BSHM Bulletin, 2016
Vol. 31, No. 2, 90–103, http://dx.doi.org/10.1080/17498430.2015.1046038

Commutativity and collinearity: a historical case study of the interconnection of mathematical ideas. Part II

ADRIAN RICE
Randolph-Macon College, USA

EZRA BROWN
Virginia Tech, USA

This paper investigates the discovery of an intriguing and fundamental connection between the famous but apparently unrelated mathematical work of two late third-century mathematicians. This link went unnoticed for well over 1500 years until the publication of two groundbreaking but again ostensibly unrelated works by two German mathematicians at the close of the nineteenth century. In this, the second and final part of the paper, we continue our examination of the chain of mathematical events and the related development of mathematical disciplines, without which the connection might never have been noticed in the first place.

Introduction

This paper concerns a previously overlooked connection between two mathematical achievements of late antiquity: Pappus' Theorem on the collinearity of line intersections in plane geometry and an identity used by Diophantus involving products of sums of two squares. In Part I of our paper (Rice and Brown 2015), we discussed how Diophantus' two-squares identity

$$\left(a^2+b^2\right)\left(c^2+d^2\right)=(ac\mp bd)^2+(bc\pm ad)^2 \tag{1}$$

was extended, first to four squares by Euler in the mid-eighteenth century, and then to eight squares by Degen in the early nineteenth. This eight-squares identity was subsequently re-discovered by John Graves and Arthur Cayley in the context of their creation of the eight-dimensional normed algebra of octonions in the mid-1840s. Given that two- and four-dimensional normed algebras over $\mathbb{R}$ were also known (namely the complex numbers and Hamilton's quaternions) and since their rules of multiplication were intimately connected to the identities of Diophantus and Euler, respectively, it was realized by the mid-nineteenth century that questions concerning the existence of higher-dimensional normed algebras over the reals and higher-order identities for products of sums of n squares were actually co-dependent. What was not immediately recognized was that a new mathematical area, then just in its formative stages, would provide a link between these algebraic/number-theoretic concerns and the foundations of projective geometry, and that this link would ultimately reveal a fundamental connection between Diophantus' two-squares identity and Pappus' Theorem.

Combinatorics: the (7, 3, 1) block design

In 1844, at exactly the same time that Hamilton, Graves, and Cayley were investigating the ramifications of their new normed algebras, a certain Wesley Woolhouse (1809–93), former Deputy Secretary of the British Nautical Almanac, was appointed the new editor of the *Lady's and Gentleman's Diary*. This popular publication had a long tradition of posing and solving enigmas, puzzles, and mathematical questions since, according to its title page, it was 'designed principally for the amusement and instruction of students in mathematics: comprising many useful and entertaining particulars, interesting to all persons engaged in that delightful pursuit'. In his first edition as editor, Woolhouse posed the following 'Prize Question' for 1844:

> Determine the number of combinations that can be made out of n symbols, p symbols in each; with this limitation, that no combination of q symbols, which may appear in any one of them shall be repeated in any other.

Being dissatisfied with the only two solutions he received, Woolhouse re-posed the question in the *Diary* for 1846, this time limiting the parameters to the specific case when $p = 3$ and $q = 2$:

> How many triads can be made out of n symbols, so that no pair of symbols shall be comprised more than once amongst them?

The eventual solution to this question was not ultimately featured in the *Lady's and Gentleman's Diary*. Instead, it appeared in the *Cambridge and Dublin Mathematical Journal* as the very first mathematical publication by a forty-year-old vicar from the north of England. The Reverend Thomas Pennington Kirkman (1806–95) was a gifted mathematician who, like Hamilton and Graves, had received his undergraduate mathematical training at Trinity College, Dublin; but unlike them he spent the rest of his career in the church. From 1839 until his retirement in 1892, he was Rector of the tiny parish of Croft-with-Southworth in Lancashire. Since his parish was so small and his duties relatively undemanding, this left plenty of time for family life (he had seven children) and mathematics. Like John Graves, then, he was an amateur, but only in the sense that mathematics was not his paid profession; as Norman Biggs (1981, 97) put it: 'One fact beyond dispute is that he was no amateur'. His numerous research publications on algebraic and combinatorial topics, such as group theory, partitions, projective geometry, polyhedra and knot theory, plus his election as a Fellow of the Royal Society, attest to considerable mathematical creativity, despite his geographic and professional isolation.

Kirkman's (1847) ground-breaking paper, 'On a problem in combinations', effectively laid the foundation for what we know today as combinatorial design theory, but which in the nineteenth century went by the name of 'Tactic' (Biggs 1981, 102). In it, he focused, not just on Woolhouse's question about triples, but also on the converse: for which values of n do triple systems exist? Using a simple counting argument, he showed that triple systems can occur only when n, the total number of symbols, is one of the numbers in the sequence 7, 9, 13, 15, 19, 21, 25, …

An example of such a representation when $n = 9$, featuring triples of numbers where each pair of symbols only appears once, is:

$$\{1, 2, 3\}, \{1, 4, 7\}, \{1, 5, 9\}, \{1, 6, 8\}, \{2, 4, 9\}, \{2, 5, 8\},$$
$$\{2, 6, 7\}, \{3, 4, 8\}, \{3, 5, 7\}, \{3, 6, 9\}, \{4, 5, 6\}, \{7, 8, 9\}.$$

In such a system, several things are important to point out. Firstly, each number appears a total of $m = (n - 1)/2$ times, since the other $n - 1$ numbers appear with it in pairs. Secondly, if t is the number of triples, then $3t = mn$, giving a total of $t = n(n - 1)/6$ triples. Since by definition t is an integer, it follows that $n \equiv 1$ or

3(mod 6). Kirkman also demonstrated the much harder result that for each such n there always exists a triple system, and he gave an explicit construction for forming them. He would go on to develop these ideas in subsequent papers (Kirkman 1850a, 1850b, 1852, 1853a, 1853b, 1862).

Today, triple systems such as those discussed by Kirkman are known to be special cases of *balanced incomplete block designs* with parameters (v, k, λ). These are arrangements of v objects into a sequence of 'blocks', such that every block contains exactly k objects, and every pair of objects appears together in exactly λ blocks. As Kirkman showed, if $k = 3$ and $\lambda = 1$, the smallest non-trival arrangement occurs when $v = 7$, giving the (7, 3, 1) block design:

$$\{1, 2, 3\}, \{1, 4, 5\}, \{1, 7, 6\}, \{2, 4, 6\}, \{2, 5, 7\}, \{3, 4, 7\}, \{3, 6, 5\}.$$

Thus far, the reader might be forgiven for questioning the relevance of Kirkman's combinatorics to sums of squares and normed algebras. But there is a connection. In 1848, in paper entitled 'On pluquaternions and homoid products of sums of squares', drawing on 'Professor Sir W. R. Hamilton's elegant theory of quaternions, and on a pregnant hint kindly communicated to me, without proof, by Arthur Cayley, Esq.' (Kirkman 1848, 447), Kirkman revealed that his (7, 3, 1) triple is closely related to the algebra of the octonions (Kirkman 1848, 448–449). Recall Graves' and Cayley's rules for multiplying base units in $\mathbb{O}$:

$$\begin{aligned}
&{i_1}^2 = {i_2}^2 = {i_3}^2 = {i_4}^2 = {i_5}^2 = {i_6}^2 = {i_7}^2 = -1\\
&i_1 = i_2 i_3 = i_4 i_5 = i_7 i_6 = -i_3 i_2 = -i_5 i_4 = -i_6 i_7\\
&i_2 = i_3 i_1 = i_4 i_6 = i_5 i_7 = -i_1 i_3 = -i_6 i_4 = -i_7 i_5\\
&i_3 = i_1 i_2 = i_4 i_7 = i_6 i_5 = -i_2 i_1 = -i_7 i_4 = -i_5 i_6\\
&i_4 = i_5 i_1 = i_6 i_2 = i_7 i_3 = -i_1 i_5 = -i_2 i_6 = -i_3 i_7\\
&i_5 = i_1 i_4 = i_7 i_2 = i_3 i_6 = -i_4 i_1 = -i_2 i_7 = -i_6 i_3\\
&i_6 = i_2 i_4 = i_1 i_7 = i_5 i_3 = -i_4 i_2 = -i_7 i_1 = -i_3 i_5\\
&i_7 = i_6 i_1 = i_2 i_5 = i_3 i_4 = -i_1 i_6 = -i_5 i_2 = -i_4 i_3
\end{aligned}$$

Notice that, for distinct $\alpha, \beta \in \{1, \dots 7\}$, $i_\alpha i_\beta = \pm i_\gamma$, where $\{\alpha, \beta, \gamma\}$ is one of the seven triples in the (7, 3, 1) block design. The sign of i_γ is determined by cyclically ordering the triples as follows: {1, 2, 3}, {2, 4, 6}, {3, 4, 7}, {4, 5, 1}, {5, 7, 2}, {6, 5, 3}, {7, 6, 1}. Then $i_\alpha i_\beta = i_\gamma$ or $i_\alpha i_\beta = -i_\gamma$ according to whether α does or does not directly precede β in the unique ordered triple containing α and β. Thus, 6 precedes 1 in the triple {7, 6, 1}, so $i_6 i_1 = i_7$; but 6 does not directly precede 4 in {2, 4, 6}, so $i_6 i_4 = -i_2$. It follows that Kirkman's (7, 3, 1) triple system is simply equivalent to the multiplication table for the octonion units.[1]

In 1853, six years after Kirkman's initial work, the Swiss geometer Jakob Steiner (1796–1863) published a short paper on triple systems in *Crelle's Journal* (Steiner 1853) in which he noted correctly that triple systems with n points only exist when $n \equiv 1$ or 3(mod 6). He also asked for which values of n such systems can be constructed, clearly unaware of Kirkman's solution to this problem in 1847.[2] Moreover, when the

[1]For more information on the various algebraic, combinatorial and topological relationships of the (7, 3, 1) triple system see Brown 2002.

[2]To quote Robin Wilson (2003, 271): 'this lack of awareness probably arises from the fact that the *Cambridge and Dublin Mathematical Journal*, though well known in Britain, was little known on the Continent.'

German mathematician M Reiss published an answer to Steiner's question in 1859, his methods were quite similar to those used by Kirkman twelve years earlier.[3]

Yet today, the triple block designs that Kirkman pioneered do not bear his name. Instead they are called *Steiner triple systems*, a term coined by the twentieth-century algebraist Ernst Witt (1911–91) (Witt 1938). Thus, like John Graves and Ferdinand Degen before him, Kirkman failed to receive due recognition for a mathematical innovation, with the credit going to the subsequent work of a more famous mathematician. To add insult to injury for Kirkman, this was not the only instance of his priority being overlooked in this way. In modern-day graph theory, 'Hamiltonian circuits' are a well-known feature and are, of course, named after William Rowan Hamilton, since Hamilton used them in his Icosian Calculus of 1856 and made them famous with his Icosian Game; but in so naming them, mathematicians again ignored the fact that they first appeared (and in a more general form) in a paper by Kirkman written in 1855.[4]

So, if Kirkman received inspiration for his work on triples from Woolhouse's Prize Problem in the *Lady's and Gentleman's Diary*, from where did Steiner receive the impetus—and how, for that matter, did Woolhouse come to consider triple systems? While no definitive answer exists, Robin Wilson conjectures that the source was the same in both cases: the work of the German projective geometer Julius Plücker (1801–68). Indeed, in the first case in particular, given Steiner's mathematical background, it is more than likely that the source of his interest was geometrical; in the second case Wilson speculates 'it is possible that James Joseph Sylvester, who had a life-long interest in combinatorial systems and wrote on "combinatorial aggregation" in 1844, knew of Plücker's work and mentioned it to Woolhouse' (Wilson 2003, 269).

This brings us neatly to the subject of projective geometry, and the work of Julius Plücker.

Projective geometry: Plücker, von Staudt, and Fano

After the initial seventeenth-century forays of Desargues and Pascal, the subject of projective geometry had languished for well over a century, before enjoying a sudden resurgence of interest in the early nineteenth century. The major impetus came from the publication of *Traité des propriétés projectives des figures* by Jean-Victor Poncelet (1788–1867) in 1822. Poncelet had studied mathematics at the prestigious École Polytechnique under Gaspard Monge (1746–1818), who advocated a greater role for geometry in mainstream mathematics, feeling that it had become increasingly sidelined with the growth of analytic methods throughout the eighteenth century. Through the work of Poncelet, and other French mathematicians such as Joseph Servois and Michel Chasles, projective geometry emerged as a major mathematical discipline in its own right, stimulating a huge quantity of geometrical research throughout the nineteenth century.[5]

[3]This famously prompted Kirkman's sarcastic retort: '… how did the *Cambridge and Dublin Mathematical Journal*, Vol. II, p. 191, contrive to steal so much from a later paper in *Crelle's Journal*, Vol. LVI, p. 326, on exactly the same problem in combinations?'—Kirkman 1887.

[4]The discovery for which Kirkman is best remembered today, known as 'Kirkman's schoolgirls problem', arose from his work on the (15, 3, 1) block design—see Biggs 1981, Wilson 2003, Brown and Mellinger 2009.

[5]For more detail on the history of geometry during the nineteenth century, see Gray 2010.

Interest in this new field travelled quickly to Germany and, whereas French projective geometry had been largely synthetic in style, German geometers, such as Augustus Ferdinand Möbius (1790–1868) and particularly Julius Plücker, began to apply algebraic methods to the subject with great success. This inaugurated a fruitful period in the theory of algebraic curves (and later surfaces) when curves of degree n, denoted C_n, would be defined algebraically and studied projectively. Of particular interest at this time were plane curves of degree $n > 2$, particularly cubics and quartics. One of the earliest major discoveries in this area was made by Plücker in his *Theorie der algebraischen Curven*, where he famously showed that all plane quartic curves C_4 contain 28 real and imaginary bitangents (Plücker 1839, 245–248).

Four years earlier in his *System der analytischen Geometrie*, Plücker had shown that, for $n \geq 2$, all projective plane curves C_n have $3n(n-2)$ points of inflection (Plücker 1835, 264). Observing, as a consequence of this, that a general plane cubic curve has nine points of inflection, he showed that these points lie on four sets of three lines, with three points on each line, such that exactly one of the 12 lines must pass through any two inflection points (Plücker 1835, 283–284). He continued:

> ...if we denote the nine inflection points as P, Q, R, P_1, P_2, Q_1, Q_2, R_1 and R_2, we get the following twelvefold collection [of lines], corresponding to the following scheme:[6]
>
> $\{P, Q, R\}$, $\{P, P_1, P_2\}$, $\{Q, Q_1, Q_2\}$, $\{R, R_1, R_2\}$,
> $\{P, Q_1, R_2\}$, $\{P, Q_2, R_1\}$, $\{Q, P_1, R_2\}$, $\{Q, P_2, R_1\}$,
> $\{R, P_1, Q_2\}$, $\{R, P_2, Q_1\}$, $\{P_1, Q_1, R_1\}$, $\{P_2, Q_2, R_2\}$.

This system of lines, now known as the *nine-point affine plane*, is of course none other than the Kirkman–Steiner triple system (9, 3, 1), a fact further reinforced by a footnote immediately below Plücker's (1835, 284n) example:

> Not all numbers of elements can be grouped in threes so that the different groups contain all combinations and each pair of elements occurs only once. If m is the number of such elements and n any non-zero integer, from the evidence above it is easy to see that m needs to be of the form $6n + 3$. The number of different groups amounts then to a third of the number of combinations of m elements in two, consequently $\frac{1}{3} \cdot \frac{m(m-1)}{1 \cdot 2}$, and each element is found in $\frac{m-1}{2}$ different groups.[7]

Thus, just as Kirkman was to do the following decade, Plücker found that the number of possible triples is $t = n(n-1)/6$ and that each element occurs with frequency $(n-1)/2$. The one mistake he made, of course, was to deduce only that $n \equiv 3 (\mathrm{mod}\, 6)$. However, he rectified that mistake four years later by adding that n could also be of the form $6m + 1$ (Plücker 1839, 246n), pre-dating the publication of

[6]'*Wir erhalten hiernach eine zwölffache Zusammenstellung, der, wenn wir die neun Wendungspuncte durch P, Q, R,* P_1, P_2, Q_1, Q_2, R_1 *und* R_2 *bezeichnen, das folgende Schema entspricht*'.

[7]'*Nicht jede Anzahl von Elementen ist von der Art, dass sie sich so zu drei gruppieren lassen, dass in den verschiedenen Gruppen alle Combinationen zweier Elemente vorkommen und jede derselben nur ein einziges Mal. Wenn* m *die Anzahl solcher Elemente und* n *irgend eine ganze Zahl, Null nicht ausgeschlossen, bedeutet, so überzeugt man sich leicht, dass* m *von der Form* 6n + *3 sein muss. Die Anzahl der verschiedenen Gruppen beträgt alsdann ein Drittel der Anzahl der Combinationen von* m *Elementen zu zwei, mithin* $\frac{1}{3} \cdot \frac{m(m-1)}{1 \cdot 2}$, *und jedes Element kommt in* $\frac{m-1}{2}$ *verschiedenen Gruppen vor.*'

Kirkman's first work on triples by eight years and Steiner's by fourteen. Perhaps then, those who object to such systems bearing only the name of Steiner might well argue that they should be renamed *Plücker–Kirkman–Steiner triples*.

Mathematics in the nineteenth-century was characterized, among other things, by an increased interest in refining and codifying the first principles of various branches of the discipline. This manifested itself, for example, in the intense rigorization of analysis, a process which had begun in the eighteenth century, but which was brought to fruition in the nineteenth by Cauchy, Riemann, Weierstrass, and others. In algebra, certain basic laws of operation were isolated and named as being worthy of particular attention (such as the *commutative* and *distributive* laws—terms coined by Servois (1814, 98)—and the *associative* law—first stated by Hamilton in 1843 (Hamilton 1967, 114)); the gradual loosening of these laws, via the development of new algebras and algebraic structures, led to the eventual axiomatization of various concepts, the first of which was Heinrich Weber's axiomatic definition of a group in 1882. And in geometry, the example of projective geometry and the subsequent discovery and development of various non-Euclidean geometries by Bolyai, Lobachevski, Gauss and Riemann gave further credence to the growing belief that the axiomatic system of Euclid's *Elements* might not be the ideal logical foundation on which to build the subject.

The first steps towards the axiomatization of projective geometry were taken by Karl Georg Christian von Staudt (1798–1867), a professor at the University of Erlangen. In his *Geometrie der Lage* of 1847, he set out to rid the subject of its reliance on the Euclidean concepts of distance and length, showing not only a connection between projective and Euclidean geometry, but also revealing the former to be more logically fundamental than the latter. He developed these ideas in his three-volume *Beiträge zur Geometrie der Lage* (1856–60), which featured his now well-known 'Wurftheorie' or 'algebra of throws', whereby he assigned numerical coordinates to points on a line without reference to any concept of length. Von Staudt then defined arithmetical operations (+, ×, etc.) by specific geometrical constructions carried out on his geometrically-defined 'numbers' (see Figure 1), whose equality was interpreted as congruence of line segments. He was thus able to construct a coherent algebraic system obeying the usual laws of arithmetic.

Although von Staudt's work was by no means the last word on the subject, and lacked rigour in places, it was to prove highly influential. In particular, his ideas played a major role in the work of Felix Klein in the 1870s, in which Klein crystallized the fundamental connections between Euclidean, non-Euclidean and projective geometries to each other as well as to the emerging theory of groups. Von Staudt and

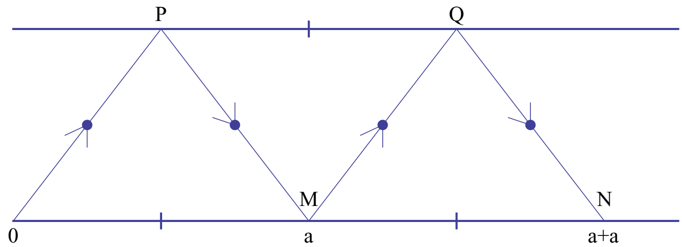

Figure 1. Von Staudt's geometrical definition of addition

Klein also had a profound influence on a growing community of geometers outside Germany.

The second half of the nineteenth century witnessed the development of a formidable school of algebraic geometry in Italy. In Pisa, Eugenio Bertini extended the work of his erstwhile professor Luigi Cremona, studying geometrical properties invariant under Cremona transformations and using the results to resolve the singularities of curves. In Padua, Giuseppe Veronese worked on higher-dimensional projective geometries and introduced the notion of non-Archimedean geometry. Veronese's student Guido Castelnuovo, on his arrival as a newly-appointed professor in Rome, embarked on an extensive collaboration with his Bologna colleague Federigo Enriques which lasted more than two decades and eventually produced a classification of algebraic surfaces.

In Turin, Enrico D'Ovidio used techniques from projective geometry to derive metric functions for non-Euclidean *n*-space, and with his student Corrado Segre built up an impressive mathematics department, where at the same time another former student Giuseppe Peano focused on the axiomatization of arithmetic and the development of formal logic. Two of Segre's students, Mario Pieri (1860–1913) and Gino Fano (1871–1952), were to produce important work on the foundations of geometry. In 1890, Pieri published an Italian translation of von Staudt's *Geometrie der Lage*. The influence of von Staudt, along with that of the foundational work of Peano and the German geometer Moritz Pasch, resulted in Pieri's seminal text *I principii della geometria di posizione composti in un sistema logico-deduttivo* (1898).[8]

Meanwhile, at Segre's suggestion, Fano had translated Klein's *Erlanger Programm* into Italian in 1889 and, in common with several Italian mathematicians at this time, spent a year studying with Klein in Göttingen. In 1892, he published an important memoir on the foundations of projective geometry (Fano 1892), in which he pioneered the idea of a finite geometry, of which Plücker's system of 9 points and 12 lines is an example, being a finite *affine* plane of order 3 (see Figure 2).

Although in his paper, Fano does not deal with affine planes, he does discuss finite *projective* planes. In general, a finite projective plane of order n is an arrangement of $n^2 + n + 1$ points and $n^2 + n + 1$ lines such that exactly $n + 1$ points lie on each line and $n + 1$ lines pass through each point. The simplest non-trivial example of such a geometry is the finite projective plane of order 2, as seen in Figure 3.

This 7-point plane, known today as the *Fano plane*, features seven points and seven lines with exactly three points on each line and three lines through each point. Because no two pairs of points can lie on more than one line, this arrangement of groups of three out of seven points is, in fact, the (7, 3, 1) triple system in disguise, as Figure 4 illustrates.

Not surprisingly, then, the Fano plane also gives a diagrammatic representation of the multiplication of the octonion units, as Hans Freudenthal pointed out in Freudenthal (1951) (see Figure 5).

Thus the geometry of Julius Plücker was intrinsically connected to the combinatorics of Thomas Kirkman, while the work of Gino Fano in 1892 provided a link from projective geometry, not only to the realm of Steiner triples, but via (7, 3, 1) to the normed algebra of octonions, and consequently to the Diophantine problem of products of sums of squares.

But this still leaves two outstanding questions:

[8]For more on Pieri, see Marchisotto 2006 and especially Marchisotto and Smith 2007.

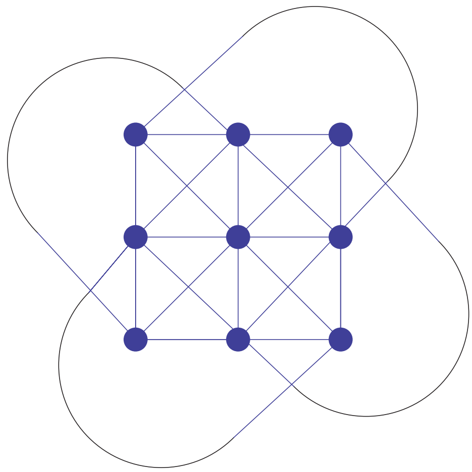

Figure 2. The nine-point affine plane of order 3

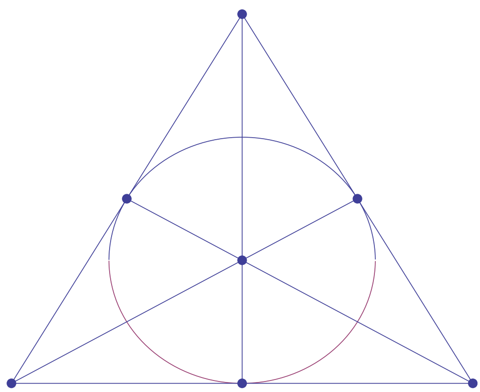

Figure 3. The Fano plane

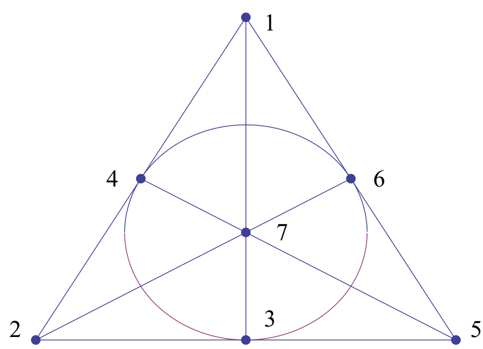

Figure 4. The Fano plane as a (7, 3, 1) block design

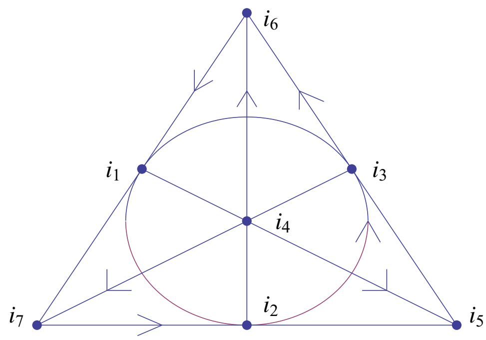

Figure 5. The Fano plane as the multiplication rule for the octonion units

- When was the n-squares/normed algebra question finally resolved?
- What is its ultimate connection with Pappus's Theorem?

Denouement

As events transpired, both questions were answered at approximately the same time. The answer to the first question was provided in an eight-page paper published in 1898 by Adolf Hurwitz (1859–1919), a German mathematician then working at the University of Zurich. It concerned a particular problem in the algebraic theory of quadratic forms in any number of variables, namely the possibility of representing the product of two quadratic forms, $\varphi(x_1, x_2, \ldots, x_n)$ and $\psi(y_1, y_2, \ldots, y_n)$, as a third such form, $\chi(z_1, z_2, \ldots, z_n)$, where $z_1, z_2, \ldots, z_n$ were suitably chosen bilinear functions of $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$. Hurwitz chose to focus his attention on the case where $\varphi = x_1^2 + x_2^2 + \ldots + x_n^2$, $\psi = y_1^2 + y_2^2 + \ldots + y_n^2$, and $\chi = z_1^2 + z_2^2 + \ldots + z_n^2$, that is

$$\left(x_1^2 + x_2^2 + \cdots + x_n^2\right)\left(y_1^2 + y_2^2 + \cdots + y_n^2\right) = z_1^2 + z_2^2 + \cdots + z_n^2, \qquad (2)$$

thus drawing a deliberate and explicit connection to the sums of n squares problem. 'In the following lines', he wrote, 'I will show that this is possible only in the cases $n = 2, 4, 8$',[9] going on to observe that 'by this proof, in particular, the old question of whether the known product formulas for sums of 2, 4 and 8 squares can be extended to sums of more than 8 squares will finally be answered in the negative'[10] (Hurwitz 1898, 309).

Hurwitz's paper appears to have been motivated, at least in part, by some rather unsatisfactory attempts by British mathematicians to solve the n-squares problem in the intervening years. For example, in the 1870s and 1880s Cayley and his younger contemporary Samuel Roberts had endeavoured to answer the question when $n = 16$, but had only succeeded in providing corroborative data through a series of increasingly complicated special cases. As Hurwitz wrote, although Roberts (1879) and

[9] '*In den folgenden Zeilen will ich zeigen, dass dieses nur in den Fällen* n = 2, 4, 8 *möglich ist ...*'

[10] '*Durch diesen Nachweis wird dann insbesondere auch die alte Streitfrage, ob sich die bekannten Produktformeln für Summen von 2, 4 und 8 Quadraten auf Summen von mehr als 8 Quadraten ausdehnen lassen, endgültig, und zwar in verneinendem Sinne entschieden.*'

Cayley (1881) 'used evidence to show that the product of two sums of 16 squares cannot be represented as a sum of 16 squares[, t]heir extremely laborious attempts have no probative value, however, since they are justified only by specific assumptions with respect to the bilinear forms $z_1, z_2\ldots$'[11] (Hurwitz 1898, 309n). In contrast, via an elegant general argument relying essentially only on notions of linear independence and symmetric matrices, Hurwitz was able to prove, firstly that identity (2) could never hold for odd $n > 1$, secondly, if it did hold then $2^{n-2} \leq n^2$, or in other words, $n \leq 8$, and finally that (2) was impossible when $n = 6$.[12] At a stroke, the n-squares problem was finally resolved, and with it came corroboration that no further normed algebras beyond the octonions could possibly exist: the only finite-dimensional normed division algebras over the reals (up to isomorphism) were $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$.[13]

In a sense, Hurwitz's theorem marks the end of a story that began with Book III, Problem 22 of Diophantus's *Arithmetica*—the search for formulae for products of sums of squares and their corresponding normed algebras was finally over. Yet at the same time, another story was only just beginning. For within the space of a year, a landmark publication was to shed new light on the foundational underpinnings of both algebra and geometry, revealing in the process hitherto unrealized connections between them. It is fitting, then, that this work was produced by a lifelong friend of Hurwitz, on whom he had exercised a huge mathematical influence (Reid 1996, 13–14) and whose work he held in the highest regard. That mathematician was David Hilbert (1862–1943), and the work in question was Hilbert's monumental *Grundlagen der Geometrie* (1899).

Although better known at this time for his work in algebra and number theory, Hilbert had been studying the foundations of geometry since at least 1891 (Toepell 1986), and his knowledge of projective geometry went back even further to his days as a gymnasium student in 1879. Notes for the lecture courses on geometry he delivered in the 1890s reveal a familiarity with the foundational work of von Staudt, Pasch, Peano, and Veronese, outlined in the previous section (Hilbert 2004). After a three-year mid-decade hiatus, his geometric interests were reignited by the communication of a letter from Friedrich Schur to Felix Klein in 1898 which, as Hilbert wrote to Hurwitz 'has given me the inspiration to take up again my old ideas about the foundations of Euclidean geometry' (Toepell 1985, 641).

The result was a masterpiece, and a constantly evolving one. From its first edition, published in June of 1899, to its seventh in 1930, the *Grundlagen* was a work in progress, being continuously revised and updated by its author. Although interest in the logical refinement of geometry was clearly *à la mode* at the time of its initial composition, and while Hilbert was far from the first to attempt to re-cast the subject as a purely theoretical deductive system, his re-axiomatization of Euclidean geometry set a new standard for logical precision in mathematics and established the foundations of geometry as a research area in its own right. Its significance for the subject of this paper lies in its revelation of a fundamental connection between the apparently distinct worlds of geometry and algebra.

[11]'*Roberts und Cayley haben sich im 16. und 17. Bande des Quarterly Journal mit dem Nachweis beschäftigt, dass ein Produkt von zwei Summen von je 16 Quadraten nicht als Summe von 16 Quadraten darstellbar sei. Ihre äusserst mühsamen, auf Probieren beruhenden Betrachtungen besitzen indessen keine Beweiskraft, weil ihnen bezüglich der bilinearen Formen* $z_1, z_2\ldots$ *spezielle Annahmen zugrunde liegen, die durch nichts gerechtfertigt sind.*'

[12]For an expanded and more detailed exposition of Hurwitz's proof, see Dickson 1919.

[13]For a proof of this, see Curtis 1963.

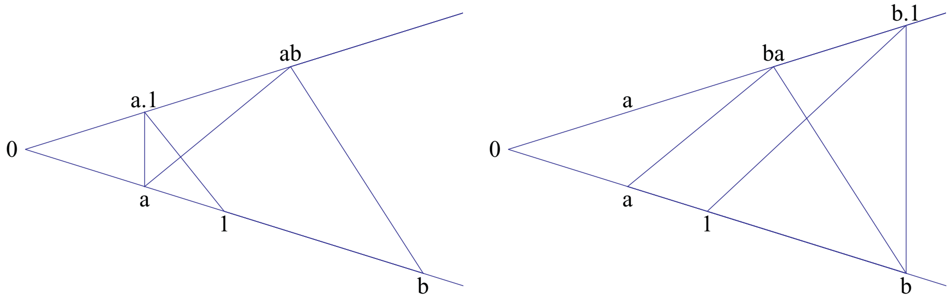

Figure 6. Hilbert's constructions of $a \cdot 1$ and ab (left) and $b \cdot 1$ and ba (right)

Recall von Staudt's 'algebra of throws', whereby 'numbers' and operations of addition and multiplication were constructed geometrically. For example, given a base unit 1 and numbers a and b, the products $a \cdot 1$, ab, $b \cdot 1$ and ba were all constructed as in Figure 6.

What Hilbert made explicit for the first time was that, depending on which number system one chose to base one's geometry ($\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$), each projective plane could be 'coordinatized' by either real or complex numbers, quaternions, or octonions, with the behaviour of each geometry corresponding to the fundamental axioms of its underlying algebra. In particular, he proved that Pappus' Theorem would hold in a geometric system if and only if the algebra on which that system was based was commutative. Indeed, as he said, if we combine the two constructions in Figure 6, it becomes immediately clear that 'the commutative law of multiplication is none other than Pascal's Theorem' (Hilbert 1899, 76).[14] And indeed, a look at Figure 7 confirms that the geometrical construction which forces the equivalence of the points ab and ba likewise necessitates the validity of Pappus' Theorem. And vice versa.[15]

This momentous discovery is mathematics at its most beautiful, and its significance was recognized immediately. But no one appears to have drawn a further connection that Hilbert's result and Hurwitz's theorem now suggested. Since the only normed algebras that are commutative with respect to multiplication are $\mathbb{R}$ and $\mathbb{C}$, it follows that these are the only two algebras upon which so-called Pappian geometries may be based. Moreover, as we saw in the third section of Part I of this paper, the rule for the multiplication of two complex numbers is explicitly contained within Diophantus' two-squares identity (1). It therefore follows that Pappus' Theorem can only hold in a geometric system whose axioms are equivalent to those of the real or complex numbers, or in other words, where Diophantus' two-squares identity holds. Thus we finally see that the geometry of Pappus and the number theory of Diophantus are inextricably linked. It had taken over one-and-a-half millennia and the evolution of mathematical disciplines never imagined in antiquity, but mathematics had

[14]'*...das commutative Gesetz der Multiplikation zweier Strecken auch hier nichts anderes als den Pascalschen Satz*'. Note that throughout the *Grundlagen* Hilbert referred to Pappus's theorem by its more general name of Pascal's theorem.

[15]Although the connection of Pappus's theorem with commutativity of multiplication had also been considered by Friedrich Schur in the first edition of his 1898 *Lehrbuch der Analytischen Geometrie* (Schur 1898, 11), the link was not stated explicitly until the second edition of 1912 (Schur 1912, 11), thirteen years after the publication of Hilbert's *Grundlagen*.

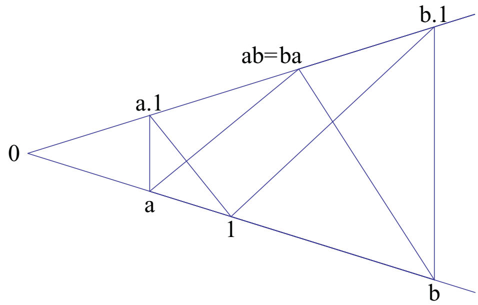

Figure 7. The equivalence of Pappus's Theorem and the commutative law of multiplication

finally undergone sufficient development for a hitherto unrealized connection between Pappus and Diophantus to be revealed at last.[16]

Bibliography

Biggs, N L, 'T. P. Kirkman, mathematician', *Bulletin of the London Mathematical Society*, 13 (1981), 97–120.

Brown, E, 'The many names of (7, 3, 1)', *Mathematics Magazine*, 75 (2002), 83–94.

Brown, E, and Mellinger, K E, 'Kirkman's schoolgirls wearing hats and walking through fields of numbers', *Mathematics Magazine*, 82 (2009), 3–15.

Cayley, A, 'On the theorems of the 2, 4, 8, and 16 squares', *Quarterly Journal of Pure and Applied Mathematics* 17 (1881), 258–276; *Collected Mathematical Papers*, 11, 294–313.

Curtis, C W, 'The four and eight square problem and division algebras', in A A Albert (ed), *Studies in Modern Algebra*, Washington DC: Mathematical Association of America, 1963, 100–125.

Dickson, L E, 'On quaternions and their generalization and the history of the eight squares theorem', *Annals of Mathematics, Second Series*, 20 (1919), 155–171.

Fano, G, 'Sui postulati fondamentali della geometria proiettiva in uno spazia lineare a un numero qualunque di dimensioni', *Giornale di matematiche*, 30 (1892), 106–132.

Freudenthal, H, *Oktaven, Ausnahmegruppen und Oktavengeometrie*, Utrecht: Mathematisch Instituut der Rijksuniversiteit te Utrecht, 1951. Reprinted in *Geometriae Dedicata*, 19 (1985), 7–63.

Gray, J, *Worlds out of nothing: a course in the history of geometry in the 19th century*, New York: Springer, 2010.

Hamilton, W R, *The mathematical papers of Sir William Rowan Hamilton* (eds H Halberstam and R E Ingram), Cambridge: University Press, 1967, vol 3.

Hilbert, D, *Grundlagen der Geometrie*, published as Part I of *Festschrift zur Feier der Enthüllung des Gauss-Weber-Denkmals in Göttingen*, Leipzig: Teubner, 1899.

[16]Of course, this was far from the end of the matter. The *Grundlagen* also contained Hilbert's construction of a geometry in which Desargues' Theorem fails to hold (Hilbert 1899, 51–55), corresponding to an absence of associativity in multiplication. Thus the real numbers, complex numbers, and quaternions give rise to Arguesian geometries, but the octonions do not. In other words, Desargues' theorem can only hold in a geometric system whose axioms are equivalent to those of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, that is, where Euler's four-squares theorem holds. As a consequence of this discovery, the early twentieth century saw the creation of a variety of non-Arguesian and non-Pappian geometries by F R Moulton (1902) and Oswald Veblen and Joseph Wedderburn (1907) among others. This culminated in the work of Ruth Moufang in the 1930s (Moufang 1933), inaugurating the systematic study of non-Arguesian planes and their associated non-associative algebras (or 'Moufang loops'), which via their connections with ternary rings, cohomology sets, and Jordan algebras, continue to exert an influence on mathematics to this day (see Weibel 2007).

Hilbert, D, *David Hilbert's lectures on the foundations of geometry, 1891–1902* (eds Michael Hallett and Ulrich Majer), Berlin: Springer-Verlag, 2004.
Hurwitz, A, 'Über die Komposition der quadratischen Formen von beliebig vielen Variablen', *Göttinger Nachrichten*, (1898), 309–316; *Mathematische Werke*, 2, 565–571.
Kirkman, T P, 'On a problem in combinations', *Cambridge and Dublin Mathematical Journal*, 2 (1847), 191–204.
Kirkman, T P, 'On pluquaternions and homoid products of sums of squares', *Philosophical Magazine, Series 3*, 33 (1848), 447–459, 494–509.
Kirkman, T P, 'On an unanswered prize question', *Cambridge and Dublin Mathematical Journal*, 5 (1850a), 255–262.
Kirkman, T P, 'On the triads made with fifteen things', *Philosophical Magazine, Series 3*, 37 (1850b), 169–171.
Kirkman, T P, 'Theorems in the doctrine of combinations', *Philosophical Magazine, Series 4*, 4 (1852), 209.
Kirkman, T P, 'Theorems on combinations', *Cambridge and Dublin Mathematical Journal*, 8 (1853a), 38–45.
Kirkman, T P, 'Note on combinations', *Philosophical Magazine, Series 4*, 5 (1853b), 11–12.
Kirkman, T P, 'On the enumeration of x-edra having triedral summits and an $(x-1)$-gonal base', *Philosophical Transactions of the Royal Society London*, 146 (1856), 399–412 (received 13 June 1855).
Kirkman, T P, 'On the problem of the fifteen young ladies', *Philosophical Magazine, Series 4*, 23 (1862), 199–204.
Kirkman, T P, 'Appendix 1 [two questions on polyhedra]', *Mathematical Questions with their Solutions from the Educational Times,* 47 (1887), 125–127.
Marchisotto, E A, The projective geometry of Mario Pieri: a legacy of Georg Karl Christian von Staudt, *Historia Mathematica*, 33 (2006), 277–314.
Marchisotto, E A, and Smith, J T, *The legacy of Mario Pieri in geometry and arithmetic*, Boston: Birkhäuser, 2007.
Moufang, R, 'Alternativkörper und der Satz von vollständigen Vierseit (D_9)', *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 9 (1933), 207–222.
Moulton, F R, 'A simple non-Desarguesian plane geometry', *Transactions of the American Mathematical Society*, 3 (1902), 192–195.
Plücker, J, *System der analytischen Geometrie, auf neue Betrachtungsweisen gegründet, und insbesondere eine ausführliche Theorie der Curven dritter Ordnung enthalend*, Berlin: Duncker und Humboldt, 1835.
Plücker, J, *Theorie der algebraischen Curven, gegründet auf eine neue Behandlungensweise der analytischen Geometrie*, Bonn: Marcus, 1839.
Reid, C, *Hilbert*, New York: Copernicus, 1996.
Reiss, M, 'Über eine Steinersche combinatorische Aufgabe, welche in 45sten Bande dieses Journals, seite 181, gestellt worden ist', *Journal für die reine und angewandte Mathematik*, 56 (1859), 326–344.
Rice, A, and Brown, E, 'Commutativity and collinearity: a historical case study of the interconnection of mathematical ideas. Part I', *BSHM Bulletin* (2015), http://dx.doi.org/10.1080/17498430.2015.1046037
Roberts, S, 'On the impossibility of the general extension of Euler's theorem', *Quarterly Journal of Pure and Applied Mathematics*, 16 (1879), 159–170.
Schur, F, *Lehrbuch der Analytischen Geometrie*, Leipzig: Von Veit, 1898.
Schur, F, *Lehrbuch der Analytischen Geometrie*, Leipzig: Von Veit, 2nd edition, 1912.
Servois, F-J, 'Essai sur un nouveau mode d'exposition des principes du calcul differentiel', *Annales de mathématiques pures et appliquées*, 5(4) (1814), 93–140.
Steiner, J, 'Combinatorische Aufgabe', *Journal für die reine und angewandte Mathematik*, 45 (1853), 181–182; *Gesammelte Werke*, 2, 435–438.

Sylvester, J J, 'Elementary researches in the analysis of combinatorial aggregation', *Philosophical Magazine, Series 3*, 24 (1844), 285–296.
Toepell, M, 'Zur Schlüsselrolle Friedrich Schurs bei der Entstehung von David Hilberts "Grundlagen der Geometrie"', in *Mathemata. Festschrift für Helmuth Gericke*, Stuttgart: Franz Steiner, 1985, 637–649.
Toepell, M, *Über die Entstehung von David Hilberts 'Grundlagen der Geometrie'*, Göttingen: Vandenhoeck & Ruprecht, 1986.
Veblen, O, and Wedderburn, J H M, 'Non-Desarguesian and non-Pascalian geometries', *Transactions of the American Mathematical Society*, 8 (1907), 379–388.
Weber, H, 'Beweis des Satzes, dass jede eigentlich primitive quadratische Form unendlich viele Primzahlen darzustellen fahig ist', *Mathematische Annalen*, 20 (1882), 301–329.
Weibel, C, 'Survey of non-Desarguesian planes', *Notices of the American Mathematical Society*, 54(10) (2007), 1294–1303.
Wilson, R, 'The early history of block designs', *Rendiconti del Seminario Matematico di Messina, Serie 11*, 25(9) (2003), 267–276.
Witt, E, 'Über Steinersche Systeme', *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 12 (1938), 265–275.
Woolhouse, W S B, 'Prize question 1733', *Lady's and Gentleman's Diary*, London: Company of Stationers, 1844.