Why Hamilton Couldn't Multiply Triples

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Ask any professional mathematician what a quaternion is and you will be sure to get an answer: a quaternion q is a four-dimensional "number" consisting of one real and three imaginary components such that

$$q = a + bi + cj + dk, \text{ where } a, b, c, d \in \mathbb{R}, \text{ and } i^2 = j^2 = k^2 = ijk = -1. \quad (1)$$

The real component a is called the *scalar* part, while the imaginary section $bi + cj + dk$ is known as the *vector* part. It is therefore perhaps not surprising that quaternions have many applications in physics (where, for example, their non-commutativity has deep consequences in quantum mechanics), engineering (for instance, in robotics), and computing (where their most widespread use to date is in 3D graphic animation).

Mathematically, of course, quaternions are most likely to be encountered by students in the context of an undergraduate course in abstract algebra, in which they will likely be exhibited as a good example of a non-abelian group. And indeed, much is often made of their non-commutativity, which is a direct consequence of their fundamental equation

$$i^2 = j^2 = k^2 = ijk = -1.$$

Multiply this on the left by $-i$ and we obtain $i = jk$; multiplying this new equation on the right by $-k$ gives us $-ik = j$, which when multiplied on the left by i gives $k = ij$. Similarly, if we now take $k = ij$ and multiply on the right by $-j$, we will get $-kj = i$, which when multiplied on the left by k gives $j = ki$. This final equation, multiplied on the right by i, will result in $-ji = k$. We thus have the following identities:

$$\begin{aligned} ij &= -ji = k \\ jk &= -kj = i \\ ki &= -ik = j. \end{aligned}$$

The immediate consequence is that, given any two quaternions, p and q, their products pq and qp will not, in general, be equal. Since most students will have probably multiplied vectors or matrices—and thus experienced non-commutativity—before their first encounter with quaternions, this violation of the commutative law of multiplication does not usually strike most of them today as particularly unusual or noteworthy.

But the significance of quaternions lies in the fact that they were one of the very first non-commutative algebras to be discovered, and their consistency and utility paved the way for the subsequent development of new and even more unorthodox algebraic systems.

Along with the mathematics itself, the story of the discovery of quaternions is also very well known [**8**]. Following the gradual acceptance of complex numbers during the 18th century, the early years of the 19th century saw mathematicians becoming increasingly familiar with their algebraic and geometric properties. In particular, just as you could add, subtract, multiply, and divide complex numbers, you could also represent such operations geometrically in two-dimensional space. The question mathematicians now began to ask was: is it possible to come up with a higher form of complex number algebra that could represent numbers in *three*-dimensional space?

This question intrigued several mathematicians of the time, one of whom was the Irish mathematician and physicist Sir William Rowan Hamilton (1805–1865). In common with many of his contemporaries, Hamilton started by constructing a new algebra in analogy to that of the complex numbers. So, while a complex number looks like $a + bi$, where $a, b \in \mathbb{R}$ and $i^2 = -1$, Hamilton's triples would be of the form

$$z = a + bi + cj, \text{ where } a, b, c \in \mathbb{R}, i \neq j, \text{ and } i^2 = j^2 = -1. \tag{2}$$

Clearly for such a number to exist, its arithmetical operations must be well-defined, and for addition and subtraction this was no problem. However, Hamilton and his contemporaries quickly found that they could not multiply two triples together to form another triple—the multiplication just didn't work. Then, after having worked on the problem on and off for about a decade, on October 16, 1843, Hamilton suddenly realized that its solution lay not with number triples, but with quadruples of the form in (1). Thus was born his system of quaternions, a consistent algebra with addition and subtraction, plus most importantly, well-defined multiplication and division.

From this point, as they say, the rest is history: Hamilton devoted the rest of his life to the development and application of his new algebra to problems in mathematical physics, further non-commutative (and even non-associative) algebras were soon created [**5**, pp. 415–426], such as matrices and octonions, while vector algebra arose from the study of the three-dimensional imaginary part of quaternions [**2**]—hence the use of the symbols **i**, **j**, and **k** in vector calculus to this day.

But one mathematical aspect of the story has received far less attention, namely, the question of *why* Hamilton was unable to create a coherent system of algebraic triples in the first place. Once he had discovered quaternions, and consequently found an answer to his question, Hamilton never looked at triples again. And it took over half a century before anyone was able to prove that no such triple system actually existed. So just why was Hamilton unable to find a consistent three-dimensional linear algebra with well-defined multiplication? Why doesn't such a system of complex numbers exist? The purpose of this article is to find out.

Normed algebras over the reals

In modern terminology, Hamilton and his contemporary mathematicians were trying to find a *normed algebra over the real numbers*. In other words, they wanted to see if they could construct an n-dimensional vector space, A, with the following basic properties. First and foremost, they wanted the elements of A to behave just like "proper" numbers; so, for example, properties like distribution of multiplication over addition, and closure under addition and multiplication, should still hold true. But in addition to this, since their new algebra was intended to be representable geometrically, they needed to define a function that would give a well-defined measure of distance. This mapping $N\colon A \to \mathbb{R}$, called the *norm*, was defined as $N(x) = x \cdot \overline{x}$, where $\overline{x}$ is the complex conjugate of x, and for the algebra to be consistent, it needed to be such that, for any x and y in A,

$$N(x)N(y) = N(xy). \tag{3}$$

Given that, for every real number, $x = \overline{x}$, it should be clear that such a rule trivially holds for the reals—hence the real numbers are a normed algebra. When we extend $\mathbb{R}$ into two dimensions to form the complex numbers, the norm function also exists. For example, if $x = a + bi$ and $y = c + di$ are two complex numbers, their respective conjugates are $\overline{x} = a - bi$ and $\overline{y} = c - di$. Since we know that $xy = (ac - bd) + (ad + bc)i$, it is easy to show that

$$(x \cdot \overline{x})\,(y \cdot \overline{y}) = (a^2 + b^2)\,(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2 = xy \cdot \overline{xy}$$

which, of course, is equation (3). Thus the complex numbers are also a normed algebra, and since they are comprised of real number coefficients, we call them a normed algebra over the reals.

The reason that Hamilton and friends had such a hard time coming up with a normed extension of $\mathbb{C}$ into three dimensions was that they could not find a system in which equation (3) worked. More than that, they couldn't even find a system which was closed under multiplication. For example, given two complex triples of the form in (2), multiplication would give

$$\begin{aligned}(a_0 + a_1 i + a_2 j)(b_0 + b_1 i + b_2 j) = {} & (a_0 b_0 - a_1 b_1 - a_2 b_2) + (a_0 b_1 + a_1 b_0)i \\ & + (a_0 b_2 + a_2 b_0)j + (a_1 b_2 + a_2 b_1)ij.\end{aligned}$$

But what was this mystery term in ij? Up to this point, all that was known was that i and j were independent square roots of -1, but whether their product ij should be interpreted as real or imaginary, or even nonzero, was anybody's guess.

Not surprisingly, finding a three-dimensional normed algebra was a thankless task, since even the norm looked wrong: If $z = a + bi + cj$, then clearly $\overline{z} = a - bi - cj$, but when multiplied together, they give, not $N(z) = a^2 + b^2 + c^2$ as we would hope, but the rather ugly $a^2 + b^2 + c^2 - 2ijbc$. This last rogue term was made up from two non-cancelling components: $-bcij$ and $-bcji$. It was in an attempt to force the cancellation of these terms that Hamilton was prompted to violate the commutative law by making $ij = -ji$. What followed was a realization that

$$(ij)^2 = (ij)(ij) = i(ji)j = i(-ij)j = (-ii)(jj) = -(i^2)(j^2) = -1$$

and hence that ij was in fact a third independent square root of -1. Calling this square root k resulted in the fundamental formula in (1) and the birth of the four-dimensional algebra of quaternions.

But was this algebra well-defined? And most importantly, was it a normed algebra? Well, first of all, given any two quaternions $z = a_0 + a_1 i + a_2 j + a_3 k$ and $w = b_0 + b_1 i + b_2 j + b_3 k$, their product

$$\begin{aligned} zw &= (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3) + (a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2)i \\ &\quad + (a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1)j + (a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0)k \end{aligned}$$

is another quaternion. Hence, the quaternions are closed under multiplication. That's a good start, but things got even better when Hamilton turned to the norm. Observing that the conjugate of a quaternion z is $\overline{z} = a_0 - a_1 i - a_2 j - a_3 k$ led to the satisfying result that $N(z) = a_0^2 + a_1^2 + a_2^2 + a_3^2$. Even more pleasing was the fact that

$$\begin{aligned} N(z)N(w) &= (a_0^2 + a_1^2 + a_2^2 + a_3^2)(b_0^2 + b_1^2 + b_2^2 + b_3^2) \\ &= (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3)^2 + (a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2)^2 \\ &\quad + (a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1)^2 + (a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0)^2 \\ &= N(zw) \end{aligned}$$

which, since this is equation (3), meant that he had found a third normed algebra over the reals.

It was not long after Hamilton's 1843 discovery of quaternions that a deep connection was drawn between the problem of finding real normed algebras and a completely different problem in number theory, namely to find the values of n for which a product of two sums of n squares could itself be expressed as a sum of n squares. As well as being trivially true for $n = 1$ (since $a^2b^2 = (ab)^2$), in the $n = 2$ case it had long been known that the product of two sums of two squares could be written as a sum of two squares, namely,

$$(a^2 + b^2)(c^2 + d^2) = (ac \mp bd)^2 + (ad \pm bc)^2$$

one case of which corresponds to the norm equation (3) for complex numbers. Meanwhile, in 1748, no less a mathematician than Euler had announced the $n = 4$ case [**7**, p. 6], that any product of two sums of four squares could be written as a sum of four squares, or

$$(a_0^2 + a_1^2 + a_2^2 + a_3^2)(b_0^2 + b_1^2 + b_2^2 + b_3^2)$$
$$= (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3)^2 + (a_0b_1 + a_1b_0 \pm a_2b_3 \mp a_3b_2)^2$$
$$+ (a_0b_2 \mp a_1b_3 + a_2b_0 \pm a_3b_1)^2 + (a_0b_3 \pm a_1b_2 \mp a_2b_1 + a_3b_0)^2$$

which this time yields equation (3) for quaternions. Thus we see that Euler had found the above formula 95 years before Hamilton, but without the accompanying achievement of discovering quaternions.

This naturally raises the question of whether an equivalent equation exists for products of sums of n squares when $n = 3$. But any mathematician who knew their number theory at this time would have known that, in 1798, the French mathematician Legendre had given a counterexample to answer the question in the negative. He showed that although both 3 and 21 can be written as the sum of three squares (i.e. $3 = 1^2 + 1^2 + 1^2$ and $21 = 4^2 + 2^2 + 1^2$), their product 63 cannot be partitioned into anything less than four square numbers, and thus cannot be expressed as a sum of three squares [**6**, p. 200]. Hence, a general formula of the form $(a_0^2 + a_1^2 + a_2^2)(b_0^2 + b_1^2 + b_2^2) = c_0^2 + c_1^2 + c_2^2$ cannot hold over the integers.

Hamilton, however, was not a number theorist and had never seen the work of Euler or Legendre in this area. Had he done so, he would probably have realized that his search for triple systems was hopeless and he might never have discovered quaternions. But luckily for the development of algebra, history turned out differently. In any case, a counterexample is all very well, but it doesn't shed any light on the true underlying reason Hamilton would never have been able to find a 3-dimensional normed algebra. For this, we need a proof, and for that we need a brief reminder of a few important concepts from linear algebra.

A quick linear algebra review

First and foremost, we need to remind ourselves what a *vector space* is. Basically, a vector space V is a set of objects—let's just call them "numbers"—that, when added together or multiplied by a scalar quantity (for our purposes a real number), simply give another member of V. Examples of vector spaces featured in this article are $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$, the set of quaternions. In addition to obeying many of the usual laws of arithmetic, such as having an identity element and additive inverses, a key property is that, regardless of whether the individual elements of V are commutative with respect to multiplication, scalar multipliers do commute. So, if α is a scalar, then even if x and y are members of the (non-commutative) algebra built on the vector space $\mathbb{H}$ of quaternions, for example,

$$\alpha(xy) = (\alpha x)y = (x\alpha)y.$$

Now let's consider a function $m\colon V \to W$ that maps elements from one vector space to another. If such a mapping preserves the operations of addition and scalar multiplication, in other words, given x and y in V and any scalar α, if both

- $m(x + y) = m(x) + m(y)$, and
- $m(\alpha x) = \alpha m(x)$,

then the function m is known as a *linear transformation*. One of the simplest examples of a linear transformation is the mapping $m\colon V \to V$ defined by $m(x) =$

$4x$, for which it is easy to see that $m(x+y) = 4(x+y) = 4x+4y = m(x) + m(y)$ and $m(\alpha x) = 4(\alpha x) = \alpha(4x) = \alpha m(x)$. Note however that the norm function $N\colon V \to \mathbb{R}$, which when $V = \mathbb{R}$ is $N(x) = x^2$, is nonlinear, since $N(x+y) = (x+y)^2 \neq x^2 + y^2 = N(x) + N(y)$.

It's a standard result in linear algebra that if a function mapping an n-dimensional vector space to itself is a linear transformation, then it can be represented by an $n \times n$ matrix M over the real numbers, so that for any $x \in V$, $m(x) = Mx$. So, if $V = \mathbb{R}^3$, then in our example above $m(x)$ would be equivalent to multiplication by the 3×3 matrix

$$M = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

Given this (or any $n \times n$) matrix, together with the identity matrix, I, and some unknown real number λ, we can construct the determinant

$$f(\lambda) = \det(\lambda I - M),$$

which, in the case of the matrix M above, gives

$$\begin{vmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda - 4 & 0 \\ 0 & 0 & \lambda - 4 \end{vmatrix} = \lambda^3 - 12\lambda^2 + 48\lambda + 64.$$

In general, for an $n \times n$ matrix this will give a polynomial of degree n with real coefficients, called the *characteristic polynomial*. Setting this equal to zero and solving gives the value (or values) of λ, known as the matrix's *eigenvalues*, $\lambda = 4$ in the above case. This means that multiplying some nonzero x by the matrix M has the same effect as scalar multiplication by λ, or

$$Mx = \lambda x,$$

and, since multiplication by the matrix M is equivalent to the effect of the linear transformation m,

$$m(x) = \lambda x.$$

But how can we guarantee that we will always be able to find a value of λ? Is it true that every characteristic equation $f(\lambda) = 0$ will have at least one root? Fortunately, thanks to the *Fundamental Theorem of Algebra*, the answer to this question is yes. Every polynomial equation

$$f(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0 = 0$$

where $a_n, a_{n-1}, \ldots, a_1, a_0 \in \mathbb{R}$ can be written as a product of linear and/or irreducible quadratic factors over the real numbers. So, for example, the cubic equation

$$\lambda^3 - 1 = 0$$

can be written as

$$(\lambda - 1)(\lambda^2 + \lambda + 1) = 0,$$

yielding the real solution $\lambda = 1$. This now gives us everything we need to prove that there was no way that Hamilton could ever have found a consistent algebra of complex number triples.

The "You can't multiply triples" Theorem

Theorem 1. *A normed algebra of dimension 3 over the real numbers $\mathbb{R}$ does not exist.*

Proof. Let A be a three-dimensional real algebra with norm N. We identify $\mathbb{R}$ as a one-dimensional subspace of A and let $b \in A - \mathbb{R}$. Define the mapping $m_b \colon A \to A$ by $m_b(x) = bx$. Since multiplication distributes over addition, we see that

$$m_b(x+y) = b(x+y) = bx + by = m_b(x) + m_b(y).$$

Since multiplication of elements of A is associative and commutes with (scalar) multiplication, then if r is a real number,

$$m_b(rx) = b(rx) = (br)x = r(bx) = rm_b(x).$$

Hence m_b is a linear transformation, and so it has a representation as a 3×3 matrix M over $\mathbb{R}$.

The characteristic polynomial $f(\lambda) = \det(\lambda I - M)$ has real coefficients and is of degree 3. By the Fundamental Theorem of Algebra, it can therefore be written as a product of linear and irreducible quadratic factors over the real numbers. Since $f(\lambda)$ is a cubic, this product can only comprise one linear and one quadratic factor or three linear factors. Either way, $f(\lambda)$ must have a linear factor $\lambda - r$ for some nonzero real number r.

Thus, r is an eigenvalue of M and so $bv = rv$ for some nonzero $v \in A$. Hence, $(b-r)v = 0$.

But $b - r$ is nonzero (recall that $b \in A - \mathbb{R}$), and so is v. Hence, by the definition of the norm function, $N(b-r)$ and $N(v)$ must both be nonzero. Finally, by use of equation (3),

$$0 = N(0) = N((b-r)v) = N(b-r)N(v) \neq 0,$$

contrary to the assumption that A is a normed algebra. ■

Final remarks

We finish with a few brief closing remarks and observations.

- Firstly, we leave it as an exercise for the reader to extend the above proof to show that, for $n \geq 3$, odd-dimensional real normed algebras do not exist.
- Secondly, it should probably be pointed out that, although the linear algebraic concepts used in our proof all existed in various forms in Hamilton's time, the subject of linear algebra was still many years away from the fully rigorous and systematized discipline it is today. Therefore, even if he had been motivated to do so, it is highly unlikely that Hamilton would have been able to come up with a proof as succinct as the one we have presented.
- As well as real numbers, complex numbers and quaternions, a further consistent normed algebra over the reals can be found with $n = 8$ dimensions. As well as being non-commutative with respect to multiplication, this algebra, known as the octonions, was the first example of an algebra whose multiplication is non-associative [**1**]. And since it is normed, it also provided a version of equation (3) expressing any product of two sums of eight squares as a sum of eight squares [**3**].

- The question of when a product of two sums of n squares could itself be expressed as a sum of n squares was finally resolved in 1898 by the German mathematician Adolf Hurwitz [**4**], who proved that such expressions exist if and only if $n = 1, 2, 4,$ or 8. Thus, consistent n-dimensional normed algebras over the reals can only occur for those same values.
- If you would like to know why n cannot have any other values, please see our accompanying paper, "An Accessible Proof of Hurwitz's Sums of Squares Theorem" in *Mathematics Magazine*, in which we give a proof, intelligible to any student who has taken a first course in linear algebra, that n must equal 1, 2, 4, or 8.

But our proof for the $n = 3$ case is perhaps the simplest proof of all and provides possibly the best explanation for just why it was that Hamilton couldn't multiply triples.

Summary. The history of the discovery of the 4-dimensional algebra of quaternions by William Rowan Hamilton is very well known, but one aspect of the story has received far less attention. This is the question of why he was unable to create a coherent system of complex numbers in 3-space. In fact, even after Hamilton discovered quaternions in 1843, it still took over half a century before anyone was able to prove that no such triple system actually existed. But what is the mathematical reason that Hamilton was unable to find a consistent three-dimensional linear algebra? The purpose of this article is to find out.

References

1. John C. Baez, The octonions. *Bull. Amer. Math. Soc.* **39** (2002) 145–205.
2. Michael J. Crowe, *A history of vector analysis: the evolution of the idea of a vectorial system*, University of Notre Dame Press, Notre Dame, 1967.
3. Leonard E. Dickson, On quaternions and their generalization and the history of the eight square theorem, *Ann. Math.* **20** (1919) 155–171.
4. Adolf Hurwitz, Über die Komposition der quadratischen Formen von beliebig vielen Variablen, *Göttinger Nachrichten* (1898) 309–316 [= *Mathematische Werke* **2** 565–571].
5. Victor J. Katz and Karen H. Parshall, *Taming the unknown: A history of algebra from antiquity to the early twentieth century*, Princeton University Press, Princeton and Oxford, 2014.
6. Adrien-Marie Legendre, *Essai sur la théorie des nombres*, Duprat, Paris, 1798.
7. Adrian Rice and Ezra Brown, Commutativity and collinearity: a historical case study of the interconnection of mathematical ideas. Part I, *BSHM Bulletin: Jour. Brit. Soc. Hist. Math.* **31** (2016) 1–14.
8. Bartel L. van der Waerden, Hamilton's discovery of quaternions. *Math. Mag.* **49** (1976) 227–234.