}{\text{distance to the fixed stars}}$$

The consequences of this assertion are that the diameter of Aristarchus' universe is approximately 10^4 earth–sun distances, and that the volume of his universe is about 10^{12} times the volume of the geocentric universe. As a result, it would take about $10^{51} \times 10^{12}$, or about 10^{63} grains of sand to fill Aristarchus' universe.

Question 3. How does the number of arrangements of an ordinary deck of 52 playing cards compare with the number of grains of sand filling either of Archimedes' two universes?

Question 4. How many neutrons (radius approximately $1/10^{15}$ meters) would it take to fill the visible universe as we know it (radius 30 billion light-years)?

How Many Cattle?

The previous numbers are indeed large. But those are just warm ups for Archimedes' main event. In a letter to his friend Eratosthenes he posed the problem of finding the size of a certain herd of cattle, whose size pitifully dwarfs all of the previous numbers. This document was translated from a Latin source into German by Lessing in the eighteenth century; its beginning and end follow.

> If thou art diligent and wise, O stranger, compute the number of cattle of the Sun, who once upon a time grazed on the fields of the Thrinacian isle of Sicily, divided into four herds of different colours, one milk white, another a glossy black, a third yellow and the last dappled…If thou art able, O stranger, to find out all these things and gather them together in your mind, giving all the relations, thou shalt depart crowned with glory and knowing that thou hast been adjudged perfect in this species of wisdom!

Here are the particulars. Archimedes asks us to find the numbers W, X, Y, and Z of white, black, yellow, and dappled bulls, and the numbers w, x, y, and z of white, black, yellow, and dappled cows, subject to the following nine conditions:

1. $W = (1/2 + 1/3)X + Z$,

2. $X = (1/4 + 1/5)Y + Z$,
3. $Y = (1/6 + 1/7)W + Z$,
4. $w = (1/3 + 1/4)(X + x)$,
5. $x = (1/4 + 1/5)(Y + y)$,
6. $y = (1/5 + 1/6)(Z + z)$,
7. $z = (1/6 + 1/7)(W + w)$,
8. $W + X$ is a square, and
9. $Y + Z$ is a triangular number.

This problem has fascinated many mathematicians for the past two hundred years, and there are a number of accounts of its solution. The conditions amount to nine equations in eight unknowns, so that it is not obvious that a solution even exists. But it does, and we can do it in stages.

First, we notice that equations 1 through 7 are linear in the eight variables W, X, Y, Z, w, x, y, and z. Transposing the variables, we see that we have a homogeneous linear system with one fewer equation than unknown, so that there are infinitely many solutions. To solve this by hand is tedious but straightforward; with a computer algebra system, finding the solution takes much less time than entering the equations!

It turns out that the first seven variables are rational multiples of z with common denominator 5439213; if we put $z = 5439213v$, we obtain integer values for all eight variables, namely

$$W = 10366482v,\ X = 7460514v,\ Y = 7358060v,\ Z = 4149387v,$$
$$w = 7206360v,\ x = 4893246v,\ y = 3515820v,\ z = 5439213v,$$

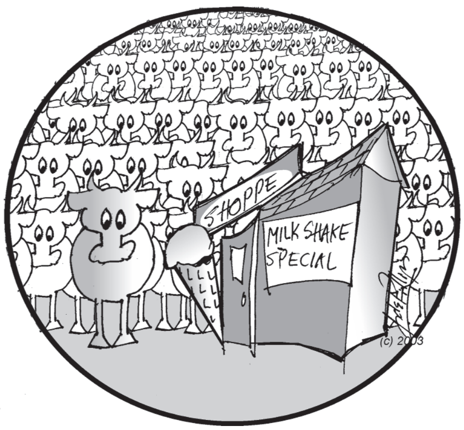
where v is an integer-valued parameter.

If you want to keep score, Archimedes now has about 50 million head of cattle.

To satisfy condition 8, we want $W + X = 17826996v$ to be a square. Since $17826996 = 4 \cdot 4456749$, with the latter factor square free, this will occur if we put $v = 4456749s^2$. This yields $W = 46200808287018s^2$ with similarly magnified values for the other seven variables. At this point, with about 225 trillion head of cattle in the herd, Archimedes has clearly overrun the planet—and he still must satisfy condition 9, namely that $Y + Z$ must be triangular.

Now, the triangular numbers are 1, 3, 6, 10, 15, … and have the general form $n(n + 1)/2$, so this means that $Y + Z = 51285802909803s^2 = n(n + 1)/2$ for some integer n. Multiplying this equation by 8 and adding 1 yields the equation $410286423278424s^2 + 1 = (2n + 1)^2$. So, if we set $t = 2n + 1$, we conclude that satisfying conditions 1–9 amounts to solving the equation $t^2 - 410286423278424s^2 = 1$, for nonzero integers s and t.

It is apparent madness to prove that s exists, let alone ever find it. But it does, and we can, by means of continued fractions. In fact, if d is a positive nonsquare integer, then the equation $t^2 - ds^2 = 1$ always has a solution for which $s \neq 0$—in fact, infinitely many such solutions.



Cartoons by John de Pillis

The bottom line is that the total number of cattle satisfying all conditions is about 7.7×10^{206544}. This pitifully dwarfs the number of neutrons filling the visible universe!

Question 5. How large would a spherical universe have to be in order to hold the entire herd of Archimedes' cattle?

Question 6. How many grains of sand would fill a full-grown cow?

How Big Did Archimedes REALLY Think?

Expressing huge numbers using Greek numerals (similar to Roman numerals) was unwieldy and complicated. As we have seen, in *The Sandreckoner*, Archimedes described a concise system similar to our use of powers—the system of periods, orders, and *octads*—for creating and expressing very large numbers. The largest of these is the myriad-myriadth order of the myriad-myriadth period, which works out to $10^{80,000,000,000,000,000}$, or 1 followed by eighty quadrillion zeros. So, Archimedes really did think big!

Further Reading

See *A Contextual History of Mathematics*, Ronald Calinger, pp. 124–130, for some of the details surrounding the founding of the Museum and the establishment of Alexandria as the principal cultural/economic center of the Mediterranean world. You can find a detailed description of how to use continued fractions to find the solution to the Cattle Problem in "Three connections to continued fractions," *Pi Mu Epsilon Journal*, 2002, pp.353–362. Another approach is taken in "Archimedes' Cattle Problem" by Ilan Vardi in *The American Mathematical Monthly*, 1998, pp. 305–319. To learn more about Archimedes' life and work, a good place to start is Sherman Stein's book *Archimedes: What Did He Do Besides Cry Eureka?*, published by the MAA. Finally, an excellent source on the web is `http://www.mcs.drexel.edu/~crorres/Archimedes/contents.html`.