Structure-Preserving Generalized Balancing for Nonlinear Port-Hamiltonian Systems

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Overview

Model reduction

System:

$$\dot{x}(t) = f(t, x, u)$$
$$y(t) = h(t, x, u), \quad x(t) \in \mathbb{R}^n, y(t) \in \mathbb{R}^p$$

Approximated/reduced order system:

$$\dot{\hat{x}}(t) = \hat{f}(t, \hat{x}, u)$$
$$\hat{y}(t) = \hat{h}(t, \hat{x}, u), \quad \hat{x}(t) \in \mathbb{R}^r, \hat{y}(t) \in \mathbb{R}^p$$

where $r << n$.

Criteria

- Small approximation error.
- Possible error bound.
- Preservation of structural properties e.g. stability, passivity, symmetry.
- Computational efficiency.

Linear balancing

Linear system :

$$\dot{x} = Ax + Bu \qquad (1)$$
$$y = Cx$$

Controllability Gramian:

$$W_{gc} = \int_0^\infty e^{A\tau} BB^\top e^{A^\top \tau} d\tau$$

Observability Gramian:

$$W_{go} = \int_0^\infty e^{A^\top \tau} C^\top C e^{A\tau} d\tau$$

$$A^\top W_{go} + W_{go} A + C^\top C = 0 \qquad (2)$$
$$AW_{gc} + W_{gc} A^\top + BB^\top = 0$$

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Linear balancing

Let $T \in \mathbb{R}^{n\times n}$ is an invertible matrix s.t.

$$T^\top W_{gc} T = T^{-1} W_{go} T^{-\top} = \Sigma, \tag{3}$$

$$\Sigma = \{\sigma_1, \sigma_2, \cdots, \sigma_n\},$$

where $\sigma_1 \geq \sigma_2 \geq, \cdots, \geq \sigma_n$.

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} u$$

$$\bar{y} = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

Reduced-order model:

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u \\ \hat{y} &= \hat{C}\hat{x}, \end{aligned} \tag{4}$$

where, $\hat{A} = \bar{A}_{11}$, $\hat{B} = \bar{B}_1$ and $\hat{C} = \bar{C}_1$.

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Controllability and Observability functions

Nonlinear system:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{5}$$

Controllability function :

$$L_c(x_0) := \min_{\substack{u \in L_2(-\infty, 0), \\ x(-\infty)=0, x(0)=x_0}} \frac{1}{2} \int_{-\infty}^{0} ||u(t)||^2 dt,$$

Observability function :

$$L_o(x_0) := \frac{1}{2} \int_0^{\infty} ||y(t)||^2 dt, x(0) = x_0, u(t) \equiv 0, 0 \leq t < \infty.$$

For linear system →

$$L_c(x_0) = \frac{1}{2} x_0^\top W_{gc}^{-1} x_0 \quad L_o(x_0) = \frac{1}{2} x_0^\top W_{go} x_0 \tag{6}$$

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Controllability and Observability functions

Theorem

[a] *If 0 is an asymptotically stable equilibrium of $f(x)$ on a neighbourhood U of 0, then for all $x \in U$, $L_o(x)$ is the unique smooth solution of*

$$\frac{\partial L_o}{\partial x} f(x) + \frac{1}{2} h^\top(x) h(x) = 0, \quad L_o(0) = 0 \tag{7}$$

Moreover, for all $x \in U$, $L_c(x)$ is the smooth solution of

$$\frac{\partial L_c(x)}{\partial x} f(x) + \frac{1}{2} \frac{\partial L_c(x)}{\partial x} g(x) g^\top(x) \frac{\partial^\top L_c(x)}{\partial x} = 0, \quad L_c(0) = 0 \tag{8}$$

such that 0 is an asymptotically stable equilibrium of $-(f(x) + g(x)g^\top(x)\frac{\partial^\top L_c(x)}{\partial x})$ on U.

[a] J.M.A. Scherpen, "Balancing for nonlinear systems", In Systems and Control Letters, vol. 21, no. 2, pp. 143-153, 1993.

Generalized observability and controllability functions

If $\tilde{L}_o(x) > 0$ is a smooth solution of

$$\frac{\partial \tilde{L}_o(x)}{\partial x}(x) f(x) + \frac{1}{2} h^\top(x) h(x) \leq 0, \quad \tilde{L}_o(0) = 0, \tag{9}$$

$$L_o(x_0) \leq \tilde{L}_o(x_0). \tag{10}$$

If $\tilde{L}_c(x) > 0$ is a smooth solution of

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$$L_c(x_0) \geq \tilde{L}_c(x_0). \tag{12}$$

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Port-Hamiltonian structure

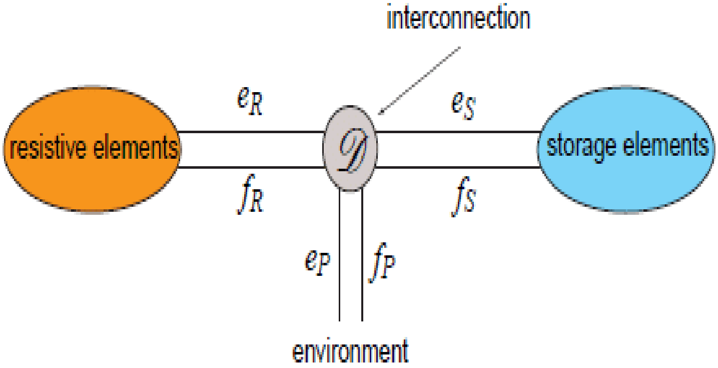

Figure 1: Port-Hamiltonian structure

Port-Hamiltonian Systems

Input-state-output nonlinear port-Hamiltonian (PH) system :

$$\Sigma_{PH} : \begin{cases} \dot{x} = (J(x) - R(x))\dfrac{\partial \mathcal{H}(x)}{\partial x} + g(x)u \\ y = g^\top(x)\dfrac{\partial \mathcal{H}(x)}{\partial x} \end{cases} \tag{13}$$

What is the goal now?

Propose a balanced realization of the port-Hamiltonian system in which the generalized controllability function $\tilde{L}_c$ and generalized observability function $\tilde{L}_o$ are truly balanced and the Hamiltonian $\mathcal{H}$ is also in diagonal form as well.

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Literature

- K. Fujimoto, Balanced realization and model order reduction for port-Hamiltonian systems, Journal of System Design and Dynamics 2 (3) (2008) 694–702.
- C. Beattie and S. Gugercin, "Structure-preserving model reduction for nonlinear port-Hamiltonian systems," 2011 50th IEEE Conference on Decision and Control and European Control Conference, Orlando, FL, USA, 2011, pp. 6564-6569, doi: 10.1109/CDC.2011.6161504.
- Y. Kawano and J. M. A. Scherpen, "Structure Preserving Truncation of Nonlinear Port Hamiltonian Systems," in IEEE Transactions on Automatic Control, vol. 63, no. 12, pp. 4286-4293, Dec. 2018, doi: 10.1109/TAC.2018.2811787.
- Beattie, C., Gugercin, S., Mehrmann, V. (2022). Structure-Preserving Interpolatory Model Reduction for Port-Hamiltonian Differential-Algebraic Systems. In: Beattie, C., Benner, P., Embree, M., Gugercin, S., Lefteriu, S. (eds) Realization and Model Reduction of Dynamical Systems. Springer, Cham.

Necessary assumptions

We define $F(x) := (J(x) - R(x))$

Assumption

For a nonlinear port-Hamiltonian system, assume the following holds

- 0 *is an asymptotically stable equilibrium of* $F(x)\frac{\partial \mathcal{H}(x)}{\partial x}$ *on some neighbourhood* U *of* 0.
- *The linearized system at the origin is asymptotically stable.*
- 0 *is an asymptotically stable equilibrium of* $-(F(x)\frac{\partial \mathcal{H}(x)}{\partial x} + g(x)g^\top(x)\frac{\partial^\top \tilde{L}_c(x)}{\partial x})$ *on* U.
- $\tilde{L}_o$ *and* $\tilde{L}_c$ *are smooth on* U.
- $\frac{\partial^2 \tilde{L}_c}{\partial x^2}(0) \succ 0$, $\frac{\partial^2 \tilde{L}_o}{\partial x^2}(0) \succ 0$ *and* $\frac{\partial^2 \mathcal{H}}{\partial x^2}(0) \succ 0$.

PH structure-preserving generalized balancing

Lemma

There exists a coordinate transformation $\phi : \mathbb{R}^n \to \mathbb{R}^n$ *such that* $x = \phi(z)$, $\phi(0) = 0$ *(defined on a neighbourhood of the origin) and* $\tilde{L}_c(x)$ *in the new coordinates has the following form*

$$\tilde{L}_c(\phi(z)) = \frac{1}{2} z^\top z. \tag{14}$$

Moreover, we can also write $\tilde{L}_o(x)$ *and* $\mathcal{H}(x)$ *in the new coordinates in the following forms*

$$\begin{aligned} \tilde{L}_o(\phi(z)) &= \frac{1}{2} z^\top M(z) z, \quad M(0) = \frac{\partial^2 \tilde{L}_o}{\partial x^2}(0), \\ \mathcal{H}(\phi(z)) &= \frac{1}{2} z^\top H(z) z, \quad H(0) = \frac{\partial^2 \mathcal{H}}{\partial x^2}(0), \end{aligned} \tag{15}$$

where $M(z)$ *and* $H(z)$ *are* $n \times n$ *symmetric matrices with entries which are smooth functions of* z.

PH structure-preserving generalized balancing

Assumption

There exists $\tilde{L}_c(x), \tilde{L}_o(x) > 0$ *such that the eigenvalues of* $\frac{\partial^2 \tilde{L}_c}{\partial x^2}(0)^{-1}\frac{\partial^2 \tilde{L}_o}{\partial x^2}(0)$ *as well as the eigenvalues of* $\frac{\partial^2 \mathcal{H}}{\partial x^2}(0)^{-1}\frac{\partial^2 \tilde{L}_c}{\partial x^2}(0)^{-1}\frac{\partial^2 \tilde{L}_o}{\partial x^2}(0)$ *are distinct.*

Lemma

If there exists a neighbourhood V *of* 0 *where the number of distinct eigenvalues of* $M(z)$, $H(z)$ *and* $H^{-1}(z)M(z)$ *is constant for* $z \in V$, *then on* V *the eigenvalues of* $M(z)$ *and* $H(z)$ *are smooth functions of* z *along with the associated eigenvectors.*

PH structure-preserving generalized balancing

Assumption

There exists $\tilde{L}_c(x), \tilde{L}_o(x) > 0$ *such that the eigenvalues of* $\frac{\partial^2 \tilde{L}_c}{\partial x^2}(0)^{-1}\frac{\partial^2 \tilde{L}_o}{\partial x^2}(0)$ *as well as the eigenvalues of* $\frac{\partial^2 \mathcal{H}}{\partial x^2}(0)^{-1}\frac{\partial^2 \tilde{L}_c}{\partial x^2}(0)^{-1}\frac{\partial^2 \tilde{L}_o}{\partial x^2}(0)$ *are distinct.*

Lemma

If there exists a neighbourhood V *of* 0 *where the number of distinct eigenvalues of* $M(z)$, $H(z)$ *and* $H^{-1}(z)M(z)$ *is constant for* $z \in V$, *then on* V *the eigenvalues of* $M(z)$ *and* $H(z)$ *are smooth functions of* z *along with the associated eigenvectors.*

PH structure-preserving generalized balancing

Theorem

Consider the system (13) *and assume that Assumption 2 and Lemma 3 are fulfilled. Then, on a neighbourhood U of* 0 *there exists a transformation $\psi : \mathbb{R}^n \to \mathbb{R}^n$ such that $x = \psi(\bar{z}), \psi(0) = 0$ and $\tilde{L}_c(x)$ in the new coordinates $\bar{z} \in W := \psi^{-1}(U)$ takes the form*

$$\tilde{L}_c(\psi(\bar{z})) = \frac{1}{2}\bar{z}^\top \bar{z}. \tag{16}$$

Moreover, $\tilde{L}_o$ and $\mathcal{H}$ in the new coordinates are of the following form

$$\begin{aligned} \tilde{L}_o(\phi(\bar{z})) &= \frac{1}{2}\bar{z}^\top \begin{bmatrix} \bar{\lambda}_1(\bar{z}) & & 0 \\ & \ddots & \\ 0 & & \bar{\lambda}_n(\bar{z}) \end{bmatrix} \bar{z}, \\ \mathcal{H}(\phi(\bar{z})) &= \frac{1}{2}\bar{z}^\top \begin{bmatrix} \bar{\kappa}_1(\bar{z}) & & 0 \\ & \ddots & \\ 0 & & \bar{\kappa}_n(\bar{z}) \end{bmatrix} \bar{z}, \end{aligned} \tag{17}$$

where $\bar{\lambda}_1(\bar{z}) \geq \bar{\lambda}_2(\bar{z}) \geq \cdots \geq \bar{\lambda}_n(\bar{z})$ and $\bar{\kappa}_1(\bar{z}) \geq \bar{\kappa}_2(\bar{z}) \geq \cdots \geq \bar{\kappa}_n(\bar{z})$ are smooth functions of z.

PH structure-preserving generalized balancing

Theorem

Suppose that the assumptions hold. Then there exists a transformation $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ *on a neighbourhood U of the origin such that* $x = \Phi(z), \Phi(0) = 0$ *which converts the system into a new realization where the following holds*

$$\begin{aligned} \tilde{L}_c(\Phi(z)) &= \frac{1}{2} z^\top z, \\ \tilde{L}_o(\Phi(z)) &= \frac{1}{2} \sum_{i=1}^{n} (z_i \sigma_i(z_i))^2, \\ \mathcal{H}(\Phi(z)) &= \frac{1}{2} \sum_{i=1}^{n} z_i^2 \eta_i(z_i), \end{aligned} \tag{18}$$

where $\sigma_1(z_1) \geq \sigma_2(z_2) \geq \cdots \geq \sigma_n(z_n)$ *and* $\eta_1(z_1) \geq \eta_2(z_2) \geq \cdots \geq \eta_n(z_n)$ *are smooth functions.*

Proof$\rightarrow$ We can prove that there exist a smooth transformation for n=2. Then we can generalize for any n via mathematical induction.

PH structure-preserving generalized balancing

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Proof$\rightarrow$ We can prove that there exist a smooth transformation for n=2. Then we can generalize for any n via mathematical induction.

PH structure-preserving generalized balancing

Theorem

Suppose that assumptions hold. Then there exists a transformation $\bar{\Phi} : \mathbb{R}^n \to \mathbb{R}^n$ on a neighbourhood U of the origin such that $x = \bar{\Phi}(\bar{z}), \bar{\Phi}(0) = 0$ which converts the system into a new realization where the following holds

$$\begin{aligned} \tilde{L}_c(\bar{\Phi}(\bar{z})) &= \frac{1}{2}\sum_{i=1}^{n} \frac{\bar{z}_i^2}{\bar{\sigma}_i(\bar{z}_i)}, \\ \tilde{L}_o(\bar{\Phi}(\bar{z})) &= \frac{1}{2}\sum_{i=1}^{n} \bar{z}_i^2 \bar{\sigma}_i(\bar{z}_i), \\ \mathcal{H}(\bar{\Phi}(\bar{z})) &= \frac{1}{2}\sum_{i=1}^{n} z_i^2 \frac{\bar{\eta}_i(\bar{z}_i)}{\bar{\sigma}_i(\bar{z}_i)}, \end{aligned} \tag{19}$$

where $\bar{\sigma}_1(\bar{z}_1) \geq \bar{\sigma}_2(\bar{z}_2) \geq \cdots \geq \bar{\sigma}_n(\bar{z}_n)$ and $\bar{\eta}_1(\bar{z}_1) \geq \bar{\eta}_2(\bar{z}_2) \geq \cdots \geq \bar{\eta}_n(\bar{z}_n)$ are smooth functions.

Proof$\to$ We can apply a coordinate transformation $z = \Phi^{-1} \circ \bar{\Phi}(\bar{z}) := (\bar{\phi}_1(\bar{z}_1), \bar{\phi}_1(\bar{z}_2), \cdots, \bar{\phi}_1(\bar{z}_n))$ with $\bar{z}_i = \bar{\phi}_i^{-1}(z_i) := z_i\sqrt{\sigma_i(z_i)}$ to the system in z coordinates..

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Suppose that assumptions hold. Then there exists a transformation $\bar{\Phi} : \mathbb{R}^n \to \mathbb{R}^n$ on a neighbourhood U of the origin such that $x = \bar{\Phi}(\bar{z}), \bar{\Phi}(0) = 0$ which converts the system into a new realization where the following holds

$$\begin{aligned} \tilde{L}_c(\bar{\Phi}(\bar{z})) &= \frac{1}{2}\sum_{i=1}^{n} \frac{\bar{z}_i^2}{\bar{\sigma}_i(\bar{z}_i)}, \\ \tilde{L}_o(\bar{\Phi}(\bar{z})) &= \frac{1}{2}\sum_{i=1}^{n} \bar{z}_i^2 \bar{\sigma}_i(\bar{z}_i), \\ \mathcal{H}(\bar{\Phi}(\bar{z})) &= \frac{1}{2}\sum_{i=1}^{n} z_i^2 \frac{\bar{\eta}_i(\bar{z}_i)}{\bar{\sigma}_i(\bar{z}_i)}, \end{aligned} \tag{19}$$

where $\bar{\sigma}_1(\bar{z}_1) \geq \bar{\sigma}_2(\bar{z}_2) \geq \cdots \geq \bar{\sigma}_n(\bar{z}_n)$ and $\bar{\eta}_1(\bar{z}_1) \geq \bar{\eta}_2(\bar{z}_2) \geq \cdots \geq \bar{\eta}_n(\bar{z}_n)$ are smooth functions.

Proof$\rightarrow$ We can apply a coordinate transformation $z = \Phi^{-1} \circ \bar{\Phi}(\bar{z}) := (\bar{\phi}_1(\bar{z}_1), \bar{\phi}_1(\bar{z}_2), \cdots, \bar{\phi}_1(\bar{z}_n))$ with $\bar{z}_i = \bar{\phi}_i^{-1}(z_i) := z_i\sqrt{\sigma_i(z_i)}$ to the system in z coordinates..

Reduced order PH model

In $\bar{z}$ coordinates, the state-space representation of (13) reads

$$\bar{\Sigma}_{PH}: \begin{cases} \dot{\bar{z}} = (J_{\bar{z}}(\bar{z}) - R_{\bar{z}}(\bar{z}))\dfrac{\partial \mathcal{H}_{\bar{z}}}{\partial \bar{z}} + g_{\bar{z}}(\bar{z})u, \\ y = g_{\bar{z}}^{\top}(\bar{z})\dfrac{\partial \mathcal{H}_{\bar{z}}}{\partial \bar{z}}, \end{cases} \tag{20}$$

where

$$\mathcal{H}_{\bar{z}}(\bar{\Phi}^{-1}(x)) = \mathcal{H}(x), \quad J_{\bar{z}}(\bar{\Phi}^{-1}(x)) = \frac{\partial^{\top}\bar{\Phi}^{-1}(x)}{\partial x} J(x) \frac{\partial \bar{\Phi}^{-1}(x)}{\partial x},$$

$$R_{\bar{z}}(\bar{\Phi}^{-1}(x)) = \frac{\partial^{\top}\bar{\Phi}^{-1}(x)}{\partial x} R(x) \frac{\partial \bar{\Phi}^{-1}(x)}{\partial x},$$

$$g_{\bar{z}}(\bar{\Phi}^{-1}(x)) = \frac{\partial^{\top}\bar{\Phi}^{-1}(x)}{\partial x} g(x).$$

Note that $J_{\bar{z}}(\bar{z}) = -J_{\bar{z}}^{\top}(\bar{z})$, $R_{\bar{z}}(\bar{z}) = R_{\bar{z}}^{\top}(\bar{z}) \succeq 0$, $\mathcal{H}_{\bar{z}}(\bar{z}) \succ 0$, $\frac{\partial \mathcal{H}_{\bar{z}}}{\partial \bar{z}}(0) = 0$ and $\frac{\partial^2 \mathcal{H}_{\bar{z}}}{\partial \bar{z}^2}(0) \succ 0$.

Reduced order PH model

We split $\bar{z} = [\bar{z}_r^\top, \bar{z}_t^\top]^\top$, where $\bar{z}_r = [\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_k]^\top \in \mathbb{R}^n$ and $\bar{z}_t = [\bar{z}_{k+1}, \bar{z}_{k+2}, \cdots, \bar{z}_n]^\top \in \mathbb{R}^{n-k}$. Similarly,

$$J_{\bar{z}} = \begin{bmatrix} J_{\bar{z},rr}(\bar{z}_r, \bar{z}_t) & J_{\bar{z},rt}(\bar{z}_r, \bar{z}_t) \\ -J_{\bar{z},rt}^\top(\bar{z}_r, \bar{z}_t) & J_{\bar{z},tt}(\bar{z}_r, \bar{z}_t) \end{bmatrix},$$

$$R_{\bar{z}} = \begin{bmatrix} R_{\bar{z},rr}(\bar{z}_r, \bar{z}_t) & R_{\bar{z},rt}(\bar{z}_r, \bar{z}_t) \\ R_{\bar{z},rt}(\bar{z}_r, \bar{z}_t) & R_{\bar{z},tt}(\bar{z}_r, \bar{z}_t) \end{bmatrix},$$

$$g_{\bar{z}} = \begin{bmatrix} g_{\bar{z},r}(\bar{z}_r, \bar{z}_t) \\ g_{\bar{z},t}(\bar{z}_r, \bar{z}_t) \end{bmatrix},$$

where $J_{\bar{z},rr}(\bar{z}_r, \bar{z}_t)$ and $J_{\bar{z},tt}(\bar{z}_r, \bar{z}_t)$ are skew-symmetric, $R_{\bar{z},rr}(\bar{z}_r, \bar{z}_t)$ and $R_{\bar{z},tt}(\bar{z}_r, \bar{z}_t)$ are symmetric and positive semidefinite.

Reduced-order PH model

Theorem

Consider a continuous-time nonlinear input-state-output port-Hamiltonian system Σ_{PH}*. Suppose that the assumptions are satisfied and we obtain a balanced realization of the system as in* (19)*. Then a reduced-order model can be represented as follows*

$$\Sigma_r : \begin{cases} \dot{\bar{z}}_r = (J_{\bar{z},rr}(\bar{z}_r, 0) - R_{\bar{z},rr}(\bar{z}_r, 0))\dfrac{\partial \mathcal{H}_{\bar{z}}(\bar{z}_r, 0)}{\partial \bar{z}_r} + g_{\bar{z},r}(\bar{z}_r, 0)u, \\ y_r = g_{\bar{z},r}^{\top}(\bar{z}_r, 0)\dfrac{\partial \mathcal{H}_{\bar{z}}(\bar{z}_r, 0)}{\partial \bar{z}_r}, \end{cases} \tag{21}$$

which is also a port-Hamiltonian system with the Hamiltonian $\mathcal{H}_{\bar{z}}(\bar{z}_r, 0)$*.*

Special case

Let us consider the following nonlinear port-Hamiltonian system

$$\Sigma_{PH} : \begin{cases} \dot{x} = (J(x) - R(x)) \dfrac{\partial \mathcal{H}(x)}{\partial x} + Bu, \\ y = B^\top \dfrac{\partial \mathcal{H}(x)}{\partial x}, \end{cases} \tag{22}$$

Consider $\mathcal{H}(x) = \frac{1}{2} x^\top H x$, where $H = H^\top \succ 0$.
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Generalized balancing via linear transformation

If there exist constant matrices $P \succ 0$ and $Q \succ 0$ which satisfy

$$QF(x)H + HF^\top(x)Q + HBB^\top H \preceq 0 \tag{23}$$

and

$$F(x)HP + PHF^\top(x) + BB^\top \preceq 0 \tag{24}$$

respectively for all $x \in \mathbb{R}^n$, then we can find an invertible matrix $W \in \mathbb{R}^n$ which transforms the system to generalized balanced coordinates in which

$$W^\top PW = W^{-1}QW^{-\top} = \Lambda_{PQ} \tag{25}$$

such that $\Lambda_{PQ} = \text{diag}\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$, where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$. Note that $\tilde{L}_o(x) = \frac{1}{2}x^\top Qx$ and $\tilde{L}_c(x) = \frac{1}{2}x^\top P^{-1}x$ satisfy (9) and (11) respectively.

PH structure-preserving balancing via linear transformation

Let $P \succ 0$ be a solution to (24). Consider a full rank matrix $\phi_P \in \mathbb{R}^{n \times n}$ such that

$$P = \phi_P^\top \phi_P,$$
$$\phi_P H \phi_P^\top = U_{HP} \Lambda_{HP} U_{HP}^\top,$$

Define

$$\mathcal{F}_c(x) := U_{HP}^\top \phi_P^{-\top} F(x) \phi_P^{-1} U_{HP},$$
$$\mathcal{B}_c := U_{HP}^\top \phi_P^{-\top} B. \tag{26}$$

If

$$\Lambda_{PQ}^2 \Lambda_{HP}^{-1} \mathcal{F}_c(x) + \mathcal{F}_c^\top(x) \Lambda_{HP}^{-1} \Lambda_{PQ}^2 + \mathcal{B}_c \mathcal{B}_c^\top \preceq 0 \tag{27}$$

holds for a diagonal matrix Λ_{PQ} for all $x \in \mathbb{R}^n$, then

$$Q = \phi_P^{-1} U_{HP} \Lambda_{PQ}^2 U_{HP}^\top \phi_P^{-\top} \tag{28}$$

is a solution of (23). Moreover, the linear transformation

$$W_{spc} = \phi_P^\top U_{HP} \Lambda_{PQ}^{-\frac{1}{2}} \tag{29}$$

balances the system and diagonalizes H.

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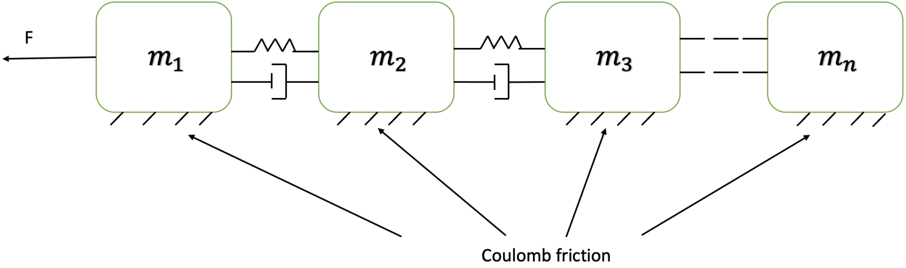
Algorithm

1. Find a positive definite matrix P.
2. Find a diagonal matrix Λ_{PQ}.
3. Compute Q.
4. Transform the system into the balanced coordinates using W_{spc} in which $\bar{H}$ is positive definite and diagonal.
5. Discard the states corresponding to small values of σ_is i.e. the diagonal entries of Λ_{PQ} to arrive at a reduced order model which is also a port-Hamiltonian system.

Example



Frictional nonlinearity

Coulomb friction :

$$F_i = \begin{cases} -\delta_i, & \dot{q}_i > 0, \\ [-\delta_i, \delta_i], & \dot{q}_i = 0, \\ \delta_i, & \dot{q}_i < 0. \end{cases}$$

Smooth approximation :

$$F_i = -\frac{\delta_i \dot{q}_i}{\sqrt{\gamma_i + \dot{q}_i^2}}$$

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Dynamics of the nonlinear mass-spring-damper system

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \underbrace{\begin{bmatrix} 0 & I_n \\ -I_n & -(D+R) \end{bmatrix}}_{\text{(J - R)}} \underbrace{\begin{bmatrix} K & 0 \\ 0 & M^{-1} \end{bmatrix}}_{\text{H}} \underbrace{\begin{bmatrix} q \\ p \end{bmatrix}}_{\text{x}} + \underbrace{\begin{bmatrix} 0 \\ G \end{bmatrix}}_{\text{B}} u \\ y &= \begin{bmatrix} 0 & G^\top \end{bmatrix} \begin{bmatrix} K & 0 \\ 0 & M^{-1} \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} \end{aligned} \tag{30}$$

$$R = \text{diag}\left\{ \frac{\delta_i m_i}{\sqrt{\gamma_i m_i^2 + p_i^2}} \right\} \geq 0,$$

for $i = 1, 2, \cdots, n$.

How to solve the inequalities?

Let $\mathbf{F} = \{F_0, F_1, \cdots, F_n\}$ be a finite set of matrices with

$$F_0 = \begin{bmatrix} 0_n & I_n \\ -I_n & -D \end{bmatrix},$$
$$F_i = \begin{bmatrix} 0_n & I_n \\ -I_n & -(D + \text{diag}(0, \cdots, \frac{\delta_i}{m_i\sqrt{\gamma_i}}, \cdots, 0)) \end{bmatrix},$$

where $i = 1, 2, \cdots, n$. The i^{th} diagonal entry of F_i should be $\frac{\delta_i}{m_i\sqrt{\gamma_i}}, i = \{1, 2, \cdots, n\}$ and every other diagonal entry is zero. Now, we can consider $F(x) = (J(x) - R(x))$ satisfies the following inclusion

$$F(x) \in \text{ConvexHull}(\mathbf{F})$$

for all $x \in \mathbb{R}^n$.

Simulation Results

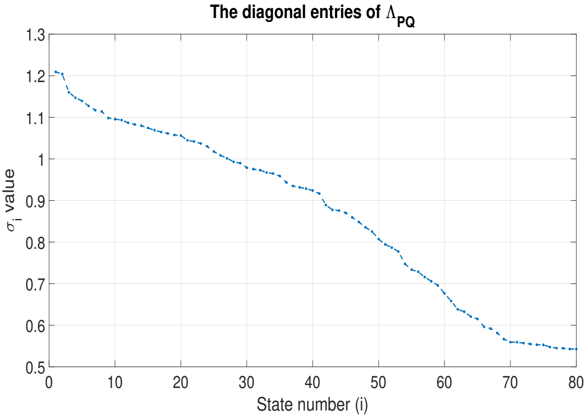

Figure 2: Diagonal entries of Λ_{PQ} depicting the importance of the state variables in balanced coordinates

Simulation results

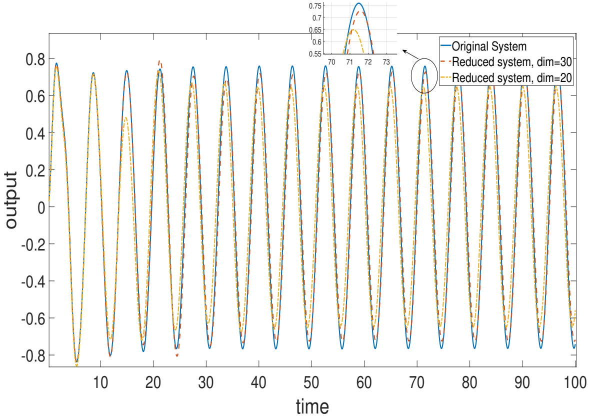

Figure 3: Comparison of output trajectories of original and reduced order model

Outlook & Conclusion

- Generalized balancing is a framework for nonlinear balancing which provides flexibility to preserve port-Hamiltonian structure using the generalized controllability and observability functions.
- For special cases of nonlinear pH systems, the algorithm can also computationally tractable.

Future directions:

- Computational tractability of the approach for the generic case.
- Possible apriori error bound based on Lipschitz type of assumptions on the drift vector field.

Ongoing:

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- Utilization of generalized differential balancing to preserve monotonicity(generalization of positivity for nonlinear systems) of nonlinear systems.

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Thank You!