Universität Stuttgart

SimTech Cluster of Excellence

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On multi-objective model reduction of port-Hamiltonian systems

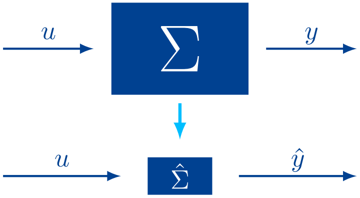
joint work with P. Schwerdtner and B. Unger

Nonlinear Model Reduction for Control
May 25, 2023

System theoretical model order reduction

$$\Sigma \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \qquad A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m}, \quad C \in \mathbb{R}^{p \times n}$$

System theoretical model order reduction



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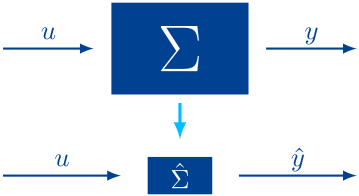
$$\hat{\Sigma} \begin{cases} \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u & \hat{A} \in \mathbb{R}^{r \times r}, \quad \hat{B} \in \mathbb{R}^{r \times m}, \\ \hat{y} = \hat{C}\hat{x} & \hat{C} \in \mathbb{R}^{p \times r} \end{cases}$$

System theoretical model order reduction



$$\Sigma \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \qquad A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \quad C \in \mathbb{R}^{p\times n}$$

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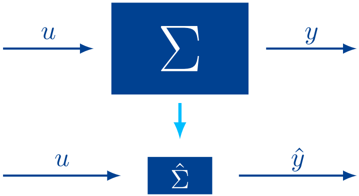
$$\mathcal{L}(\Sigma) \begin{cases} sX = AX + BU \\ Y = CX \end{cases} \quad \rightsquigarrow X = (sI_n - A)^{-1}BU$$

$$Y = \underbrace{C(sI_n - A)^{-1}B}_{\Sigma(s)}U$$

System theoretical model order reduction



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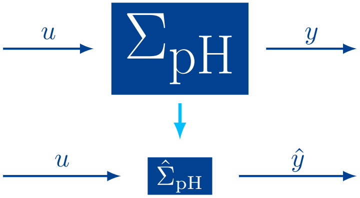
$$\Sigma(s) := C(sI_n - A)^{-1}B \qquad \hat{\Sigma}(s) := \hat{C}(sI_r - \hat{A})^{-1}\hat{B}$$

$$\left\| \Sigma - \hat{\Sigma} \right\|_{\mathcal{H}_2}^2 := \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\| \Sigma(\imath\omega) - \hat{\Sigma}(\imath\omega) \right\|_{\mathrm{F}}^2 \mathrm{d}\omega$$

Structure-preserving model order reduction



Port-Hamiltonian systems

- close to physics
- invariant under coupling
- use energy as lingua franca

V. Mehrmann and B. Unger.
Control of port-Hamiltonian differential-algebraic systems and applications.
ArXiv e-print 2201.06590, 2022.

Introduction to pH systems

1

Derivation of the pH class

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t)$$

Key properties - part I

- use energy as lingua franca
- close to physics

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Derivation of the pH class

$$\dot{x}(t) = J\nabla\mathcal{H}(x(t)) + Bu(t),$$
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Key properties - part I

- use energy as lingua franca ✓
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- Structure operator: $J^\mathsf{T} = -J$

Derivation of the pH class

$$\dot{x}(t) = (J - R)\nabla\mathcal{H}(x(t)) + Bu(t),$$
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- Dissipation operator: $R^\mathsf{T} = R \geq 0$

Derivation of the pH class

$$\dot{x}(t) = (J - R)\nabla\mathcal{H}(x(t)) + Bu(t),$$
$$y(t) = B^{\mathsf{T}}\nabla\mathcal{H}(x(t))$$

Key properties - part I

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- (Quadratic) Hamiltonian: $\mathcal{H}\colon \mathbb{R}^n \to \mathbb{R}$, $x \mapsto \frac{1}{2}x^{\mathsf{T}}Qx$ with $Q^{\mathsf{T}} = Q \geq 0$
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- Dissipation operator: $R^{\mathsf{T}} = R \geq 0$
- Dissipation inequality: $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(x(t)) \leq \langle y(t), u(t)\rangle$

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Derivation of the pH class

port-Hamiltonian system

$$\dot{x}(t) = (J - R)\nabla\mathcal{H}(x(t)) + Bu(t),$$
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Derivation of the pH class

port-Hamiltonian system

$$\dot{x}(t) = (J - R)Qx(t) + Bu(t),$$
$$y(t) = B^\mathsf{T}Qx(t)$$

Key properties - part I

- use energy as lingua franca ✓
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Close to physics - passivity

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad \dot{x}(t) = (J - R)Qx(t) + Bu(t),$$
$$y(t) = Cx(t), \qquad y(t) = B^{\mathsf{T}}Qx(t)$$

Definition

A system is called passive if there exists a state-dependent storage function $\mathcal{H}\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that the dissipation inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(x(t)) \leq \langle y(t), u(t) \rangle$$

is satisfied for any $t > 0$.

Close to physics - passivity

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$$y(t) = Cx(t),$$

$$\dot{x}(t) = (J - R)Qx(t) + Bu(t),$$
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The Hamiltonian

$$\mathcal{H}(x) := \frac{1}{2}x^{\mathsf{T}}Qx$$

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The Hamiltonian

$$\mathcal{H}(x) := \tfrac{1}{2}x^{\mathsf{T}}Qx$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(x(t)) = x(t)^{\mathsf{T}}Q\dot{x}(t)$$

Close to physics - passivity

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$$y(t) = Cx(t),$$

$$\dot{x}(t) = (J - R)Qx(t) + Bu(t),$$
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The Hamiltonian

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$$\dot{x}(t) = Ax(t) + Bu(t), \qquad \dot{x}(t) = (J - R)Qx(t) + Bu(t),$$
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The Hamiltonian

$$\mathcal{H}(x) := \tfrac{1}{2}x^{\mathsf{T}}Qx$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(x(t)) &= x(t)^{\mathsf{T}}Q\dot{x}(t) = x(t)^{\mathsf{T}}Q(J-R)Qx(t) + x(t)^{\mathsf{T}}QBu(t) \\ &= -x(t)^{\mathsf{T}}QRQx(t) + y(t)^{\mathsf{T}}u(t)\end{aligned}$$

Close to physics - passivity

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The Hamiltonian

$$\mathcal{H}(x) := \tfrac{1}{2} x^\mathsf{T} Q x$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(x(t)) &= x(t)^\mathsf{T} Q\dot{x}(t) = x(t)^\mathsf{T} Q(J - R)Qx(t) + x(t)^\mathsf{T} QBu(t) \\
&= -x(t)^\mathsf{T} QRQx(t) + y(t)^\mathsf{T} u(t) \\
&\leq y(t)^\mathsf{T} u(t)
\end{aligned}$$

Close to physics - Lyapunov stability

$$\dot{x}(t) = (J - R)Qx(t) + Bu(t)$$
$$y(t) = B^{\mathsf{T}}x(t)$$

The Hamiltonian

$$\mathcal{H}(x) := \tfrac{1}{2}x^{\mathsf{T}}Qx \geq 0$$

Close to physics - Lyapunov stability

$$\dot{x}(t) = (J - R)Qx(t)$$

The Hamiltonian

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Close to physics - Lyapunov stability

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As before: $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(x(t)) = -x(t)^{\mathsf{T}}QRQx(t) \leq 0$

Close to physics - Lyapunov stability

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As before: $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(x(t)) = -x(t)^{\mathsf{T}}QRQx(t) \leq 0$

$\rightsquigarrow$ the Hamiltonian is a Lyapunov function

Port-Hamiltonian systems

Definition (port-Hamiltonian system)

A dynamical system of the form

$$\Sigma_{\mathrm{pH}} \begin{cases} \dot{x} = (J - R)Qx + Bu \\ y = B^{\mathsf{T}}Qx \end{cases}$$

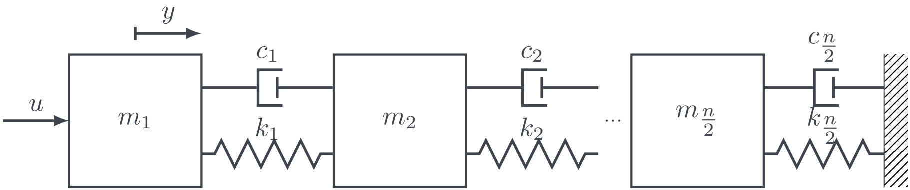
together with Hamiltonian

$$\mathcal{H} : \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto \tfrac{1}{2}x^T Qx$$

is called a *port-Hamiltonian system* if

$$J^{\mathsf{T}} = -J, \qquad R^{\mathsf{T}} = R \geq 0, \qquad Q^{\mathsf{T}} = Q \geq 0.$$

A mass-spring-damper example



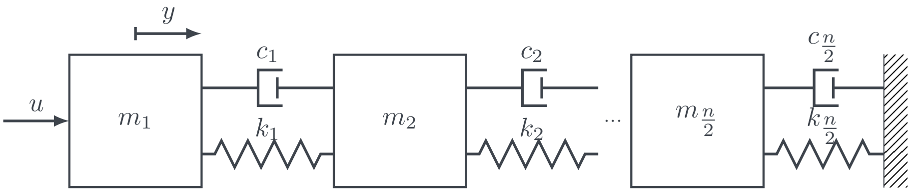
Parameter:

- $n = 6, m = p = 1$

$$Q = \begin{bmatrix} k_1 & 0 & -k_1 & 0 & 0 & 0 \\ 0 & \frac{1}{m_1} & 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_1+k_2 & 0 & -k_2 & 0 \\ 0 & 0 & 0 & \frac{1}{m_2} & 0 & 0 \\ 0 & 0 & -k_2 & 0 & k_2+k_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{m_3} \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix},$$

$$R = \operatorname{diag}(0, c_1, 0, c_2, 0, c_3), \qquad B^\mathsf{T} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A mass-spring-damper example



FOM Parameter:

- $n = 100$, $m = p = 2$, $c_i = 1$, $m_i = 4$, $k_i = 4$

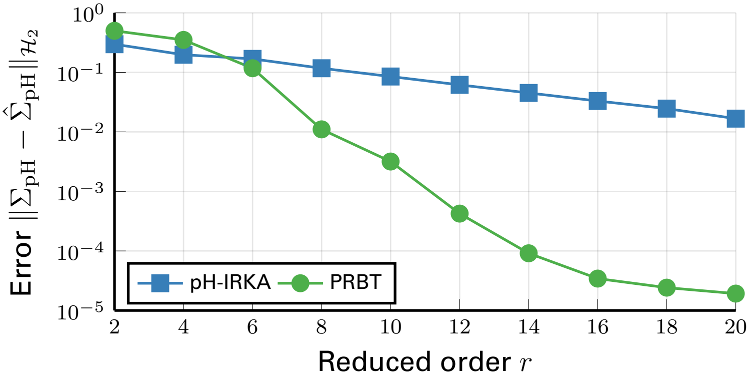
$$Q = \begin{bmatrix} k_1 & 0 & -k_1 & 0 & 0 & 0 \\ 0 & \frac{1}{m_1} & 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_1+k_2 & 0 & -k_2 & 0 \\ 0 & 0 & 0 & \frac{1}{m_2} & 0 & 0 \\ 0 & 0 & -k_2 & 0 & k_2+k_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{m_3} \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix},$$

$$R = \operatorname{diag}(0, c_1, 0, c_2, 0, c_3), \qquad B^\mathsf{T} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

MOR for pH sys-tems

2

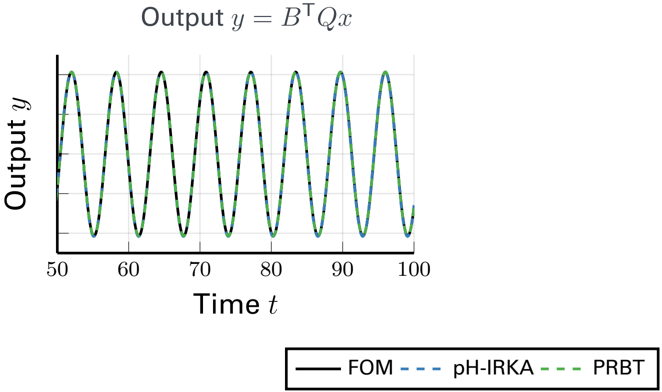
PH-MOR in action for a mass-spring-damper example



S. Gugercin, et al.
Structure-preserving tangential interpolation for model reduction of port-Hamiltonian systems.
Automatica J. IFAC, 48, 2012.

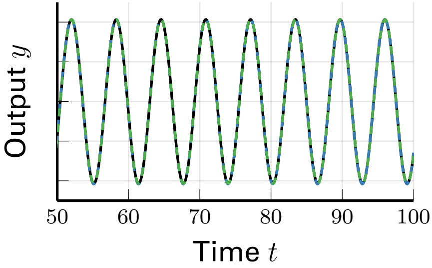
U. Desai and D. Pal
A transformation to stochastic model reduction.
IEEE Tran. Automat. Control, 29, 1984.

PH-MOR in action for a mass-spring-damper example



PH-MOR in action for a mass-spring-damper example

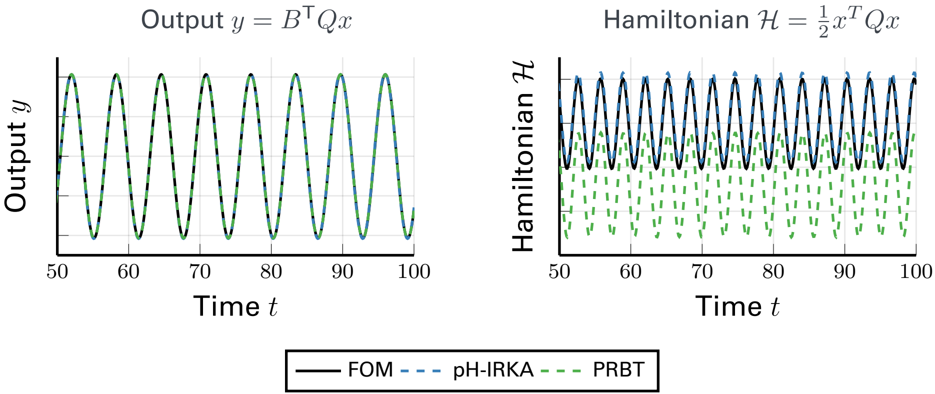
Output $y = B^\mathsf{T} Q x$



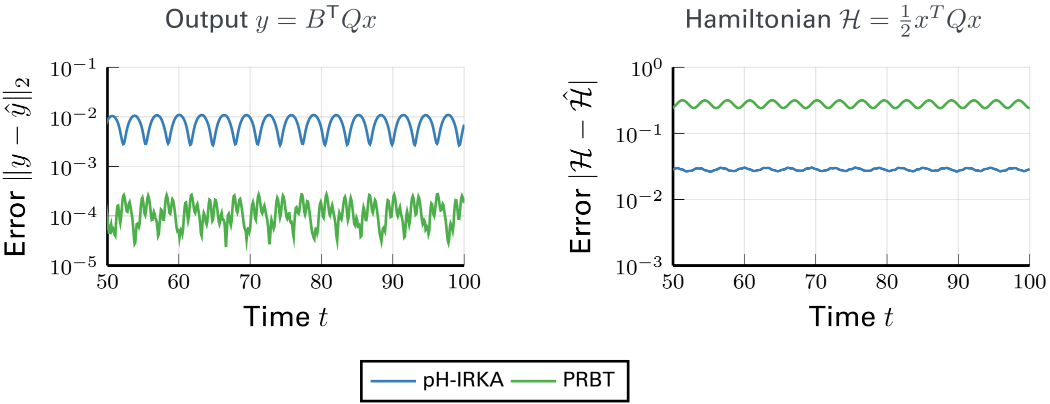
Hamiltonian $\mathcal{H} = \frac{1}{2} x^T Q x$

FOM pH-IRKA PRBT

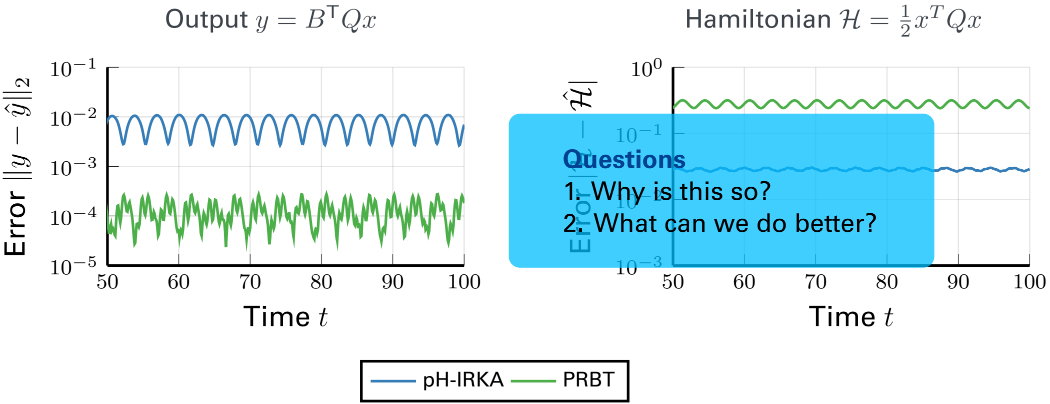
PH-MOR in action for a mass-spring-damper example



PH-MOR in action for a mass-spring-damper example



PH-MOR in action for a mass-spring-damper example



Questions

1. Why is this so?
2. What can we do better?

The pH objectives

Input-output dynamics

$$\Sigma_{\mathrm{pH}} \begin{cases} \dot{x} = (J - R)Qx + Bu \\ y = B^{\mathsf{T}}Qx \end{cases}$$

The pH objectives

Input-output dynamics

$$\Sigma_{\mathrm{pH}} \begin{cases} \dot{x} = (J - R)Qx + Bu \\ y = B^{\mathsf{T}} Q x \end{cases}$$

Classical $\mathcal{H}_2$-norm

$$\|\Sigma_{\mathrm{pH}}\|_{\mathcal{H}_2} := \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \|\Sigma_{\mathrm{pH}}(\imath\omega)\|_{\mathrm{F}}^2 \,\mathrm{d}\omega}$$

The pH objectives

Input-output dynamics

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Hamiltonian dynamics

$$\Sigma_{\mathcal{H}} \begin{cases} \dot{x} = (J-R)Qx + Bu \\ y_{\mathcal{H}} = \frac{1}{2}x^{\mathsf{T}}Qx \end{cases}$$

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Which norm to use?

The pH objectives

Input-output dynamics

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LTIQO dynamics

$$\Sigma_{\mathrm{QO}} \begin{cases} \dot{x} = Ax + Bu \\ y = x^{\mathsf{T}}Mx \end{cases}$$

Linear dynamical system with quadratic output (LTIQO)

The pH objectives

Input-output dynamics

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Linear dynamical system with quadratic output (LTIQO)

- Gramians

$$A\mathcal{P} + \mathcal{P}A^{\mathsf{T}} + BB^{\mathsf{T}} = 0, \qquad A^{\mathsf{T}}\mathcal{O}_{\mathrm{QO}} + \mathcal{O}_{\mathrm{QO}}A + M\mathcal{P}M = 0$$

- $\mathcal{H}_2$-norm

$$\|\Sigma_{\mathrm{QO}}\|_{\mathcal{H}_2} := \sqrt{\mathrm{tr}(B^{\mathsf{T}}\mathcal{O}_{\mathrm{QO}}B)}$$

- output bound

$$\|y\|_{\infty} \leq \|\Sigma_{\mathrm{QO}}\|_{\mathcal{H}_2} \|u \otimes u\|_{L^2}$$

P. Benner, P. Goyal and I. Pontes Duff.
Gramians, energy functionals, and balanced truncation for linear dynamical systems with quadratic outputs.
IEEE Trans. Automat. Control, 67(2), 2022.

The pH objectives

Input-output dynamics

$$\Sigma_{\mathrm{pH}} \begin{cases} \dot{x} = (J-R)Qx + Bu \\ y = B^{\mathsf{T}}Qx \end{cases}$$

Hamiltonian dynamics

$$\Sigma_{\mathcal{H}} \begin{cases} \dot{x} = (J-R)Qx + Bu \\ y_{\mathcal{H}} = \frac{1}{2}x^{\mathsf{T}}Qx \end{cases}$$

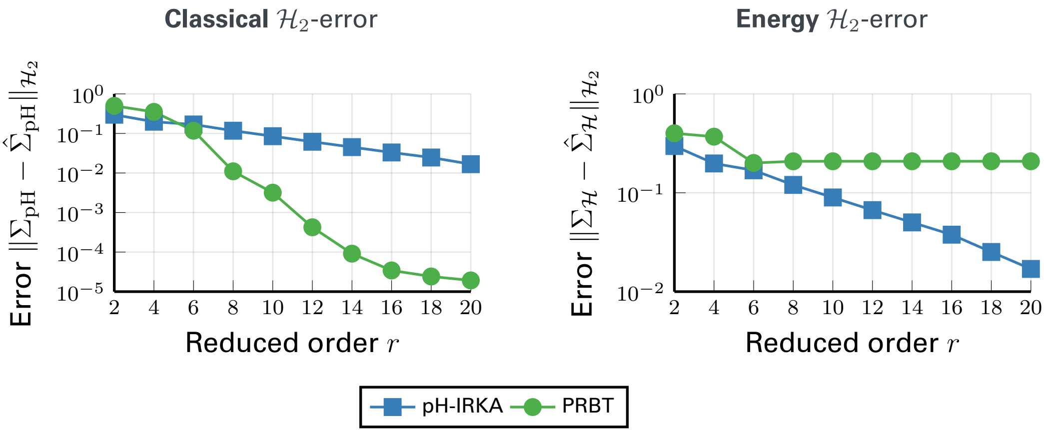
Classical $\mathcal{H}_2$-norm

$$\|\Sigma_{\mathrm{pH}}\|_{\mathcal{H}_2} := \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \|\Sigma_{\mathrm{pH}}(\imath\omega)\|_{\mathrm{F}}^2 \,\mathrm{d}\omega}$$

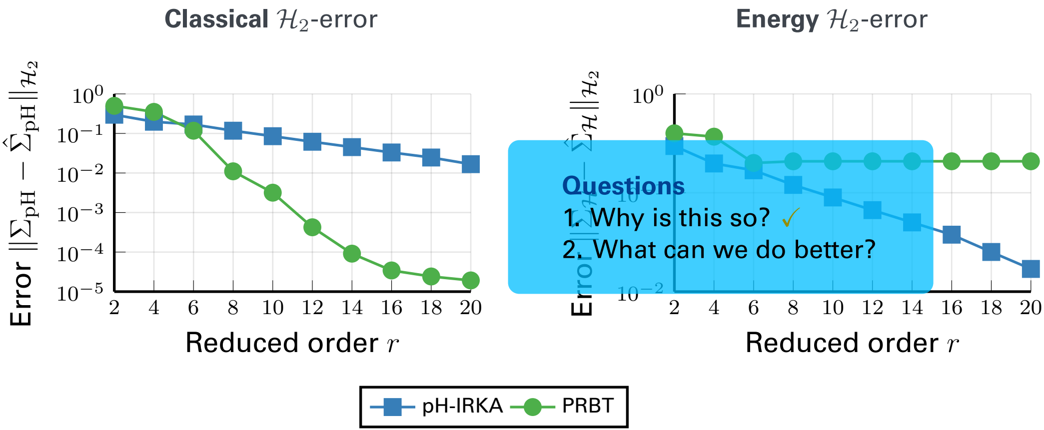
Energy $\mathcal{H}_2$-norm

$$\|\Sigma_{\mathcal{H}}\|_{\mathcal{H}_2} := \sqrt{\mathrm{tr}(B^{\mathsf{T}}\mathcal{O}_{\mathrm{QO}}B)}$$

The mass-spring-damper example, revisited



The mass-spring-damper example, revisited



Questions

1. Why is this so? ✓
2. What can we do better?

The dual-objective optimization problem

FOM

$$\Sigma_{\mathrm{pH}+\mathcal{H}} \begin{cases} \dot{x} = (J-R)Qx + Bu \\ y = B^{\mathsf{T}}Qx \\ y_{\mathcal{H}} = \frac{1}{2}x^{\mathsf{T}}Qx \end{cases}$$

ROM

$$\hat{\Sigma}_{\mathrm{pH}+\hat{\mathcal{H}}} \begin{cases} \dot{\hat{x}} = (\hat{J}-\hat{R})\hat{Q}\hat{x} + \hat{B}u \\ \hat{y} = \hat{B}^{\mathsf{T}}\hat{Q}\hat{x} \\ \hat{y}_{\hat{\mathcal{H}}} = \frac{1}{2}\hat{x}^{\mathsf{T}}\hat{Q}\hat{x} \end{cases}$$

Multi-objective model reduction

$$\min_{\hat{J},\hat{R},\hat{B},\hat{Q}} \alpha \left\| \Sigma_{\mathrm{pH}} - \hat{\Sigma}_{\mathrm{pH}} \right\|_{\mathcal{H}_2}^2 + \beta \left\| \Sigma_{\mathcal{H}} - \hat{\Sigma}_{\mathcal{H}} \right\|_{\mathcal{H}_2}^2 \quad \text{s.t.} \quad \hat{J} = -\hat{J}, \hat{R} \geq 0, \hat{Q} \geq 0$$

The dual-objective optimization problem

FOM

$$\Sigma_{\mathrm{pH}+\mathcal{H}} \begin{cases} \dot{x} = (J-R)Qx + Bu \\ y = B^{\mathsf{T}}Qx \\ y_{\mathcal{H}} = \frac{1}{2}x^{\mathsf{T}}Qx \end{cases}$$

ROM

$$\hat{\Sigma}_{\mathrm{pH}+\hat{\mathcal{H}}} \begin{cases} \dot{\hat{x}} = (\hat{J}-\hat{R})\hat{Q}\hat{x} + \hat{B}u \\ \hat{y} = \hat{B}^{\mathsf{T}}\hat{Q}\hat{x} \\ \hat{y}_{\hat{\mathcal{H}}} = \frac{1}{2}\hat{x}^{\mathsf{T}}\hat{Q}\hat{x} \end{cases}$$

Multi-objective model reduction

$$\min_{\hat{J},\hat{R},\hat{B},\hat{Q}} \alpha \left\| \Sigma_{\mathrm{pH}} - \hat{\Sigma}_{\mathrm{pH}} \right\|_{\mathcal{H}_2}^2 + \beta \left\| \Sigma_{\mathcal{H}} - \hat{\Sigma}_{\mathcal{H}} \right\|_{\mathcal{H}_2}^2 \quad \text{s.t.} \quad \hat{J} = -\hat{J}, \hat{R} \geq 0, \hat{Q} \geq 0$$

$\rightsquigarrow$ Future work

The dual-objective optimization problem

FOM

$$\Sigma_{\mathrm{pH}+\mathcal{H}} \begin{cases} \dot{x} = (J - R)Qx + Bu \\ y = B^{\mathsf{T}}Qx \\ y_{\mathcal{H}} = \frac{1}{2}x^{\mathsf{T}}Qx \end{cases}$$

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Observation: $\hat{A} = (\hat{J}_1 - \hat{R}_1)\hat{Q}_1 = (\hat{J}_2 - \hat{R}_2)\hat{Q}_2$ while $\hat{C} = \hat{B}^{\mathsf{T}}\hat{Q}_1 = \hat{B}^{\mathsf{T}}\hat{Q}_2$

Example

$$\hat{J}_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \hat{R}_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \hat{Q}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \hat{A} = \begin{bmatrix} -2 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

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$$\hat{J}_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \hat{R}_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \hat{Q}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \hat{A} = \begin{bmatrix} -2 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\hat{Q}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x \end{bmatrix}$$

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$$\hat{Q}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x \end{bmatrix} \implies \hat{J}_2 = \hat{J}_1 \quad \hat{R}_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{2}{x} \end{bmatrix} \quad \hat{A} = (\hat{J}_1 - \hat{R}_1)\hat{Q}_1 = (\hat{J}_2 - \hat{R}_2)\hat{Q}_2$$

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$$\hat{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \implies \hat{C} = \hat{B}^\mathsf{T}\hat{Q}_1 = \hat{B}^\mathsf{T}\hat{Q}_2$$

Back to the dual-objective optimization problem

FOM

$$\Sigma_{\mathrm{pH}+\mathcal{H}} \begin{cases} \dot{x} = (J-R)Qx + Bu \\ y = B^{\mathsf{T}}Qx \\ y_{\mathcal{H}} = \frac{1}{2}x^{\mathsf{T}}Qx \end{cases}$$

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Idea: For a given ROM, find the optimal $\hat{Q}$ w.r.t. **Energy** $\|\cdot\|_{\mathcal{H}_2}$ without changing the io-dynamics

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Energy matching

$$\min_{\hat{Q}} \left\| \Sigma_{\mathcal{H}} - \hat{\Sigma}_{\mathcal{H}} \right\|_{\mathcal{H}_2}^2 \quad \text{s.t.} \quad ?$$

Observation: $\hat{A} = (\hat{J}_1 - \hat{R}_1)\hat{Q}_1 = (\hat{J}_2 - \hat{R}_2)\hat{Q}_2$ while $\hat{C} = \hat{B}^\mathsf{T}\hat{Q}_1 = \hat{B}^\mathsf{T}\hat{Q}_2$

Idea: For a given ROM, find the optimal $\hat{Q}$ w.r.t. **Energy** $\|\cdot\|_{\mathcal{H}_2}$ without changing the io-dynamics

Port-Hamiltonian and KYP

Theorem e.g. Beattie et al. '22

(Technical details aside.) The following are equivalent:

- The system can be formulated as port-Hamiltonian system.
- The Kalman-Yakubovich-Popov (KYP) inequality

$$\mathcal{W}(X) := \begin{bmatrix} -A^\mathsf{T} X - XA & C^\mathsf{T} - XB \\ C - B^\mathsf{T} X & 0 \end{bmatrix} \geq 0$$

has a positive definite solution X.

C. Beattie, V. Mehrmann and H. Xu.
Port-Hamiltonian realizations of linear time invariant systems.
ArXiv e-print 2201.05355, 2022.

From ABC to JRQ

$$\mathcal{W}(X) := \begin{bmatrix} -A^\mathsf{T} X - XA & C^\mathsf{T} - XB \\ C - B^\mathsf{T} X & 0 \end{bmatrix} \geq 0$$

Step 1 Find $X = X^\mathsf{T} > 0$ satisfying $\mathcal{W}(X) \geq 0$, set $Q := X$

From ABC to JRQ

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Step 2 Define

$$J := \tfrac{1}{2}(AX^{-1} - X^{-1}A^\mathsf{T}), \qquad R := -\tfrac{1}{2}(AX^{-1} + X^{-1}A^\mathsf{T})$$

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Check:

$$(J - R)X = \tfrac{1}{2}\left(AX^{-1} - X^{-1}A^\mathsf{T} + AX^{-1} + X^{-1}A^\mathsf{T}\right) X = A$$

From ABC to JRQ

$$\mathcal{W}(X) := \begin{bmatrix} -A^\mathsf{T}X - XA & C^\mathsf{T} - XB \\ C - B^\mathsf{T}X & 0 \end{bmatrix} \geq 0$$

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Step 3 Arrive at

$$\Sigma_{\mathrm{pH}} \begin{cases} \dot{x} = (J - R)Qx + Bu \\ y = B^\mathsf{T}Qx \end{cases}$$

From ABC to JRQ

$XA \quad C^\mathsf{T} - XB\Big] \geq 0$
0

Step 1 Find $X =$ ... $:= X$

Step 2 Define ... $:= -\frac{1}{2}(AX^{-1} + X^{-1}A^\mathsf{T})$

Check: ... $+ AX^{-1} + X^{-1}A^\mathsf{T}\big) X = A$

Step 3 Arrive at

$(J - R)Qx + Bu$
$y = B^\mathsf{T}Qx$

Any solution of the KYP inequality yields a pH representation

Energy matching for reduced pH systems

Energy matching

For a given ROM, find $\hat{Q}$ that minimize

$$\min_{\hat{Q}} \left\| \Sigma_{\mathcal{H}} - \hat{\Sigma}_{\mathcal{H}}(\hat{Q}) \right\|_{\mathcal{H}_2} \quad \text{s.t.} \quad \mathcal{W}(\hat{Q}) \geq 0$$

Energy matching for reduced pH systems

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Theorem

This optimization problem is convex and has a unique solution.

Energy matching for reduced pH systems

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For a given ROM, find $\hat{Q}$ that minimize

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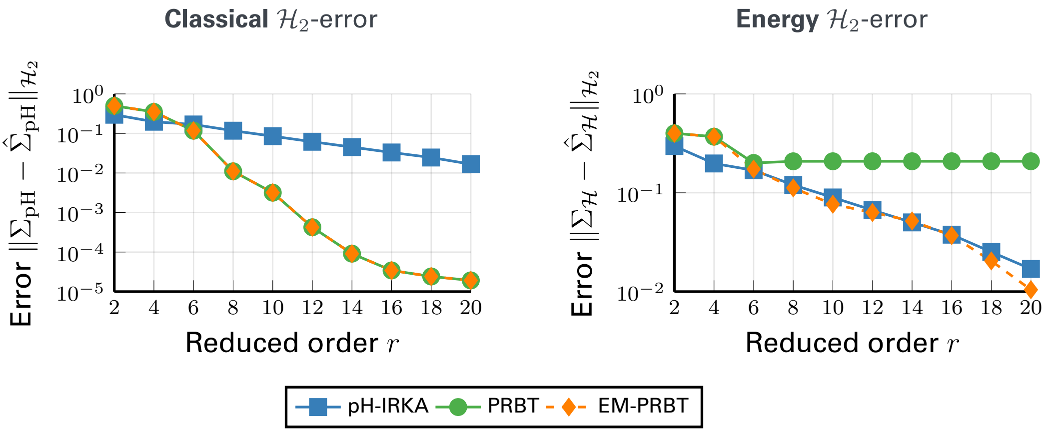
Theorem

This optimization problem is convex and has a unique solution.

Key idea of the proof.
Compute the first and second derivative, then show that the second derivative is positive definite.

The mass-spring-damper example, one last time



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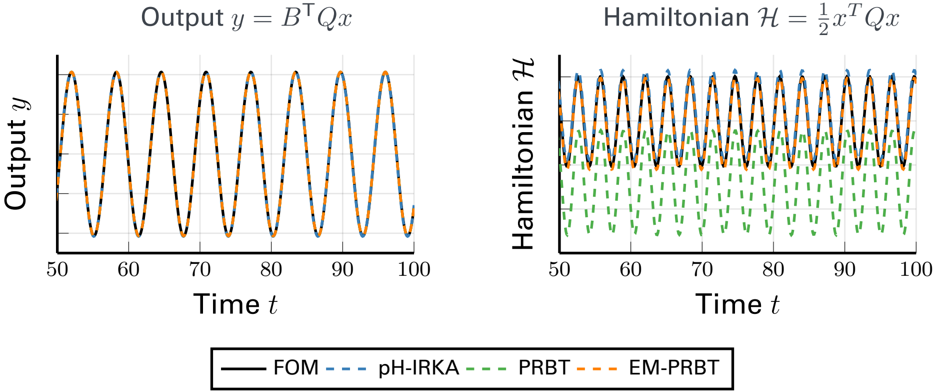

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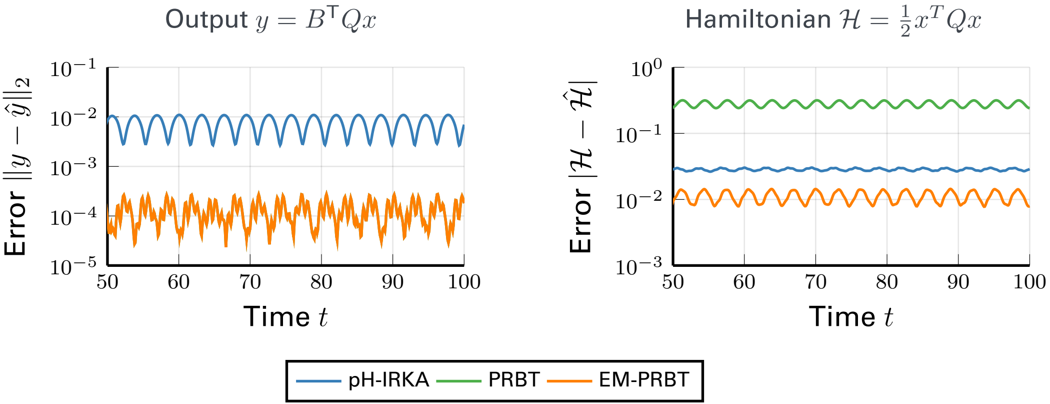

Summary

- the class of pH-systems is nice
- pH-systems have two objectives, io-dynamics and Hamiltonian dynamics
- pH MOR is a dual-objective optimization problem
- the $A = (J - R)Q$ factorization is not unique

J. Nicodemus, P. Schwerdtner, and B. Unger
Energy matching in reduced passive and port-Hamiltonian systems
In preparation, 2023

Summary

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- pH MOR is a dual-objective optimization problem
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Future work:

- characterize the pareto-front of the dual-objective optimization problem

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