A Conversation with Archimedes

Ezra Brown

Cartoon by John Johnson

In my history of mathematics class, we were studying Archimedes. The discussions were often intense, and today's class meeting—concerning his method of dividing a sphere into segments—was no exception. At the end of class I told them, "Next time, we'll talk about the thirteen Archimedean solids."

I went to my office and unlocked the door. I was startled to see a stranger standing there, staring intently at a colorful poster depicting several families of polyhedra: the Platonic and Archimedean Solids, the Prisms and Anti-Prisms, and the Johnson Solids. The stranger was oddly dressed, and I wondered how he had gotten into my office.

The Stranger: This is a beautiful poster, Friend. Did you construct it?

Math Horizons: No, it was a present from a colleague.

The Stranger: A wise colleague, I'm sure.

MH: I'll tell her you said so; and are you, by any chance, Archimedes of Syracuse?

A: Yes, I am.

MH: Sir, you are generally regarded as the greatest scientific mind of antiquity. May I ask you some questions?

A: Why, certainly.

MH: First of all, what does the name "Archimedes" mean?

A: It comes from *arche*, meaning "rule," "principle," or "first," and *medos*, meaning "mind" or "wisdom." In my day "principle" meant something like "foundation of the universe." Read it as you do now and it means something like "leading mind"; read it as we Greeks frequently did, and it means "mind of the principle." Take your pick.

MH: And you lived in Syracuse?

A: I was born in Syracuse, which was, and still is, a port city located in southern Sicily. The rulers of Syracuse were called Tyrants, and by the modern sense of the word, some were and some were not. I was a youth when Hieron II began his rule (by your reckoning, that was the year 275 B.C.E), and my father was Hieron's court astronomer. They say I was quick to learn to read and write, and even quicker at counting. The fifty years of Hieron's tyranny were years of peace, and Friend, that was important.

MH: Why so, Archimedes?

A: Think about it. Fifty years of peace and prosperity under one enlightened ruler meant years of support for artists, musicians, educators, and thinkers. It was my good fortune to live in those times, and to have Heiron's support. He provided for me so that I could spend my time thinking about mathematics and mechanics.

MH: How were you educated?

A: Tutors and family members, at first; then—when I was old enough to travel—it was off to Alexandria, to study with the Masters.

MH: Did you study with Euclid?

A: Pretty idea, Friend, but he was decades before my time. The Musaeum, including the Library, was a remarkable institution, founded about the time I was born by the equally remarkable Egyptian ruler Ptolemy Soter. He brought all the great thinkers to Alexandria. During my stay, I learned from Euclid's "Elements" all of the mathematics in the Western World: geometry, triangles, Eudoxus' method of exhaustions, numbers, proportions, your Platonic solids up there on the wall…and especially quadrature—how to describe curved regions or solids as regions or solids bounded by lines or planes. I took a special interest in quadrature, particularly those vexing compass-and-straightedge problems: the Duplication of the Cube, Trisection of the Angle, Quadrature of the Circle. Maddening!

MH: But you showed how to trisect a general angle with a compass and a marked straightedge, as well as with your spirals, didn't you?

A: Yes, that's right. I loved those spirals…such beautiful figures.

MH: I'll say.

A: The importance of construction problems might be overlooked in your time, but let me tell you, Friend, that those three problems began a mathematical tradition that has persisted for two-and-a-half millennia; namely, posing a specific question to be answered in a specific way. Construct a cube equal to twice a given cube using only compass and straightedge. Find a solution to the general cubic equation that uses algebra and not geometry. Construct a proof of the Prime Number Theorem that does not use complex analysis. Find a primality test that is deterministic and runs in polynomial time—a recent remarkable achievement, I might add. It all began with those construction problems.

MH: Why were your writings practically forgotten, while Euclid's "Elements" achieved enormous popularity in every corner of the scientific world? The "Elements" has gone through more than a thousand printed editions!

A: At the time, the "Elements" contained all of the mathematics anyone needed to know in order to be an educated person. It was beautifully written, and you could teach yourself from it. You still can. Newton supposedly found it "wanting," but his teacher Isaac Barrow pointed out that to understand Descartes properly, Newton should go back and master Euclid—and so Newton did. You see, Euclid wrote a textbook. I was interested in answering more advanced questions in mathematics and mechanics, so I wrote for myself and for a few colleagues. To be frank, it's amazing that any of it has survived.

MH: OK, so you know this question is coming: "The Crown and the Bath," tell me about that.

> **Then one day, I went to my workshop and sliced off all eight corners of a Platonic cube. With care, this truncated cube would be a new kind of solid with 24 corners, 36 edges, six regular octagonal faces, and eight equilateral triangular faces.**

A: Oh, *that* story! Hieron had commissioned a goldsmith to make a crown—it looked more like a wreath, actually—and gave him the gold to make it. For some reason, he suspected that the goldsmith kept some of the gold and made the crown out of an alloy of gold and some base metals. He asked me to figure out how to determine the composition of the crown without melting it down. That was the problem.

MH: What about shouting, "Eureka!" and running naked through the streets when you figured out a solution?

A: Pure silliness. Public baths were places you went to relax, to be sociable, and sometimes just to sit and think. Look, everyone notices that when you get into a bath the water level rises. Haven't you?

MH: Well, yes, but…

A: So I got in the bath and displaced some water. For me, the key question was not "Why do I float?" but "What holds me up?" The water I displaced was held in place by the upward force of the water around it. The displaced water weighed more than me, so my downward force on the water below was less than the upward force from the water. Consequently, I floated. Similarly, a hollow metal vessel is mostly air, and so it is also less dense than water. The volume of water it displaces weighs more than it does, and again, the force of the water holds the vessel up. That's buoyancy, don't you see?

MH: So what does this have to do with the crown?

A: The point is that while I was thinking about the crown problem, I made this much more important observation about floating bodies. Of course I was happy. Of course I shouted "Eureka!" And of course I did *not* run naked screaming through the streets of Syracuse; that would reflect poorly on the royal house of Hieron and embarrass my family.

MH: So could you have solved the problem of the crown without that insight?

A: Certainly, Friend. In my day, it was known that volumes of objects of equal mass vary inversely as their densities. A crown of gold is very dense, and a crown of the same weight made of an alloy of gold and baser metal would be less dense. The latter displaces more water than the gold crown, and a *very* careful experiment reveals this difference. But I didn't even need to do that.

MH: You didn't need to immerse the crown in water?

A: Not at all. I told a metal worker about the crown, and he said, "A crown of pure gold? You jest, Archimedes! Try bending Hieron's crown…gently, of course! If it bends easily, it's gold. If not, it's an alloy." I took his advice and tested the crown. It strongly resisted being bent, and so it was not pure gold.

MH: But why…

A: …the story? I really don't know. The earliest source is Vitruvius, the Roman author who wrote about it two hundred years later. Was he there? No. Similar stories abound about Michelangelo, Newton, Galileo, Gauss, Mozart, Beethoven, Hilbert, Van Gogh, and Einstein. Elvis, too: a very polite young man.

MH: Let's go back to the Archimedean solids. According to Pappus, you were the one who first described the semi-regular polyhedra that bear your name; is that right?

A: I wrote a manuscript about them, but was I the first? Who knows? I studied the Platonic solids, and I learned that Plato knew about the cuboctahedron, the one with six square and eight triangular faces.

MH: But Plato mostly focused on solids whose faces were one kind of regular polygon.

A: Exactly. And he found them all, so it was clear that he was thinking about the next thing. Then one day, I went to my workshop and sliced off all eight corners of a Platonic cube. With care, this truncated cube would be a new kind of solid with 24 corners, 36 edges, six regular octagonal faces, and eight equilateral triangular faces. Two octagons and a triangle would meet at each corner. With a little more slicing, the triangles expanded, the octagons became squares, and I constructed the cuboctahedron: the one Plato knew about. This was really interesting! Now I got serious about it and was able to work out the shapes of all possible polyhedra with regular polygonal faces having the same configuration of faces at each corner, and having more than one kind of face.

MH: You make no mention of them in any of your extant works; why is that?

A: Because it was practically my last investigation. War came to Syracuse and my job was to help with the defense of the city during the Roman siege. In your poster, Johnson solid J37, the elongated square gyrobicupola—a wonderful name!—is obtained from my rhombocuboctahedron by rotating the cap of four equilateral triangles and five squares one-eighth of a full turn. Three squares and one triangle meet at each corner, so technically, there should be *fourteen* Archimedean solids, not thirteen. I can fix that.

MH: Big question: did you invent calculus?

A: Me? Certainly not. It is true that some of my methods foreshadow calculus. For example, in "On Spirals" I found the area swept out by the spiral $r = a\theta$ in one complete rotation to be one-third the area of the bounding circle. To do this, I sliced the bounding circle into an equal number of sectors…

MH: Like a pizza?

A: Exactly. I approximated each segment of the spiral by a segment of a circle and added up the areas two different ways, one using the shortest ray inside the spiral segment and one using the longest such ray. I guess you would say I used polar coordinates and used upper and lower sums to bound the integral of an increasing function, but that's not how I thought of it. In "Quadrature of the Parabola" I showed that the area of a parabolic segment is equal to four-thirds the area of a special triangle inscribed in that segment. That proof involved showing that a sum of areas of certain triangles nested within the parabolic segment is equal to a particular geometric sum.

Solving all these problems boiled down to cleverly using Eudoxus' method of exhaustions because the idea of a limit had not occurred to us. All of the computations involving curvilinear surfaces and regions were my crude attempts at what eventually became the integral calculus.

MH: Which of your many achievements do you consider your best work?

A: Which of your children is your "best work"? I loved all of my works: On Floating Bodies, On Balancing Planes, Quadrature of the Parabola, On Spiral Lines, On Spheroids and Conoids, Measurement of a Circle, The Sand Reckoner, and the Cattle Problem. There are the works you don't know about, but I'm not allowed to talk about them. Peculiar rules here, you know. But my favorite is The Sphere and Cylinder, in which I use the method of exhaustions to prove that the ratio of the volume of a sphere to the volume of its circumscribing cylinder is 2/3. That's why I gave instructions that it be inscribed on my gravestone. And Cicero found it many years later.

Now, I'm afraid my time here is short, Friend, and we haven't even talked about the Method of Mechanical Theorems, or the Stomachion, or several other works. You just might find me in your office again some time.

MH: I look forward to that, Sir. Any last words of wisdom for my students?

A: Read the Masters, never lose your sense of wonder, and enjoy the beauty of mathematics.

And he disappeared.

A moment later, I looked up at my wall and stared quizzically at the poster of polyhedra. There were now *fourteen* Archimedean Solids, and the space formerly occupied by the Johnson Solid J37 was blank!

Well, he *said* he could fix it, didn't he?

Further Reading

A wonderful place to start is Reviel Netz and William Noel's *The Archimedes Codex* (Da Capo Press, 2007). Their list of Further Reading will take you far. As for textbooks, Burton's *The History of Mathematics: An Introduction* (sixth edition: McGraw-Hill, 2007) includes an introduction to Archimedes' work. Finally, you can see the Polyhedra poster at www.peda.com/posters.

About the author: *Ezra "Bud" Brown is Alumni Distinguished Professor of Mathematics at Virginia Tech. An accomplished mathematician, musician, and author, he enjoys gardening, traveling, professing, music, and the occasional encounter with a master mathematician from the past.*

email: ezbrown@math.vt.edu

DOI: 10.4169/194762110X495380