Reduced Basis Methods for Nonlinear Parametrized Partial Differential Equations

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Outline

PART I: A Reduced Basis Primer

PART II: Nonlinear (and Nonaffine) Problems

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- Introduction
- Key Ingredients
- Extensions & References

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- Empirical Interpolation Method
- Nonlinear Elliptic Problems
- Extensions & References
- Numerical Results
  - Nonlinear Reaction Diffusion Systems

Part I

A Reduced Basis Primer

Introduction

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is a **model order reduction** technique that provides

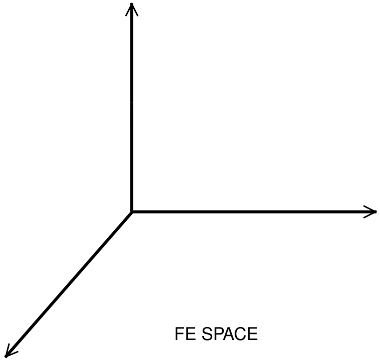
rapid and **reliable approximation** of solutions

to **parametrized** partial differential equations (μ**PDEs**)

for design and optimization, control, parameter estimation, ...

— i.e., for **parameter space exploration**.

The Reduced Basis Method



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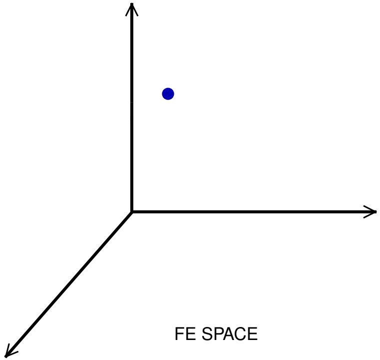

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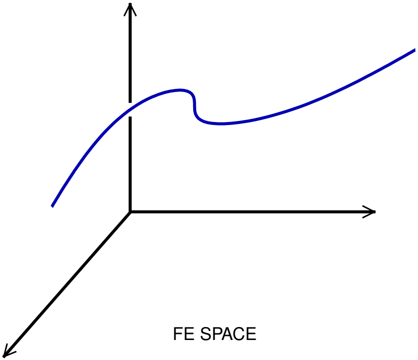

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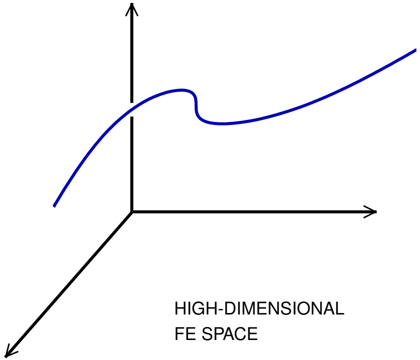

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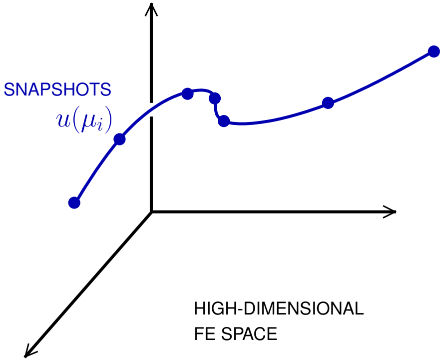

$$\boldsymbol{\mathcal{X}_N = \mathrm{span}\{\, u(\mu_i)\,,\; i = 1, \ldots, N \,\}}$$

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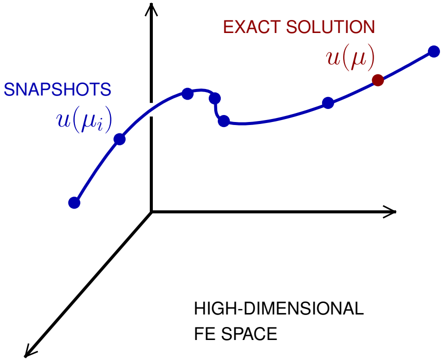

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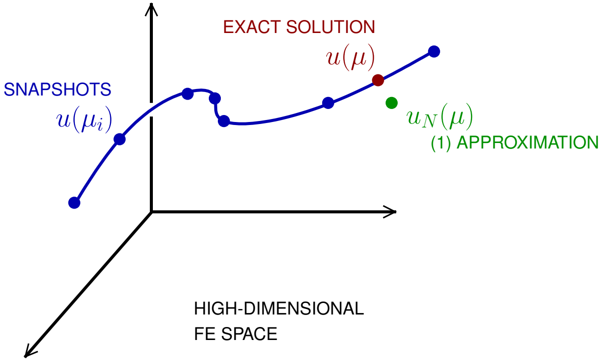
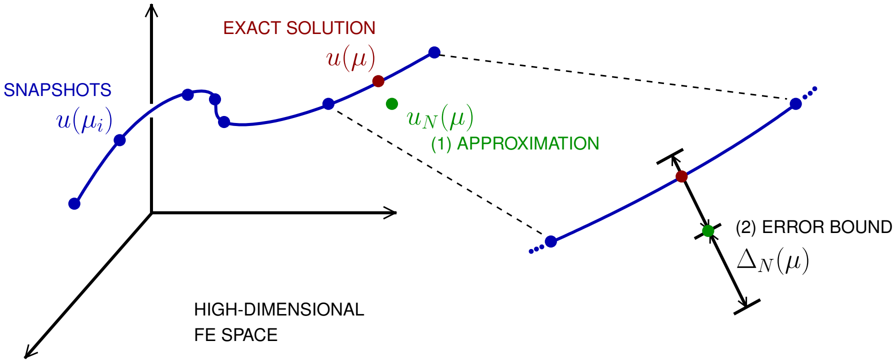


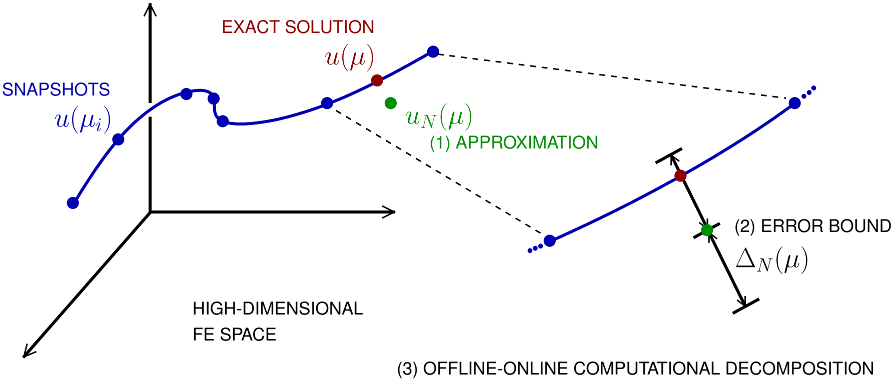
$$a(u_N(\mu), v; \mu) = f(v; \mu), \quad \text{for all } v \in \mathcal{X}_N.$$

The Reduced Basis Method



$$\|u(\mu) - u_N(\mu)\|_{\mathcal{X}} \le \Delta_N^u(\mu) := \frac{\|r_N(\cdot;\mu)\|_{\mathcal{X}'}}{\alpha_{\mathrm{LB}}(\mu)}$$

The Reduced Basis Method



$$a(w,v;\mu) = \sum_{q=1}^{Q} \underbrace{\theta^q(\mu)}_{\mu\text{-DEPENDENT COEFFICIENTS}} \underbrace{a^q(w,v)}_{\mu\text{-INDEPENDENT BILINEAR FORMS}}, \quad \forall w,v \in \mathcal{X}$$

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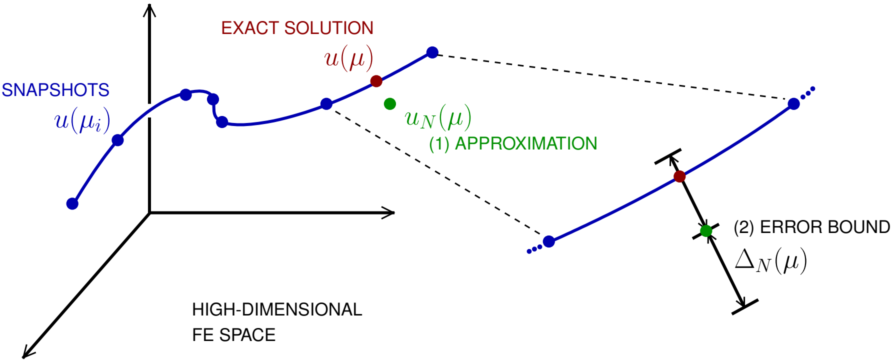

(3) OFFLINE-ONLINE COMPUTATIONAL DECOMPOSITION
(4) GREEDY ALGORITHM

$$\boldsymbol{\mu}_{n+1} = \arg\max_{\boldsymbol{\mu}\in\mathcal{D}_{\rm s}} \frac{\Delta_n^u(\boldsymbol{\mu})}{\|u_N(\boldsymbol{\mu})\|_{\mathcal{X}}}$$

Problem Statement

We wish to compute, for any parameter $\boldsymbol{\mu} \in \mathcal{D}$,

$$s(\boldsymbol{\mu}) = \ell(u(\boldsymbol{\mu}); \boldsymbol{\mu}) \qquad \text{OUTPUT}$$

where $u(\boldsymbol{\mu}) \in \mathcal{X}$ satisfies

$$a(u, v; \boldsymbol{\mu}) = f(v; \boldsymbol{\mu}), \quad \text{for all } v \in \mathcal{X}, \qquad \text{FEM}_{\mathcal{N}}(\boldsymbol{\mu})$$

and

$f(v; \boldsymbol{\mu}),\ \ell(v; \boldsymbol{\mu})$ are bounded linear functionals

$a(\cdot, \cdot; \boldsymbol{\mu}) : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is continuous and coercive

for all $\boldsymbol{\mu} \in \mathcal{D}$.

(1) Approximation

We let

$$\boldsymbol{\mathcal{X}}_N = \mathrm{span}\{ \underbrace{u(\mu_i)}_{\text{SNAPSHOTS}}, \; i = 1, \ldots, N \}$$

and compute our approximation as

$$s_N(\mu) = \ell(u_N(\mu); \mu)$$

where $u_N(\mu) \in \boldsymbol{\mathcal{X}}_N$ satisfies

$$a(u_N, v; \mu) = f(v; \mu), \quad \text{for all } v \in \boldsymbol{\mathcal{X}}_N. \qquad \mathrm{ROM}_N(\mu)$$

(1) Approximation (Algebraic Formulation)

We let

$$\boldsymbol{X}_N = [\ \underbrace{\zeta_1} \quad \underbrace{\zeta_2} \quad \cdots \quad \underbrace{\zeta_N}\ ] \quad \text{s.t.} \quad u_N(\mu) = \boldsymbol{X}_N \underline{\mathbf{u}}_N(\mu)$$

$\mathcal{X}$-ORTHONORMAL SNAPSHOTS ζ_i "=" $u(\mu_i)$

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and compute our approximation as

$$\begin{aligned} s_N(\boldsymbol{\mu}) &= \underbrace{\boldsymbol{L}(\boldsymbol{\mu})^T \boldsymbol{X}_N}\ \underline{\mathbf{u}}_N(\boldsymbol{\mu}) \\ &= \boldsymbol{L}_N^T(\boldsymbol{\mu})\ \underline{\mathbf{u}}_N(\boldsymbol{\mu}) \end{aligned}$$

where $\underline{\mathbf{u}}_N(\boldsymbol{\mu}) \in \mathbb{R}^N$ satisfies

$$\begin{aligned} \underbrace{\boldsymbol{X}_N^T \boldsymbol{A}(\boldsymbol{\mu}) \boldsymbol{X}_N}\ \underline{\mathbf{u}}_N(\boldsymbol{\mu}) &= \underbrace{\boldsymbol{X}_N^T \boldsymbol{F}(\boldsymbol{\mu})} \\ \mathbf{A}_N(\boldsymbol{\mu})\ \underline{\mathbf{u}}_N(\boldsymbol{\mu}) &= \mathbf{F}_N(\boldsymbol{\mu}) \end{aligned}$$

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How do we quantify the error?

(2) Error Estimation

Let $\alpha(\mu)$ be the **coercivity constant** of a

$$\alpha(\mu) := \inf_{v \in \mathcal{X}} \frac{a(v, v; \mu)}{\|v\|_{\mathcal{X}}^2}$$

and define **residual** $r_N(v; \mu) := f(v; \mu) - a(u_N, v; \mu), \; \forall v \in \mathcal{X}$.

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The **error**, $e_N(\mu) := u(\mu) - u_N(\mu)$, then satisfies

$$a(e_N, v; \mu) = r_N(v; \mu), \qquad \forall\, v \in \mathcal{X},$$

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$$a(e_N, v; \mu) = r_N(v; \mu), \qquad \forall\, v \in \mathcal{X},$$

and, for any $\mu \in \mathcal{D}$ and $\alpha_{\mathrm{LB}}(\mu) \leq \alpha(\mu)$, we have

$$\|e_N(\mu)\|_{\mathcal{X}} \;\leq\; \underbrace{\frac{\|r_N(\cdot; \mu)\|_{\mathcal{X}'}}{\alpha_{\mathrm{LB}}(\mu)}}_{=: \, \Delta_N^u(\mu)} \qquad \text{and} \qquad |s - s_N| \;\leq\; \underbrace{\|\ell\|_{\mathcal{X}'} \Delta_N^u}_{=: \, \Delta_N^\ell(\mu)}$$

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How do we compute $u_N, s_N, \Delta_N^u, \Delta_N^\ell$ efficiently?

(3) Offline/Online Decomposition

We assume

$$a(v, w; \mu) = \sum_{q=1}^{Q} \underbrace{\theta^q(\mu)}_{\mu\text{-DEPENDENT COEFFICIENTS}} \underbrace{a^q(v, w)}_{\mu\text{-INDEPENDENT BILINEAR FORMS}}$$

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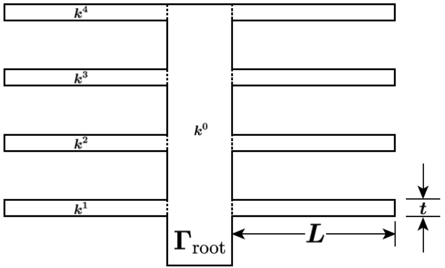
$$\underbrace{A_N(\mu)}_{\text{RB-MATRIX}} = X_N^T \underbrace{A(\mu)}_{\text{FE-MATRIX}} X_N = \sum_{q=1}^{Q} \theta^q(\mu) \underbrace{X_N^T A^q X_N}_{\mu\text{-INDEPENDENT}}$$

(3) Offline/Online Decomposition – Example

Affine Parameter Dependence: Example 1



Parameters:

$$\mu = (\underbrace{k^0, k^1, k^2, k^3, k^4}_{\text{THERMAL CONDUCTIVITIES}})$$

Governing Equation:

$$-k^i \nabla^2 y_i = 0$$

in Ω_i, $i = 0$ to 4

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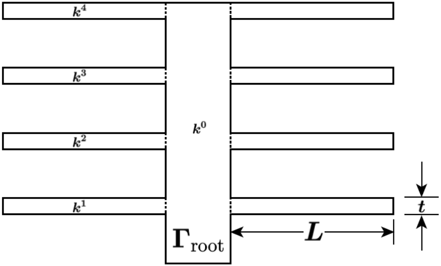

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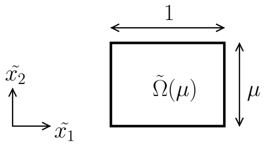
The matrix $A(\mu)$ then represents the bilinear form

$$a(v, w; \mu) = \sum_{i=0}^{4} \underbrace{k^i}_{\theta^q(\mu)} \underbrace{\int_{\Omega_i} \nabla v \cdot \nabla w}_{\text{"="}\, A^q}$$

(3) Offline/Online Decomposition – Example

Affine Parameter Dependence: Example 2



PARAMETRIZED DOMAIN

Bilinear Form on Parameter-Dependent Domain

$$\tilde{a}(\tilde{v}, \tilde{w}; \mu) = \int_{\tilde{\Omega}(\mu)} \underbrace{\left(\frac{\partial \tilde{v}}{\partial \tilde{x}_1} \frac{\partial \tilde{w}}{\partial \tilde{x}_1} + \frac{\partial \tilde{v}}{\partial \tilde{x}_2} \frac{\partial \tilde{w}}{\partial \tilde{x}_2} \right)}_{\tilde{\nabla} \tilde{v} \cdot \tilde{\nabla} \tilde{w}} d\tilde{\Omega}(\mu), \quad \tilde{v}, \tilde{w} \in H^1(\tilde{\Omega}(\mu))$$

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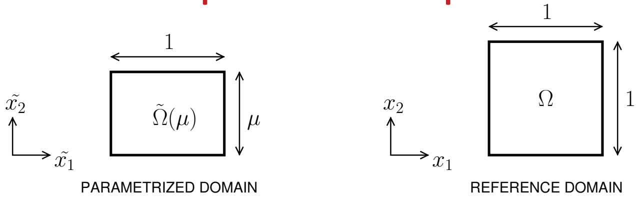
Affine Parameter Dependence: Example 2



Bilinear Form on Parameter-Dependent Domain

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After (Affine) Mapping to a Reference Domain

$$a(v,w;\mu) = \int_{\Omega} \left(\frac{\partial v}{\partial x_1}\frac{\partial w}{\partial x_1} + \frac{1}{\mu^2}\frac{\partial v}{\partial x_2}\frac{\partial w}{\partial x_2} \right) \mu d\Omega, \qquad v,w \in H^1(\Omega)$$

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We assume $a(v,w;\mu) = \sum_{q=1}^{Q} \underbrace{\theta^q(\mu)}_{\mu\text{-DEPENDENT COEFFICIENTS}} \underbrace{a^q(v,w)}_{\mu\text{-INDEPENDENT BILINEAR FORMS}}$

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Since all required quantities can be decomposed in this manner, we can

OFFLINE: Form and store μ-independent quantities at cost $O(\mathcal{N}^*)$

ONLINE: For any $\mu \in \mathcal{D}$, compute approx and error bounds at cost $O(QN^2 + N^3)$ and $O(Q^2N^2)$

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How do we choose the snapshots?

(4) (Weak) Greedy Algorithm

Given $\boldsymbol{\mathcal{X}_2} = \mathbf{span}\{\boldsymbol{u}(\boldsymbol{\mu_1}), \boldsymbol{u}(\boldsymbol{\mu_2})\}$, how do we choose $\boldsymbol{\mu_3}$?

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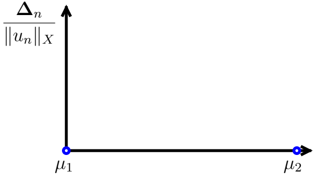

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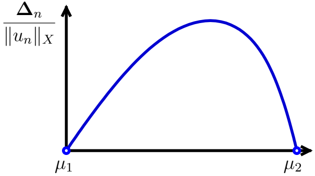

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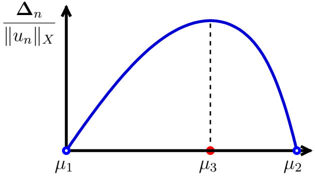

$$\boldsymbol{\mu_3} = \arg\max_{\boldsymbol{\mu} \in \boldsymbol{\mathcal{D}_s}} \frac{\boldsymbol{\Delta_2(\mu)}}{\|\boldsymbol{u_2(\mu)}\|_{\boldsymbol{\mathcal{X}}}}$$

where $\boldsymbol{\mathcal{D}_s} \subset \boldsymbol{\mathcal{D}}$ is a finite train sample

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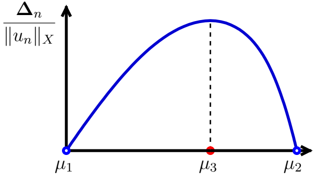

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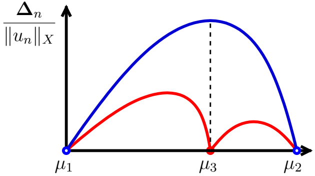

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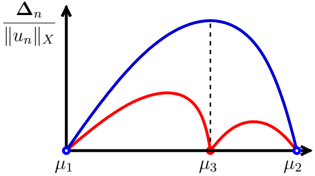

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$$\mathcal{X}_3 = \text{span}\{u(\mu_1)\ u(\mu_2)\ u(\mu_3)\}$$

Key points:

- $\Delta_n(\mu)$ is sharp and inexpensive to compute (online)
- Error bounds enable choice of good approximation spaces

Summary

The **reduced basis method** provides

accurate	$u_N \approx u$	APPROX
reliable	$\Delta_N \geq \|u - u_N\|_{\mathcal{X}}$	ERR EST
efficient surrogates	cost $O(N^*)$	DECOMP
	N small	GREEDY

to solutions of **parametrized PDEs**

for the **many-query**, **real-time**,

and **slim-computing** contexts.

Extensions & References

Review Articles: [RHP08], [QRM11], ...

Book: [HRS15]

Reduced Basis Methods for:

- **elliptic coercive** [PRV+02], [PRVP02], ...
- **elliptic noncoercive** [VPRP03], [MPR02], ...
- **saddle point (Stokes)** [Rov03], [RV07], [GV12], ...
- **parabolic** [GP05], [RMM06], [HO08], [UP14], ...
- **hyperbolic (transport-dominated)** [PR07], [HKP11], [DPW14], ...

Disclaimer:

- List contains – to the best of my knowledge – only the first papers on these topics and is thus not meant to be exhaustive
- References to nonaffine and nonlinear problems to follow ...

Extensions & References

References by topic for:

- **error bounds (adjoint techniques)** [PRV+02], [GP05], . . .
- **stability factor lower bounds** [VRP02], [HRSP07], [HKC+10], [CHMR09],
- *a priori* **convergence** [MPT02], . . .
- **greedy algorithm**
  - original papger [VPRP03]
  - convergence analysis [BMP+12], [BCD+11], [DPW12], [Haa13], . . .
  - variations (hpRB, ...) [EPR10], [BTWGvBW07], [BTWG08], . . .
- **geometry parametrization** [RHP08], [LR10], . . .
- **multiscale problems** [Ngu08]
- **stochastic problems** [BBM+09], [BBL+10]
- . . .

Disclaimer: see previous slide . . .

Part II

Nonlinear (and Nonaffine) Problems

Nonlinear Problems

Distinguish between problems involving

- **quadratic nonlinearities**

- **higher-order or nonpolynomial nonlinearities**

Nonlinear Problems

Distinguish between problems involving

- **quadratic nonlinearities**
  - (Petrov-)Galerkin projection
  - *A posteriori* error bounds: Brezzi-Rappaz-Raviart [BRR80]
  - Offline/online: dominant online cost $O(N^4)$
  - Ref.: [VPP03], [VP05], [Dep08], [DL12], [RG12], [MPU14], [Yan14], ...

- **higher-order or nonpolynomial nonlinearities**

Nonlinear Problems

Distinguish between problems involving

- **quadratic nonlinearities**
  - (Petrov-)Galerkin projection
  - *A posteriori* error bounds: Brezzi-Rappaz-Raviart [BRR80]
  - Offline/online: dominant online cost $O(N^4)$
  - Ref.: [VPP03], [VP05], [Dep08], [DL12], [RG12], [MPU14], [Yan14], ...

- **higher-order or nonpolynomial nonlinearities**
  - Additional approximation of nonlinearity (EIM)
  - *A posteriori* error estimation (in general no bounds)
  - Offline-online decomposition possible
  - References: to follow ...

Problem Statement

We wish to compute, for any parameter $\mu \in \mathcal{D}$,

$$s(\mu) = \ell(u(\mu); \mu)$$

where $u(\mu) \in \mathcal{X}$ satisfies

$$a(u, v; \mu) + \int_\Omega g^{\mathrm{nl}}(u; x; \mu)\, v = f(v), \quad \forall v \in \mathcal{X},$$

and

$f(v; \mu),\ \ell(v; \mu)$	are bounded linear functionals
$a(\cdot, \cdot; \mu) : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$	is continuous and coercive

and the nonlinearity satisfies

- $g^{\mathrm{nl}} : \mathbb{R} \times \Omega \times \mathcal{D} \to \mathbb{R}$ continuous;
- $g^{\mathrm{nl}}(u_1; x; \mu) \leq g^{\mathrm{nl}}(u_2; x; \mu),\ \forall u_1 \leq u_2$;
- $u\, g^{\mathrm{nl}}(u; x; \mu) \geq 0,\ \forall u \in \mathbb{R}$, for any $x \in \Omega,\ \mu \in \mathcal{D}$.

RB-Galerkin Approximation

We let

$$\mathcal{X}_N = \text{span}\{ \underbrace{u(\mu_i)}_{\text{SNAPSHOTS}} , \; i = 1, \ldots, n \; \}$$

and compute our approximation as

$$s_N(\mu) = \ell(u_N(\mu); \mu)$$

where $u_N(\mu) \in \mathcal{X}_N$ satisfies

$$a(u_N, v; \mu) + \int_\Omega g^{\text{nl}}(u_N; x; \mu)\, v = f(v), \quad \forall\, v \in \mathcal{X}_N .$$

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Can we still compute u_N efficiently?

RB-Galerkin Approximation

Sample Computation:

We expand $u_N(\mu) = \sum_{j=1}^{N} u_{N\,j}(\mu)\,\zeta_j,$

and obtain $\qquad (v = \zeta_i,\ 1 \le i \le \mathcal{N})$

$$\int_\Omega g^{\mathrm{nl}}(u_N(\mu); x; \mu)\,\zeta_i = \int_\Omega g^{\mathrm{nl}}\left(\sum_{j=1}^{N} u_{N\,j}(\mu)\,\zeta_j; x; \mu\right)\zeta_i$$

RB-Galerkin Approximation

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Issues

- Online assembly requires $\mathcal{N}$-dependent cost

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Issues

- Online assembly requires $\mathcal{N}$-dependent cost
- Similar to nonaffine problems: assume g^{na} is nonaffine, then

$$b(\zeta_i;\mu) = \int_\Omega g^{\mathrm{na}}(x;\mu)\,\zeta_i \qquad \text{e.g. } \mathrm{e}^{-\left(\frac{x-\mu}{\sigma}\right)^2}$$

requires $\mathcal{N}$-dependent cost.

Empirical Interpolation Method

Main Idea:

$$g^{\mathrm{na}}(x;\mu) \approx g_M^{\mathrm{na}}(x;\mu) = \sum_{m=1}^{M} \underbrace{\varphi_{M\,m}(\mu)}_{\text{EIM}} \quad \underbrace{q_m(x)}_{\text{Collateral RB}}$$

Then:
$$\begin{aligned} b(\zeta_i;\mu) &= \int_\Omega g^{\mathrm{na}}(x;\mu)\,\zeta_i \approx \int_\Omega g_M^{\mathrm{na}}(x;\mu)\,\zeta_i \\ &= \sum_{m=1}^{M} \varphi_{M\,m}(\mu) \int_\Omega q_m(x)\,\zeta_i\ , \end{aligned}$$

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Issues

- How do we choose $X_M^g = \mathrm{span}\{q_m(x),\ m = 1,\ldots,M\}$?

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$$g^{\mathrm{na}}(x;\mu) \approx g^{\mathrm{na}}_M(x;\mu) = \sum_{m=1}^{M} \underbrace{\varphi_{M\,m}(\mu)}_{\text{EIM}} \quad \underbrace{q_m(x)}_{\text{Collateral RB}}$$

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Issues

- How do we choose $X^g_M = \mathrm{span}\{q_m(x),\ m = 1,\dots,M\}$?
- Can we compute the coefficients $\varphi_{M\,m}(\mu)$ efficiently?

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Issues

- How do we choose $X^g_M = \mathrm{span}\{q_m(x),\, m = 1,\dots,M\}$?
- Can we compute the coefficients $\varphi_{M\,m}(\mu)$ efficiently?
- How do we quantify the (interpolation) error introduced?

Empirical Interpolation Method

Greedy Procedure

[MNPP07]

Initialize: Choose $\mu_1^g \in \mathcal{D}$ and set

$$x_1^T = \arg\max_{x\in\Omega} |g(x;\mu_1^g)|, \quad q_1 = \frac{g(x;\mu_1^g)}{g(x_1^T;\mu_1^g)}.$$

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For $1 \leq M \leq M_{\max} - 1$:

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For $1 \leq M \leq M_{\max} - 1$:

- Sample Set

$$\mu_{M+1}^g \equiv \arg\max_{\mu\in\Xi_{\text{train}}^g} \|g(\cdot;\mu) - g_M(\cdot;\mu)\|_{L^\infty(\Omega)}$$

Empirical Interpolation Method

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$$\mu_{M+1}^g \equiv \arg \max_{\mu \in \Xi_{\text{train}}^g} \|g(\cdot; \mu) - g_M(\cdot; \mu)\|_{L^\infty(\Omega)}$$
- Interpolation Points
$$x_{M+1}^T = \arg \max_{x \in \Omega} |r_{M+1}(x)|,$$
where $r_{M+1}(x) = g(x; \mu_{M+1}^g) - g_M(x; \mu_{M+1}^g)$

Empirical Interpolation Method

Greedy Procedure

[MNPP07]

Initialize: Choose $\mu_1^g \in \mathcal{D}$ and set

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$$x_{M+1}^T = \arg\max_{x\in\Omega} |r_{M+1}(x)|,$$

  where $r_{M+1}(x) = g(x;\mu_{M+1}^g) - g_M(x;\mu_{M+1}^g)$

- EIM Space

$$q_{M+1}(x) = \frac{r_{M+1}(x)}{r_{M+1}(x_{M+1}^T)}$$

Empirical Interpolation Method

Interpolation Points, Samples, and Spaces:

$$\begin{aligned} \boldsymbol{T}_M^g &= \{\boldsymbol{x}_1^T \in \boldsymbol{\Omega}, \ldots, \boldsymbol{x}_M^T \in \boldsymbol{\Omega}\}, \\ \boldsymbol{S}_M^g &\equiv \{\boldsymbol{\mu}_1^g \in \boldsymbol{\mathcal{D}}, \ldots, \boldsymbol{\mu}_M^g \in \boldsymbol{\mathcal{D}}\}, \text{ and associated} \\ \boldsymbol{X}_M^g &= \operatorname{span}\{q_1, \ldots, q_M\}, \quad 1 \leq M \leq M_{\max}, \end{aligned}$$

chosen by greedy procedure.

Empirical Interpolation Method

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Approximation: for given $\mu \in \mathcal{D}$

$$g(x;\mu) \approx g_M(x;\mu) = \sum_{m=1}^{M} \varphi_{M\,m}(\mu)\, q_m(x),$$

where

$$\sum_{m=1}^{M} q_m(x_n^T)\, \varphi_{M\,m}(\mu) = g(x_n^T;\mu), \quad 1 \leq n \leq M.$$

Note: $B_{nm}^M = q_m(x_n^T)$ lower triangular and invertible.

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How do we quantify the (interpolation) error?

Empirical Interpolation Method

"Next Point" Estimator

Given an approximation $g_M(x;\mu)$ for $M \leq M_{\max} - 1$, we define

$$\hat{\varepsilon}_M(\mu) \equiv |g(x^T_{M+1};\mu) - g_M(x^T_{M+1};\mu)|$$

and, if $g(\,\cdot\,;\mu) \in X^g_{M+1}$, we have

$$\|g(\,\cdot\,;\mu) - g_M(\,\cdot\,;\mu)\|_{L^\infty(\Omega)} \leq \hat{\varepsilon}_M(\mu).$$

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- The estimator $\hat{\varepsilon}_M(\mu)$ is very cheap to evaluate

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- The estimator $\hat{\varepsilon}_M(\mu)$ is very cheap to evaluate
- In general $g(\,\cdot\,;\mu) \notin X^g_{M+1}$, and hence our estimator $\hat{\varepsilon}_M(\mu)$ is indeed a lower bound
- An *a posteriori* error bound exists [EGP10], but does not extend to the nonlinear case

EIM-RB-Galerkin Approximation

Given $\boldsymbol{\mu} \in \mathcal{D}$, we evaluate

$$s_{N,M}(\boldsymbol{\mu}) = \ell(u_{N,M}(\boldsymbol{\mu}); \boldsymbol{\mu})$$

where $u_{N,M}(\boldsymbol{\mu}) \in \mathcal{X}_N$ satisfies

$$a(u_{N,M}, v; \boldsymbol{\mu}) + \int_\Omega g_M^{\mathrm{nl}, \mathrm{u}_{N,M}}(x; \boldsymbol{\mu})\, v = f(v), \quad \forall\, v \in \mathcal{X}_N.$$

and

$$g_M^{\mathrm{nl}, \mathrm{u}_{N,M}}(x; \boldsymbol{\mu}) \equiv \sum_{m=1}^{M} \varphi_{M\,m}(\boldsymbol{\mu}) q_m(x),$$

with $\sum_{j=1}^{M} B_{ij}^M \varphi_{M\,j}(\boldsymbol{\mu}) = g^{\mathrm{nl}}(u_{N,M}(x_i^T; \boldsymbol{\mu}); x_i^T; \boldsymbol{\mu}), \quad 1 \leq i \leq M.$

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Admits **offline/online** treatment:

online cost (per Newton iteration) $\mathcal{O}(M^2 + MN^2 + N^3)$

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Admits **offline/online** treatment:

online cost (per Newton iteration) $\mathcal{O}(M^2 + MN^2 + N^3)$

How do we quantify the (RB+EIM) error?

Error Estimation

Define the dual norm

$$\vartheta_M^q = \sup_{v \in \mathcal{X}} \frac{\int_\Omega q_{M+1}\, v}{\|v\|_\mathcal{X}}$$

and the **residual**, for all $v \in \mathcal{X}$,

$$r_{N,M}(v;\mu) := f(v) - a(u_{N,M}, v; \mu) - \int_\Omega g_M^{\mathrm{nl}, u_{N,M}}(x;\mu)\, v.$$

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If $g^{\mathrm{nl}}(u_{N,M}; x; \mu) \in X^g_{M+1}$, the **error**, $e_{N,M} := u - u_{N,M}$, is bounded by

$$\|e_{N,M}(\mu)\|_{\mathcal{X}} \leq \frac{1}{\alpha_{\mathrm{LB}}(\mu)} \left(\|r_{N,M}(\cdot;\mu)\|_{\mathcal{X}'} + \hat{\varepsilon}_M(\mu)\vartheta^q_M \right).$$

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If $g^{\mathrm{nl}}(u_{N,M}; x; \mu) \in X_{M+1}^g$, the **error**, $e_{N,M} := u - u_{N,M}$, is bounded by

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Note

- Admits **offline/online** treatment: online cost $\mathcal{O}(M^2 N^2)$

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Note

- Admits **offline/online** treatment: online cost $\mathcal{O}(M^2 N^2)$
- In general $g^{\mathrm{nl}}(u_{N,M}; x; \mu) \notin X_{M+1}^g$, and hence the bound is more like an estimator

Nonlinear Parabolic Problems

We wish to compute, for any $\mu \in \mathcal{D}$,

$$s(t;\mu) = \ell(u(t;\mu))$$

where $u(t;\mu) \in \mathcal{Y}$, satisfies $u(0;\mu) = 0$

$$m(\dot{u}(t;\mu), v;\mu) + a(u(t;\mu), v;\mu) + \int_\Omega g^{\mathrm{nl}}(u(t;\mu); x; \mu)\, v = b(v)u(t), \ \forall v \in \mathcal{Y}.$$

Key Ingredients:

Nonlinear Parabolic Problems

We wish to compute, for any $\mu \in \mathcal{D}$,

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Key Ingredients:

- FE[x]-FD[t] truth approximation

Nonlinear Parabolic Problems

We wish to compute, for any $\boldsymbol{\mu} \in \boldsymbol{\mathcal{D}}$,

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$$\boldsymbol{m(\dot{u}(t;\mu), v;\mu) + a(u(t;\mu), v;\mu) + \int_\Omega g^{\mathrm{nl}}(u(t;\mu);x;\mu)\, v = b(v)u(t), \ \forall v \in \mathcal{Y}.}$$

Key Ingredients:

- FE[x]-FD[t] truth approximation
- POD/Greedy algorithm to construct $\boldsymbol{\mathcal{X}_N}$ and $\boldsymbol{X_M^g}$

Nonlinear Parabolic Problems

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Key Ingredients:

- FE[x]-FD[t] truth approximation
- POD/Greedy algorithm to construct $\mathcal{X}_N$ and X_M^g
- Error bound/estimator for space-time energy norm

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Key Ingredients:

- FE$[x]$-FD$[t]$ truth approximation
- POD/Greedy algorithm to construct $\mathcal{X}_N$ and X_M^g
- Error bound/estimator for space-time energy norm
- Online cost: $\mathcal{O}(M^2 + MN^2 + N^3)^\dagger$ plus $\mathcal{O}(KM^2N^2)$.

† Cost per Newton iteration per timestep.

Extensions & References

Empirical Interpolation Method:

- **methodology** [BMNP04], [GMNP07], [MNPP07], . . .
- **error bounds** [EGP10], [EGPR13], . . .
- **generalized EIM** [MM13], [MMPY15], . . .
- **variations** [CS10], . . .

Related Methods:

- **best points interpolation** [NPP08], . . .
- **adaptive cross approximation** [Beb00], [Beb11], . . .
- **gappy POD** [ES95], [BTDW04], [Wil06], [GFWG10], . . .
- **missing point estimation** [AWWB08], . . .

For a comparison of these methods: [BMS14]

Extensions & References

RB for nonaffine problems:

- **elliptic** [BMNP04], [GMNP07], [Ngu07], [Roz09], . . .
- **parabolic** [GMNP07], [Gre12a], [KGV12], . . .

RB for nonlinear problems:

- **elliptic** [GMNP07], [NP08], [CTU09], . . .
- **parabolic** [GMNP07], [Gre12a], [DHO12], [Gre12b], . . .

Issues

- Convergence and stability of EIM-RB approximation
- Greedy EIM algorithm requires (truth) solution at all $\boldsymbol{\mu} \in \boldsymbol{\mathcal{D}}_{\mathbf{s}}$
- Simultaneous EIM + RB [DP15]

Model Problem

Given $\mu = (\mu_1, \mu_2) \in \mathcal{D} \equiv [0.01, 10]^2$, evaluate $\Omega =]0,1[^2$

$$s^k(\mu) = \int_\Omega u^k(\mu)$$

where $u^k(\mu) \in \mathcal{X}$, $1 \le k \le K$, satisfies $u^0(\mu) = 0$

$$\frac{1}{\Delta t} m(u^k(\mu) - u^{k-1}(\mu), v) + a(u^k(\mu), v) + \int_\Omega g^{\mathrm{nl}}(u^k(\mu); x; \mu)\, v = b(v)\, \sin(2\pi t^k), \quad \forall\, v \in \mathcal{X},$$

with $g^{\mathrm{nl}}(u^k(\mu); x; \mu) = \mu_1 \dfrac{e^{\mu_2\, u^k(\mu)} - 1}{\mu_2}$.

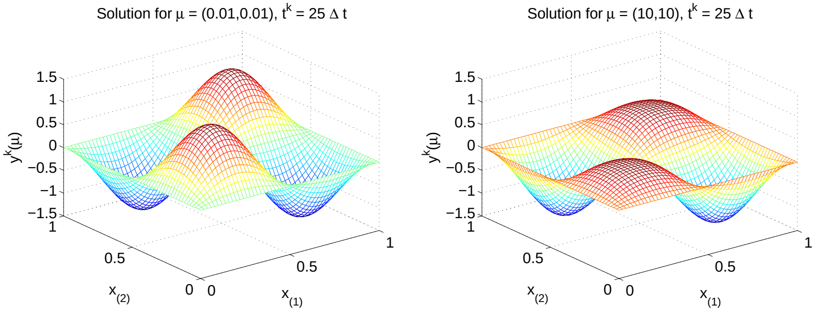
Truth Approximation:

- Space: $\mathcal{X} \subset \mathcal{X}^{\mathrm{e}} \equiv H_0^1(\Omega)$ with dimension $\mathcal{N} = 2601$;
- Time: $\bar{I} = (0, 2]$, $\Delta t = 0.01$, and thus $K = 200$.

Sample Results

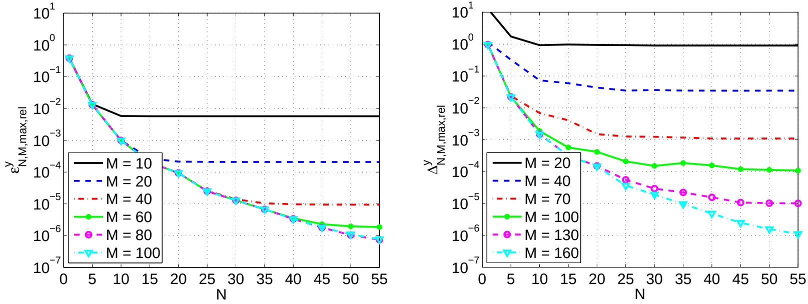
Truth solution $u(t^k;\mu)$ at time $t^k = 25\Delta t$ and

$\mu = (0.01, 0.01)$ $\mu = (10, 10)$



$$b(v) = 100 \int_\Omega v \, \sin(2\pi x_1)\sin(2\pi x_2)$$

Convergence: Energy Norm



Results for sample $\Xi_{\mathbf{test}} \in \mathcal{D}$ of size $\mathbf{225}$.

- “Plateau” in curves for M fixed.
- “Knees” reflect balanced contribution of both error terms.
- Sharp bounds require conservative choice of M.

Convergence: Energy Norm & Output

N	M	$\epsilon^y_{N,M,\max,\mathrm{rel}}$	$\Delta^y_{N,M,\max,\mathrm{rel}}$	$\bar{\eta}^y_{N,M}$
1	40	3.83 E−01	1.15 E+00	2.44
5	60	1.32 E−02	4.59 E−02	2.43
10	80	9.90 E−04	3.41 E−03	2.10
20	100	9.40 E−05	4.16 E−04	2.77
40	140	3.36 E−06	8.75 E−06	1.64

N	M	$\epsilon^s_{N,M,\max,\mathrm{rel}}$	$\Delta^s_{N,M,\max,\mathrm{rel}}$	$\bar{\eta}^s_{N,M}$
1	40	9.99 E−01	2.49 E+01	14.1
5	60	5.35 E−03	1.00 E+00	130
10	80	2.57 E−04	7.42 E−02	146
20	100	1.43 E−05	9.06 E−03	436
40	140	2.85 E−06	1.90 E−04	205

Results for sample $\Xi_{\mathrm{test}} \in \mathcal{D}$ of size $\mathbf{225}$.

Online Computational Times

N	M	$s_{N,M}(\mu, t^k)$	$\Delta^s_{N,M}(\mu, t^k)$	$s(\mu, t^k)$
1	40	5.42 E−05	9.29 E−05	1
5	60	9.67 E−05	8.58 E−05	1
10	80	1.19 E−04	9.37 E−05	1
20	100	1.71 E−04	1.05 E−04	1
40	140	3.15 E−04	1.35 E−04	1

Average CPU times for sample $\Xi_{\text{test}} \in \mathcal{D}$ of size $\mathbf{225}$.

- Computational savings $\mathcal{O}(10^3)$ for $\Delta^s_{N,M,\max,\text{rel}} < 1\%$.
- But offline stage much more expensive than for linear case.

Nonlinear Reaction-Diffusion Systems

General formulation: [Gre12b]

$$\frac{\partial \mathbf{y}(\mathbf{x},\mathbf{t};\boldsymbol{\mu})}{\partial t} = \nabla(D(\boldsymbol{\mu})\nabla\mathbf{y}(x,t;\boldsymbol{\mu})) + \mathbf{f}(\mathbf{y}(x,t;\boldsymbol{\mu});\boldsymbol{\mu})$$

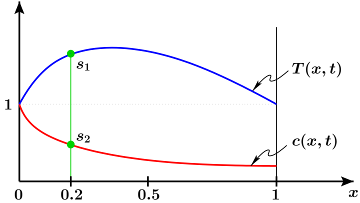
Self-ignition of a coal stockpile

$$\frac{\partial T(x,t)}{\partial t} = \nabla^2 T(x,t) + \beta\,\Phi^2\,(c(x,t)+1)\,e^{-\gamma/(T(x,t)+1)},$$

$$\frac{\partial c(x,t)}{\partial t} = \mathrm{Le}\,\nabla^2 c(x,t) - \Phi^2\,(c(x,t)+1)\,e^{-\gamma/(T(x,t)+1)},$$

where

- γ : Arrhenius number,
- β : Prater temperature,
- Le : Lewis number,
- Φ : Thiele modulus.

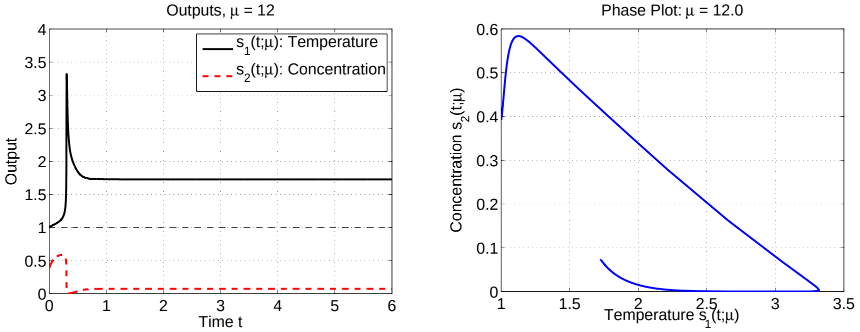


- Nonlinearity not monotonic: *a posterior* error bounds not valid.

Nonlinear Reaction-Diffusion Systems

- Very complex dynamic behavior (depending on parameters).
- Here: $\mu \equiv \gamma \in [12, 12.6]$, all other parameters fixed, $\mathcal{N} = 501$

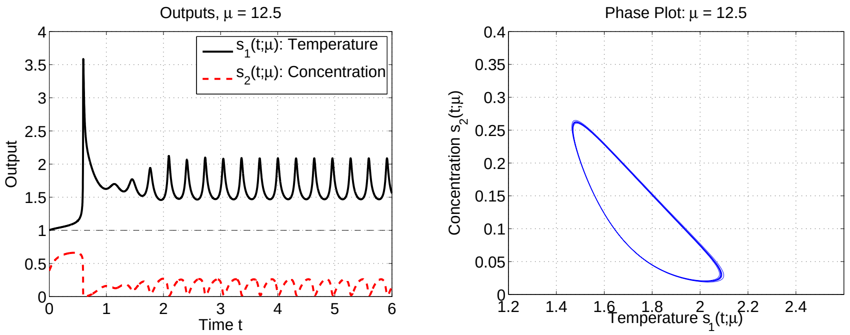
Temperature and Concentration: $\gamma = 12.0$



Nonlinear Reaction-Diffusion Systems

- Very complex dynamic behavior (depending on parameters).
- Here: $\boldsymbol{\mu} \equiv \gamma \in [\mathbf{12}, \mathbf{12.6}]$, all other parameters fixed, $\boldsymbol{\mathcal{N}} = \mathbf{501}$

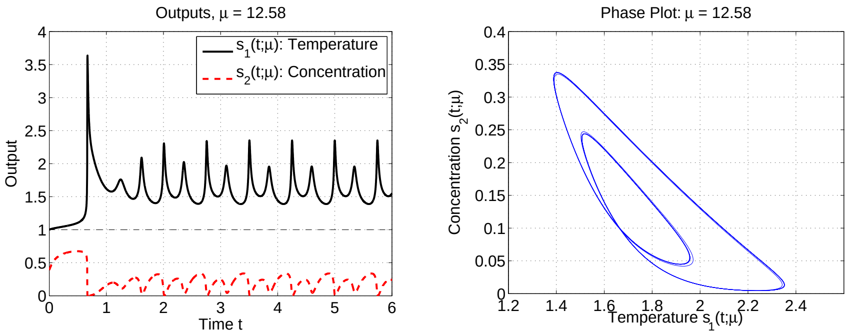
Temperature and Concentration: $\gamma = \mathbf{12.5}$



Nonlinear Reaction-Diffusion Systems

- Very complex dynamic behavior (depending on parameters).
- Here: $\boldsymbol{\mu} \equiv \gamma \in [\mathbf{12}, \mathbf{12.6}]$, all other parameters fixed, $\boldsymbol{\mathcal{N}} = \mathbf{501}$

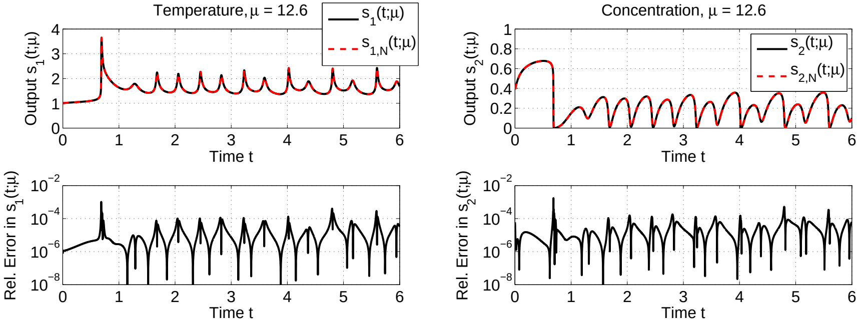
Temperature and Concenctration: $\gamma = \mathbf{12.58}$



Nonlinear Reaction-Diffusion Systems

- Very complex dynamic behavior (depending on parameters).
- Here: $\boldsymbol{\mu} \equiv \gamma \in [\mathbf{12}, \mathbf{12.6}]$, all other parameters fixed, $\boldsymbol{\mathcal{N}} = \mathbf{501}$
  $\boldsymbol{N_T} = \boldsymbol{N_c} = \mathbf{16}$, $\boldsymbol{M} = \mathbf{36}$: $\boldsymbol{t_{\mathrm{RB}}/t_{\mathrm{FEM}}} = \mathbf{5.12\,E{-}03}$

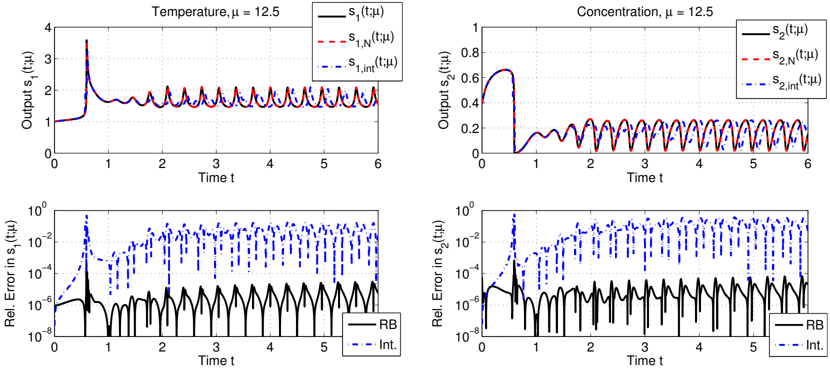
FEM solution and RB approximation: $\gamma = \mathbf{12.6}$



Nonlinear Reaction-Diffusion Systems

- Very complex dynamic behavior (depending on parameters).
- Here: $\mu \equiv \gamma \in [12, 12.6]$, all other parameters fixed, $\mathcal{N} = 501$
  $N_T = N_c = 16$, $M = 36$: $t_{\mathrm{RB}}/t_{\mathrm{FEM}} = 5.12\,\mathrm{E}{-}03$

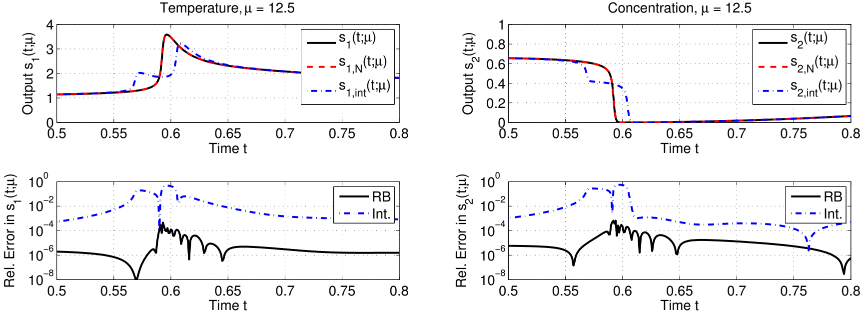
FEM, RB, and output interpolation: $\gamma = 12.5$



Nonlinear Reaction-Diffusion Systems

- Very complex dynamic behavior (depending on parameters).
- Here: $\boldsymbol{\mu \equiv \gamma \in [12, 12.6]}$, all other parameters fixed, $\boldsymbol{\mathcal{N} = 501}$
  $\boldsymbol{N_T = N_c = 16,\ M = 36}$: $\boldsymbol{t_{\mathrm{RB}}/t_{\mathrm{FEM}} = 5.12\,\mathrm{E}{-}03}$

FEM, RB, and output interpolation: $\boldsymbol{\gamma = 12.5}$



References I

[AWWB08] P. Astrid, S. Weiland, K. Willcox, and T. Backx. Missing point estimation in models described by proper orthogonal decomposition. *IEEE Transactions on Automatic Control*, 53(10):2237–2251, 2008.

[BBL+10] S. Boyaval, C. Le Bris, T. Lelièvre, Y. Maday, N.C. Nguyen, and A.T. Patera. Reduced basis techniques for stochastic problems. *Arch. Comput. Method. E.*, 17:435–454, 2010.

[BBM+09] S. Boyaval, C. Le Bris, Y. Maday, N.C. Nguyen, and A.T. Patera. A reduced basis approach for variational problems with stochastic parameters: Application to heat conduction with variable robin coefficient. *Comput. Methods Appl. Mech. Engrg.*, 198(41-44):3187 – 3206, 2009.

[BCD+11] P. Binev, A. Cohen, W. Dahmen, R. DeVore, G. Petrova, and P. Wojtaszczyk. Convergence rates for greedy algorithms in reduced basis methods. *SIAM J. Math. Anal.*, 43(3):1457–1472, 2011.

[Beb00] M. Bebendorf. Approximation of boundary element matrices. *Numer. Math.*, 86(4):565–589, 2000.

[Beb11] M. Bebendorf. Adaptive cross approximation of multivariate functions. *Constr. Approx.*, 34(2):149–179, 2011.

[BMNP04] M. Barrault, Y. Maday, N.C. Nguyen, and A.T. Patera. An 'empirical interpolation' method: application to efficient reduced-basis discretization of partial differential equations. *C. R. Math.*, 339(9):667 – 672, 2004.

[BMP+12] A. Buffa, Y. Maday, A.T. Patera, C. Prud'homme, and G. Turinici. A priori convergence of the greedy algorithm for the parametrized reduced basis method. *ESAIM: Math. Model. Num.*, 46(03):595–603, 4 2012.

[BMS14] M. Bebendorf, Y. Maday, and B. Stamm. Comparison of some reduced representation approximations. In A. Quarteroni and G. Rozza, editors, *Reduced Order Methods for Modeling and Computational Reduction*, volume 9 of *MS&A - Modeling, Simulation and Applications*, pages 67–100. Springer International Publishing, 2014.

[BRR80] F. Brezzi, J. Rappaz, and P. A. Raviart. Finite dimensional approximation of nonlinear problems. *Numer. Math.*, 38(1):1–30, 1980. 10.1007/BF01395805.

References II

[BTDW04] T. Bui-Thanh, M. Damodaran, and K. Willcox. Aerodynamic data reconstruction and inverse design using proper orthogonal decomposition. *AIAA Journal*, 42(8):1505–1516, 2004.

[BTWG08] T. Bui-Thanh, K. Willcox, and O. Ghattas. Model reduction for large-scale systems with high-dimensional parametric input space. *SIAM J. Sci. Comput.*, 30:3270–3288, October 2008.

[BTWGvBW07] T. Bui-Thanh, K. Willcox, O. Ghattas, and B. van Bloemen Waanders. Goal-oriented, model-constrained optimization for reduction of large-scale systems. *J. Comput. Phys.*, 224(2):880 – 896, 2007.

[CHMR09] Y. Chen, J.S. Hesthaven, Y. Maday, and J. Rodríguez. Improved successive constraint method based a posteriori error estimate for reduced basis approximation of 2d maxwell's problem. *ESAIM: Math. Model. Num.*, null:1099–1116, 10 2009.

[CS10] S. Chaturantabut and D.C. Sorensen. Nonlinear model reduction via discrete empirical interpolation. *SIAM J. Sci. Comput.*, 32(5):2737–2764, 2010.

[CTU09] C. Canuto, T. Tonn, and K. Urban. A-posteriori error analysis of the reduced basis method for non-affine parameterized nonlinear pde's. *SIAM J. Numer. Anal.*, 47:2001–2022, 2009.

[Dep08] S. Deparis. Reduced basis error bound computation of parameter-dependent Navier-Stokes equations by the natural norm approach. *SIAM Journal of Numerical Analysis*, 46(4):2039–2067, 2008.

[DHO12] M. Drohmann, B. Haasdonk, and M. Ohlberger. Reduced basis approximation for nonlinear parametrized evolution equations based on empirical operator interpolation. *SIAM J. Sci. Comput.*, 34(2):A937–A969, 2012.

[DL12] S. Deparis and A.E. Løvgren. Stabilized reduced basis approximation of incompressible three-dimensional Navier-Stokes equations in parametrized deformed domains. *J. Sci. Comput.*, 50:198–212, 2012. 10.1007/s10915-011-9478-2.

[DP15] C. Daversin and C. Prud'homme. Simultaneous empirical interpolation and reduced basis method for non-linear problems. Technical report, arXiv:1504.06131 [math.AP], 2015.

References III

[DPW12] R. DeVore, G. Petrova, and P. Wojtaszczyk. Greedy algorithms for reduced bases in Banach spaces. *Constr. Approx.*, 2012.

[DPW14] W. Dahmen, C. Plesken, and G. Welper. Double greedy algorithms: Reduced basis methods for transport dominated problems. *ESAIM: Mathematical Modelling and Numerical Analysis*, 48:623–663, 5 2014.

[EGP10] J.L. Eftang, M.A. Grepl, and A.T. Patera. A posteriori error bounds for the empirical interpolation method. *C. R. Math.*, 348:575–579, 2010.

[EGPR13] J.L. Eftang, M.A. Grepl, A.T. Patera, and E.M. Rønquist. Approximation of parametric derivatives by the empirical interpolation method. *Foundations of Computational Mathematics*, 13(5):763–787, 2013.

[EPR10] J.L. Eftang, A.T. Patera, and E.M. Rønquist. An "hp" certified reduced basis method for parametrized elliptic partial differential equations. *SIAM J. Sci. Comput.*, 32:3170–3200, 2010.

[ES95] R. Everson and L. Sirovich. Karhunen-Loeve procedure for gappy data. *JOSA A*, 12(8):1657–1664, 1995.

[GFWG10] D. Galbally, K. Fidkowski, K. Willcox, and O. Ghattas. Non-linear model reduction for uncertainty quantification in large-scale inverse problems. *Int. J. Numer. Meth. Engng*, 81(12):1581–1608, 2010.

[GMNP07] M.A. Grepl, Y. Maday, N.C. Nguyen, and A.T. Patera. Efficient reduced-basis treatment of nonaffine and nonlinear partial differential equations. *ESAIM: Math. Model. Num.*, 41(3):575–605, 2007.

[GP05] M.A. Grepl and A.T. Patera. A posteriori error bounds for reduced-basis approximations of parametrized parabolic partial differential equations. *ESAIM: Math. Model. Num.*, 39(1):157–181, 2005.

[Gre12a] M.A. Grepl. Certified reduced basis methods for nonaffine linear time-varying and nonlinear parabolic partial differential equations. *M3AS: Mathematical Models and Methods in Applied Sciences*, 22(3):40, 2012.

[Gre12b] M.A. Grepl. Model order reduction of parametrized nonlinear reaction-diffusion systems. *Computers and Chemical Engineering*, 43(0):33 – 44, 2012.

References IV

[GV12] A.-L. Gerner and K. Veroy. Certified reduced basis methods for parametrized saddle point problems. *SIAM J. Sci. Comput.*, 34(5):A2812–A2836, 2012.

[Haa13] B. Haasdonk. Convergence rates of the pod-greedy method. *ESAIM: Math. Model. Num.*, 47:859–873, 4 2013.

[HKC+10] D. B. P. Huynh, D. J. Knezevic, Y. Chen, J.S. Hesthaven, and A.T. Patera. A natural-norm successive constraint method for inf-sup lower bounds. *Comput. Methods Appl. Mech. Engrg.*, 199(29-32):1963–1975, 2010.

[HKP11] D. P. B. Huynh, D. J. Knezevic, and A. T. Patera. A laplace transform certified reduced basis method; application to the heat equation and wave equation. *C. R. Math.*, 349:401–405, 2011.

[HO08] B. Haasdonk and M. Ohlberger. Reduced basis method for finite volume approximations of parametrized linear evolution equations. *ESAIM: Math. Model. Num.*, 42(02):277–302, 2008.

[HRS15] J.S. Hesthaven, G. Rozza, and B. Stamm. *Certified Reduced Basis Methods for Parametrized Problems*. Springer Briefs in Mathematics. Springer, 2015.

[HRSP07] D. B. P. Huynh, G. Rozza, S. Sen, and A. T. Patera. A successive constraint linear optimization method for lower bounds of parametric coercivity and inf-sup stability constants. *C. R. Math.*, 345(8):473–478, 2007.

[KGV12] D. Klindworth, M.A. Grepl, and G. Vossen. Certified reduced basis methods for parametrized parabolic partial differential equations with non-affine source terms. *Comput. Methods Appl. Mech. Engrg.*, 209-212(0):144–155, 2012.

[LR10] T. Lassila and G. Rozza. Parametric free-form shape design with pde models and reduced basis method. *Comput. Methods Appl. Mech. Engrg.*, 199(23-24):1583–1592, 2010.

References V

[MM13] Y. Maday and O. Mula. A generalized empirical interpolation method: Application of reduced basis techniques to data assimilation. In *Analysis and Numerics of Partial Differential Equations*, volume 4 of *Springer INdAM Series*, pages 221–235. Springer Milan, 2013.

[MMPY15] Y. Maday, O. Mula, A.T. Patera, and M. Yano. The generalized empirical interpolation method: Stability theory on hilbert spaces with an application to the stokes equation. *Computer Methods in Applied Mechanics and Engineering*, 287(0):310 – 334, 2015.

[MNPP07] Y. Maday, N.C. Nguyen, A.T. Patera, and G.S.H. Pau. A general, multipurpose interpolation procedure: the magic points. *Communications on Pure and Applied Analysis (CPAA)*, 8:383 – 404, 2007.

[MPR02] Y. Maday, A. T. Patera, and D. V. Rovas. A blackbox reduced-basis output bound method for noncoercive linear problems. *Studies in Mathematics and its Applications, D. Cioranescu and J. L. Lions, eds., Elsevier Science B.V.*, 31:533–569, 2002.

[MPT02] Y. Maday, A.T. Patera, and G. Turinici. Global a priori convergence theory for reduced-basis approximations of single-parameter symmetric coercive elliptic partial differential equations. *C. R. Math.*, 335(3):289 – 294, 2002.

[MPU14] M.Yano, A.T. Patera, and K. Urban. A space-time hp-interpolation-based certified reduced basis method for burgers' equation. *Mathematical Models and Methods in Applied Sciences*, 0(0):1–33, 2014.

[Ngu07] N. C. Nguyen. A posteriori error estimation and basis adaptivity for reduced-basis approximation of nonaffine-parametrized linear elliptic partial differential equations. *J. Comput. Phys.*, 227:983–1006, December 2007.

[Ngu08] N.C. Nguyen. A multiscale reduced-basis method for parametrized elliptic partial differential equations with multiple scales. *J. Comput. Phys.*, 227(23):9807 – 9822, 2008.

[NP08] N. C. Nguyen and J. Peraire. An efficient reduced-order modeling approach for non-linear parametrized partial differential equations. *Int. J. Numer. Methods Eng.*, 76:27–55, 2008.

References VI

[NPP08] N. C. Nguyen, A. T. Patera, and J. Peraire. A "best points" interpolation method for efficient approximation of parametrized functions. *Int. J. Numer. Methods Eng.*, 73(4):521–543, 2008.

[PR07] Anthony T. Patera and Einar M. Ronquist. Reduced basis approximation and a posteriori error estimation for a boltzmann model. *Comput. Methods Appl. Mech. Engrg.*, 196(29-30):2925 – 2942, 2007.

[PRV+02] C. Prud'homme, D. V. Rovas, K. Veroy, L. Machiels, Y. Maday, A. T. Patera, and G. Turinici. Reliable real-time solution of parametrized partial differential equations: Reduced-basis output bound methods. *J. Fluid. Eng.*, 124(1):70–80, 2002.

[PRVP02] C. Prud'homme, D.V. Rovas, K. Veroy, and A.T. Patera. A mathematical and computational framework for reliable real-time solution of parametrized partial differential equations. *ESAIM: Math. Model. Num.*, 36:747–771, 2002.

[QRM11] A. Quarteroni, G. Rozza, and A. Manzoni. Certified reduced basis approximation for parametrized partial differential equations and applications. *Journal of Mathematics in Industry*, 1(3):1–49, 2011.

[RG12] M. Rasty and M.A. Grepl. Efficient reduced basis solution of quadratically nonlinear diffusion equations. In *Proceedings of 7th Vienna Conference on Mathematical Modelling – MATHMOD 2012*, 2012. accepted.

[RHP08] G. Rozza, D.B.P. Huynh, and A.T. Patera. Reduced basis approximation and a posteriori error estimation for affinely parametrized elliptic coercive partial differential equations. *Archives of Computational Methods in Engineering*, 15(3):229–275, 2008.

[RMM06] D. V. Rovas, L. Machiels, and Y. Maday. Reduced-basis output bound methods for parabolic problems. *IMA J. Numer. Anal.*, 26(3):423–445, 2006.

[Rov03] D.V. Rovas. *Reduced-Basis Output Bound Methods for Parametrized Partial Differential Equations*. PhD thesis, Massachusetts Institute of Technology, February 2003.

References VII

[Roz09] G. Rozza. Reduced basis methods for stokes equations in domains with non-affine parameter dependence. *Computing and Visualization in Science*, 12:23–35, 2009. 10.1007/s00791-006-0044-7.

[RV07] G. Rozza and K. Veroy. On the stability of the reduced basis method for Stokes equations in parametrized domains. *Comput. Methods Appl. Mech. Engrg.*, 196(7):1244 – 1260, 2007.

[UP14] K. Urban and A.T. Patera. An improved error bound for reduced basis approximation of linear parabolic problems. *Math. Comp.*, 83:1599–1615, 2014.

[VP05] K. Veroy and A. T. Patera. Certified real-time solution of the parametrized steady incompressible Navier-Stokes equations: rigorous reduced-basis a posteriori error bounds. *Internat. J. Numer. Methods Fluids*, 47:773–788, 2005.

[VPP03] K. Veroy, C. Prud'homme, and A.T. Patera. Reduced-basis approximation of the viscous burgers equation: rigorous a posteriori error bounds. *C. R. Math.*, 337(9):619–624, 2003.

[VPRP03] K. Veroy, C. Prud'homme, D. V. Rovas, and A. T. Patera. A posteriori error bounds for reduced-basis approximation of parametrized noncoercive and nonlinear elliptic partial differential equations. In *Proceedings of the 16th AIAA Computational Fluid Dynamics Conference*, 2003. AIAA Paper 2003-3847.

[VRP02] K. Veroy, D.V. Rovas, and A.T. Patera. A posteriori error estimation for reduced-basis approximation of parametrized elliptic coercive partial differential equations: "convex inverse" bound conditioners. *ESAIM: Contr. Optim. Ca.*, 8:1007–1028, 2002. Special Volume: A tribute to J.L. Lions.

[Wil06] K. Willcox. Unsteady flow sensing and estimation via the gappy proper orthogonal decomposition. *Computers and Fluids*, 35(2):208 – 226, 2006.

[Yan14] M. Yano. A space-time petrov–galerkin certified reduced basis method: Application to the boussinesq equations. *SIAM J. Sci. Comput.*, 36(1):A232–A266, 2014.

Thank you for your attention!

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Part III

Extra Slides

Offline/Online Decomposition: Error Bounds

Crucial ingredient: Dual norm of residual $\|r_N(\cdot;\mu)\|_{\mathcal{X}'}$

We expand $u_N(\mu) = \sum_{j=1}^{N} u_{N\,j}(\mu)\,\zeta_j$

and obtain from the definition of the residual and affine dependence

$$
\begin{aligned}
r_N(v;\mu) &= f(v) - a(u_N(\mu), v; \mu) \\
&= f(v) - a\Big(\sum_{n=1}^{N} u_{N\,n}(\mu)\zeta_n, v; \mu\Big) \\
&= f(v) - \sum_{n=1}^{N} u_{N\,n}(\mu)\, a\big(\zeta_n, v; \mu\big) \\
&= f(v) - \sum_{n=1}^{N} u_{N\,n}(\mu) \sum_{q=1}^{Q_a} \theta_a^q(\mu)\, a^q\big(\zeta_n, v\big)
\end{aligned}
$$

For simplicity, we assume here that $f(v)$ does not depend on μ.

Offline/Online Decomposition: Error Bounds

Riesz representation:

$$
\begin{aligned}
(\hat{e}(\mu), v)_{\mathcal{X}} &= r_N(v; \mu) \\
&= f(v) - \sum_{q=1}^{Q_a} \sum_{n=1}^{N} \theta_a^q(\mu) u_{N\,n}(\mu) a^q\big(\zeta_n, v\big),
\end{aligned}
$$

Offline/Online Decomposition: Error Bounds

Riesz representation:

$$\begin{aligned}(\hat{e}(\mu), v)_{\mathcal{X}} &= r_N(v;\mu) \\ &= f(v) - \sum_{q=1}^{Q_a}\sum_{n=1}^{N} \theta_a^q(\mu) u_{N\,n}(\mu) a^q\big(\zeta_n, v\big),\end{aligned}$$

Linear Superposition:

$$\Rightarrow \quad \hat{e}(\mu) = \mathcal{C} + \sum_{q=1}^{Q_a}\sum_{n=1}^{N} \theta_a^q(\mu) u_{N\,n}(\mu) \mathcal{A}_n^q$$

where

$$\begin{aligned}(\mathcal{C}, v)_{\mathcal{X}} &= f(v), & \forall v \in \mathcal{X}; \\ (\mathcal{A}_n^q, v)_{\mathcal{X}} &= -a^q(\zeta_n, v), & \forall v \in \mathcal{X}, \\ & & 1 \le n \le N, 1 \le q \le Q_a.\end{aligned}$$

Offline/Online Decomposition: Error Bounds

Thus

$$\|\hat{e}(\mu)\|_{\mathcal{X}}^2 = \left(\mathcal{C} + \sum_{q=1}^{Q_a} \sum_{n=1}^{N} \theta_a^q(\mu) u_{N\,n}(\mu) \mathcal{A}_n^q \,,\, \cdot \right)_{\mathcal{X}}$$

$$=$$

Offline/Online Decomposition: Error Bounds

Thus

$$\|\hat{e}(\mu)\|_{\mathcal{X}}^2 = \left(\mathcal{C} + \sum_{q=1}^{Q_a} \sum_{n=1}^{N} \theta_a^q(\mu) u_{N\,n}(\mu) \mathcal{A}_n^q \,,\ \cdot\ \right)_{\mathcal{X}}$$

$$= (\mathcal{C}, \mathcal{C})_{\mathcal{X}} + \sum_{q=1}^{Q_a} \sum_{n=1}^{N} \theta_a^q(\mu) u_{N\,n}(\mu) \Bigg\{$$

$$2(\mathcal{C}, \mathcal{A}_n^q)_{\mathcal{X}} + \sum_{q'=1}^{Q_a} \sum_{n'=1}^{N} \theta_a^{q'}(\mu) u_{N\,n'}(\mu) (\mathcal{A}_n^q, \mathcal{A}_{n'}^{q'})_{\mathcal{X}} \Bigg\}$$

Offline/Online Decomposition: Error Bounds

Offline: *once, parameter independent*

- Compute $\mathcal{C}, \mathcal{A}_n^q,\ 1 \le n \le N_{\max},\ 1 \le q \le Q_a$, from

$$\begin{aligned}
(\mathcal{C}, v)_{\mathcal{X}} &= f(v), & \forall v \in \mathcal{X}; \\
(\mathcal{A}_n^q, v)_{\mathcal{X}} &= -a^q(\zeta_n, v), & \forall v \in \mathcal{X}, \\
& & 1 \le n \le N, 1 \le q \le Q_a.
\end{aligned}$$

Offline/Online Decomposition: Error Bounds

Offline: *once, parameter independent*

- Compute $\mathcal{C}, \mathcal{A}_n^q,\ 1 \le n \le N_{\max},\ 1 \le q \le Q_a$, from

$$\begin{aligned}
(\mathcal{C}, v)_{\mathcal{X}} &= f(v), && \forall v \in \mathcal{X}; \\
(\mathcal{A}_n^q, v)_{\mathcal{X}} &= -a^q(\zeta_n, v), && \forall v \in \mathcal{X}, \\
& && 1 \le n \le N, 1 \le q \le Q_a.
\end{aligned}$$

- Form/Store $(\mathcal{C}, \mathcal{C})_{\mathcal{X}},\ (\mathcal{C}, \mathcal{A}_n^q)_{\mathcal{X}},\ (\mathcal{A}_n^q, \mathcal{A}_{n'}^{q'})_{\mathcal{X}}$,

$$1 \le n, n' \le N_{\max}, 1 \le q, q' \le Q_a.$$

Complexity depends on N, Q_a, and $\mathcal{N}$.

Offline/Online Decomposition: Error Bounds

Online: *many times, for each new* $\boldsymbol{\mu}$

(and associated solution $u_N(\mu)$)

- Evaluate

$$\|\hat{e}(\mu)\|_{\mathcal{X}}^2 = (\mathcal{C}, \mathcal{C})_{\mathcal{X}} + \sum_{q=1}^{Q_a} \sum_{n=1}^{N} \theta_a^q(\mu) u_{N\,n}(\mu) \Big\{ 2(\mathcal{C}, \mathcal{A}_n^q)_{\mathcal{X}} + \sum_{q'=1}^{Q_a} \sum_{n'=1}^{N} \theta_a^{q'}(\mu) u_{N\,n'}(\mu) (\mathcal{A}_n^q, \mathcal{A}_{n'}^{q'})_{\mathcal{X}} \Big\}$$

– $O(Q_a^2 N^2)$

Complexity depends on N, Q_a, but not $\mathcal{N}$.

Offline/Online Decomposition: Error Bounds

Summary of computational cost:

OFFLINE —

$$\underbrace{O(Q_a N_{\max} \mathcal{N}^{\bullet})}_{\text{solve Poisson problems}} \quad + \quad \underbrace{O(Q_a^2 N_{\max}^2 \mathcal{N})}_{\text{form } \mu\text{-independent inner products}} \; ;$$

ONLINE —

$$\underbrace{O(Q_a^2 N^2)}_{\text{evaluate } \|\hat{e}(\mu)\|_{\mathcal{X}}\text{-sum}} \; ;$$

Online cost is independent of $\mathcal{N}$.