Trigonometry *without* Triangles

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Suppose you know everything there is to know about calculus: You understand the Fundamental Theorem of Calculus, the Chain Rule, the Mean Value Theorem, implicit differentiation, the whole lot. But mysteriously, every trace of trigonometry has been erased from your memory. You're in luck though, because it is a little-known fact that with the clever use of calculus you can re-create the whole subject of trigonometry from the formula for the arc length of a circle, without the use of any triangles whatsoever—particularly ironic, considering that the word "trigonometry" literally means "triangle measuring"!

Another fact that deserves to be better known is that all trigonometric functions are ultimately based on the sine function:

$$\cos u = \sqrt{1-\sin^2 u},$$

$$\tan u = \frac{\sin u}{\sqrt{1-\sin^2 u}},$$

$$\sec u = \frac{1}{\sqrt{1-\sin^2 u}},$$

and so on. Thus, if you can deduce all the key properties of the sine function, pretty much all of trigonometry will follow from these foundations.

There are several ways to define the sine, of which the schoolbook "opposite over hypotenuse" version is, in the end, the least useful. One way is as the infinite power series

$$u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \cdots;$$

another is as the infinite product

$$u\left(1-\frac{u^2}{\pi^2}\right)\left(1-\frac{u^2}{2^2\pi^2}\right)\left(1-\frac{u^2}{3^2\pi^2}\right)\cdots.$$

> Of course the sine will forever be linked to its triangular heritage, which makes it all the more interesting that it is possible to define, develop, and apply trigonometry in all its abundant glory—with nary a triangle in sight!

The sine can also be defined as

$$\frac{1}{2i}\left(e^{iu} - e^{-iu}\right),$$

as well as a solution to the initial-value problem

$$\frac{d^2y}{du^2} + y = 0, \quad y(0) = 0, y'(0) = 1.$$

All these starting points have their advantages and disadvantages—each is elegant in its own way and brings a different characteristic of the sine function to the fore. Yet to establish one of the most fundamental properties of the sine function—its periodicity—is not at all straightforward under any of these definitions. By way of contrast, we will begin by constructing a new function, $s(u)$, from scratch and derive three fundamental properties that reveal it to be none other than our old friend $\sin(u)$. We will show that our function $s(u)$ is odd ($s(-u) = -s(u)$) and periodic. But the main advantage of our approach is the deduction of the addition formula

$$s(u+v) = s(u)s'(v) + s'(u)s(v).$$

This turns out to be the familiar addition formula for the sine function in disguise:

$$\sin(u+v) = \sin u \cos v + \cos u \sin v.$$

Let's get started. We'll work backwards, starting with the addition formula, which, in common with everything else in this article, will be derived without the use of a single triangle.

Defining the Function *s(u)*

Let s satisfy $-1 \le s \le 1$. The length of the arc along the unit circle between the point $P = (0,1)$ and the point $Q = (s, \sqrt{1-s^2})$ (see figure 1) is given by

$$u = \int_0^s \frac{1}{\sqrt{1-x^2}}\, dx.$$

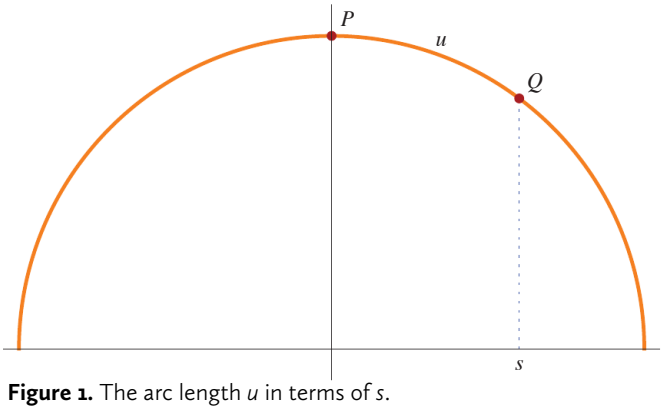

Figure 1. The arc length u in terms of s.

So, for example, if $s = 1$, then this integral gives the length of a quarter of the circle's circumference, or

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}.$$

For values of s satisfying $-1 \le s < 0$, the top limit of integration is smaller than the bottom limit, and the arc length is computed with a negative sign attached. The result is that u is an increasing, one-to-one function of s on the interval $-1 \le s \le 1$ with range $[-\pi/2, \pi/2]$. This implies that u is invertible. In other words, we may write $s = s(u)$ as a function of u, with $u \in [-\pi/2, \pi/2]$, where $s(-\pi/2) = -1$, $s(0) = 0$, and $s(\pi/2) = 1$. And this turns out to be the "right" point of view.

Now by the Fundamental Theorem of Calculus,

$$\frac{du}{ds} = \frac{1}{\sqrt{1-s^2(u)}},$$

and the Inverse Function Theorem gives

$$(1) \qquad s'(u) = \frac{ds}{du} = \frac{1}{du/ds} = \sqrt{1-s^2(u)}.$$

Squaring both sides of this equation leads to $(s'(u))^2 = 1 - s^2(u)$, and differentiating gives $2s'(u)s''(u) = -2s(u)s'(u)$. Thus, $s''(u) = -s(u)$.

Now, let C_1 be a constant between $-\pi/2$ and $\pi/2$, and set $v = C_1 - u$. (Remember: a constant minus a variable is still a variable!) Because $s''(u) = -s(u)$ and $s''(v) = -s(v)$, it follows that

$$(2) \qquad s''(u)s(v) - s''(v)s(u) = 0.$$

Intermission

We now come to a bit that seems complicated … but bear with us: it's worth it! First note that since $v = C_1 - u$,

$$\frac{dv}{du} = -1.$$

Secondly, notice that if the expression

$$(3) \qquad s(u)s'(v) + s'(u)s(v)$$

is differentiated with respect to u, we get something that at first looks intimidating—

$$\frac{d}{du}[s(u)s'(v) + s'(u)s(v)] =$$

$$s'(u)s'(v) + s(u)s''(v)\frac{dv}{du} + s''(u)s(v) + s'(u)s'(v)\frac{dv}{du},$$

—but that is in fact quite simple. Because $dv/du = -1$, this expression reduces to

$$s''(u)s(v) - s''(v)s(u),$$

which we already know equals zero from equation (2). Because the derivative of expression (3) is zero, it must be a constant, and so

$$(4) \qquad s(u)s'(v) + s'(u)s(v) = C_2,$$

for some constant C_2.

Okay. The complicated stuff is over. Everyone still with us? Good. We're almost there!

Deriving the Addition Formula

Because $s(0) = 0$, equation (1) implies $s'(0) = \sqrt{1-s^2(0)} = 1$. If we set $u = 0$, then $v = C_1$ and equation (4) reduces to

$$0 \cdot s'(C_1) + 1 \cdot s(C_1) = s(C_1) = C_2.$$

Since $u + v = C_1$, we are led to conclude that

$$s(u+v) = s(C_1) = C_2 = s(u)s'(v) + s'(u)s(v).$$

If we now define $c(u) := s'(u) = \sqrt{1 - s^2(u)}$, we obtain the addition formula

$$s(u+v) = s(u)c(v) + c(u)s(v), \tag{5}$$

and with a few extra lines of formula crunching, it is not hard to derive a similar addition formula for our newly defined function $c(u)$; namely,

$$c(u+v) = c(u)c(v) - s(u)s(v) \tag{6}$$

It should come as no great surprise that, in more familiar notation, this is really

$$\sin(u+v) = \sin u \cos v + \cos u \sin v$$

and

$$\cos(u+v) = \cos u \cos v - \sin u \sin v.$$

But we're not done yet. It turns out that these addition formulas are just what we need to prove periodicity.

Periodicity

At the moment, the addition formulas in (5) and (6) are valid as long as u, v, and $u+v$ are in the domain of s, which is currently the interval $[-\pi/2, \pi/2]$. However, if we set $v = \pi/2$ in equation (5), the resulting expression

$$s(u + \pi/2) = s(u)c(\pi/2) + s(\pi/2)c(u), \tag{7}$$

can be used to extend the domain of $s(u)$ to include the interval $(\pi/2, \pi]$ by letting u range over $(0, \pi/2]$ and treating equation (7) as the definition of s. The same trick works in the negative direction if we set $v = -\pi/2$ and let u range over $[-\pi/2, 0)$.

In a similar fashion, we can use (6) to extend the domain of $c(u)$ to $[-\pi, \pi]$, and continuing in increments of $\pi/2$, we can extend the domains of $s(u)$ and $c(u)$ to the whole real line in a way that preserves the validity of the two addition formulas.

To see that $s(u)$ is periodic on this larger domain, first observe that

$$\begin{aligned} s(u + \pi/2) &= s(u)c(\pi/2) + s(\pi/2)c(u) \\ &= s(u)\sqrt{1 - s^2(\pi/2)} + s(\pi/2)c(u) \\ &= s(u)\sqrt{1 - 1^2} + 1 \cdot c(u) = c(u). \end{aligned}$$

Then, by this result and the addition formula for $c(u+v)$,

$$\begin{aligned} s(u + \pi) &= s((u + \pi/2) + \pi/2) = c(u + \pi/2) \\ &= c(u)c(\pi/2) - s(u)s(\pi/2) \\ &= c(u)\sqrt{1 - s^2(\pi/2)} - s(u)s(\pi/2) \\ &= c(u) \cdot 0 - s(u) \cdot 1 = -s(u). \end{aligned}$$

Simple algebra then gives us

$$s(u + 2\pi) = s((u + \pi) + \pi) = -s(u + \pi) = -(-s(u)) = s(u),$$

which shows that $s(u)$ is periodic with period 2π.

Odd or Even?

Because $s(u)$ is 2π periodic, to show it is odd we just need to prove $s(-u) = -s(u)$ for all u in the one full period $[-\pi, \pi]$. On the smaller interval $[-\pi/2, \pi/2]$, the function $s(u)$ was defined via the inverse of the arc length integral

$$u = \int_0^s \frac{1}{\sqrt{1 - x^2}}\, dx,$$

and the symmetry of this integral gives us $s(-u) = -s(u)$, at least for all $-\pi/2 \le u \le \pi/2$. (See figure 2.)

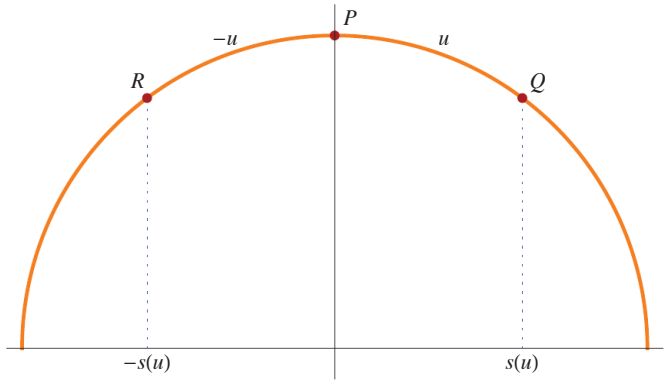

Figure 2. Why $s(-u) = -s(u)$ for $u \in [-\pi/2, \pi/2]$.

To extend this conclusion to the larger domain $[-\pi, \pi]$, we again turn to the addition formulas. Given u in $[-\pi/2, \pi/2]$,

$$\begin{aligned} s(-(u + \pi/2)) &= s(-u - \pi/2) \\ &= s(-u) \cdot 0 + c(-u)(-1) = -c(-u). \end{aligned}$$

Because $c(u) = s'(u)$, and because derivatives of odd functions are even, it follows that $c(u)$ is an even function, at least on $[-\pi/2, \pi/2]$. Therefore,

$$s(-(u + \pi/2)) = -c(-u) = -c(u) = -s(u + \pi/2),$$

so that $s(u)$ is odd (and $c(u)$ is even) on $[-\pi, \pi]$. Periodic-

ity allows us to extend this conclusion to the whole real line.

But Is Our $s(u)$ Really $\sin(u)$?

Have we convinced you that our function $s(u)$ is really just the sine function yet? If none of the above is good enough, perhaps using it to calculate a very recognizable sine, say, of 30° will do the trick.

Since we know that $s''(u) = -s(u)$, it follows that $s'''(u) = -s'(u)$ and $s^{(4)}(u) = s(u)$; thus, $s(u)$ is infinitely differentiable and has Taylor series expansion

$$s(u) = s(0) + s'(0)u + \frac{s''(0)}{2!}u^2 + \frac{s'''(0)}{3!}u^3 + \cdots$$

about the origin. Since

$$\begin{aligned} s(0) = s''(0) = s^{(4)}(0) = \cdots = 0, \\ s'(0) = s^{(5)}(0) = \cdots = 1, \\ s'''(0) = s^{(7)}(0) = \cdots = -1, \end{aligned}$$

we get the familiar expansion

$$s(u) = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \cdots$$

To calculate $\sin(30^\circ)$, set $u = \pi/6$, and our expansion becomes

$$s\left(\pi/6\right) = \left(\pi/6\right) - \frac{\left(\pi/6\right)^3}{3!} + \frac{\left(\pi/6\right)^5}{5!} - \frac{\left(\pi/6\right)^7}{7!} + \frac{\left(\pi/6\right)^9}{9!} - \cdots$$

The first five terms give a value of 0.5, correct to 10 decimal places. Not a proof, but pretty convincing. Incidentally, a pocket calculator uses precisely this method (and no triangles!) to get the same result.

Conclusions

Trigonometry, and trigonometric functions in particular, are incredibly useful in everyday life, and the most pervasive uses of trigonometry have nothing to do with triangles at all. For example, models of phenomena that involve recurrent periodicities in areas such as climatology, biology, and economics rely strongly on the periodicity of the sine and cosine functions. Moreover, the resemblance between these functions' wavelike graphs and the shape of a vibrating string results in the important role of trigonometry in music theory, particularly in the subject of acoustics. Perhaps less surprisingly, these wavelike characteristics also allow copious applications to the study of wave motion in oceanography, seismology, radiation, and radio waves.

Possibly the most common applications of the trigonometric functions occur via the use of *Fourier series*—infinite sums of sine and cosine functions, such as

$$\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots.$$

Fourier series are used, among many other things, to model heat flow and the diffusion of substances through fluid media, such as pollution in the atmosphere. They are also crucial to digital compression, by which electronic audio and video files are reduced in size to facilitate their transmission via, say, email.

To come back to the central point, although the sine and its trigonometric relatives are first encountered as the ratios between the sides of a right triangle, it should be evident that the use of triangles is not only peripheral to the subject of trigonometry, but can arguably be viewed as superfluous to it. In all these applications, it is the trigonometric *functions* that are employed, and the overriding virtue of the triangle-free approach laid out here is how quickly and effortlessly we gain access to the key properties of these functions—their periodicity, their symmetries, their derivatives, and their Taylor series.

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