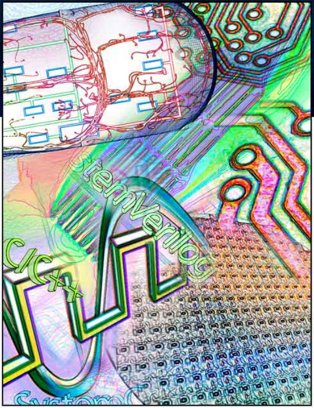
Algorithms for eigenvalue problems arising in model reduction



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MorTrans 2015, Berlin

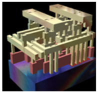
May 19, 2015

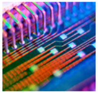
Acknowledgments

- Jan ter Maten (U Wuppertal)
- Wil Schilders (TU Eindhoven)
- Nelson Martins (CEPEL)
- Francisco Freitas (Univerity of Brasilia)
- Gerard Sleijpen (Utrecht University)
- Pascal Bolcato, Olivier Maury (Mentor Graphics)

Mentor Graphics

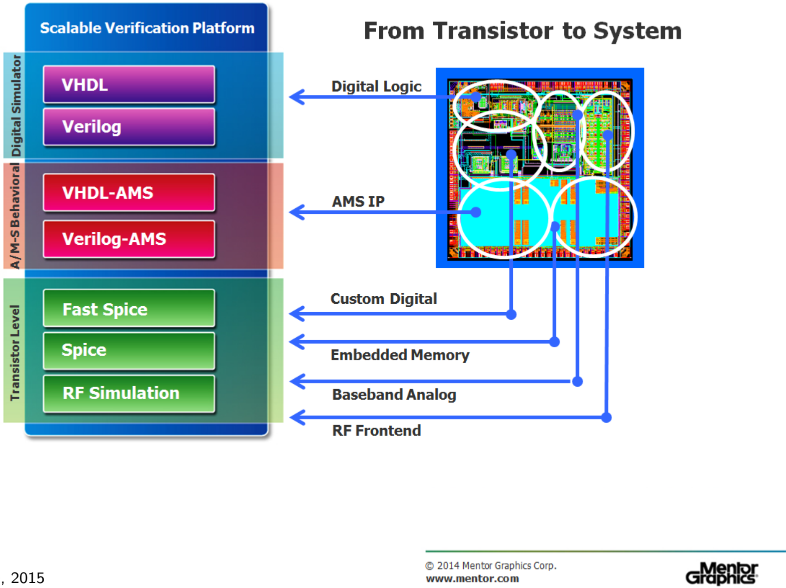
- Electronic design automation (EDA) industry pioneer and global innovator of advanced design solutions
- Founded in 1981
- Revenue - ~$1,015B
- Market Share ~24% of worldwide EDA market
- Focused on growth through internal development



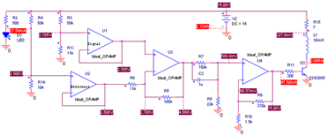


Source: EDAC Market Statistics

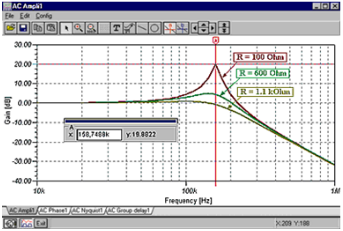
From transistor to system

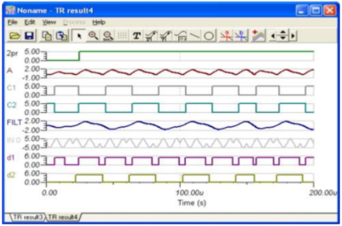


Analog simulation: basic analyses



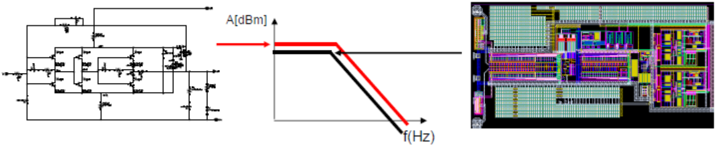
- DC: static operating point
- AC: linearized, frequency domain response





- TRAN: time domain response

Differential-Algebraic Equations (DAEs)

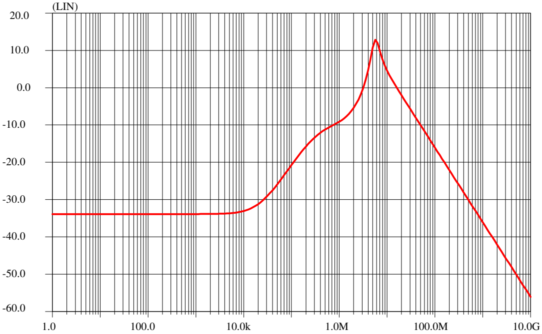


Modeled by system of differential-algebraic equations:

$$\frac{d}{dt}\mathbf{q}(t,\mathbf{x}) + \mathbf{j}(t,\mathbf{x}) = \mathbf{b}u(t)$$

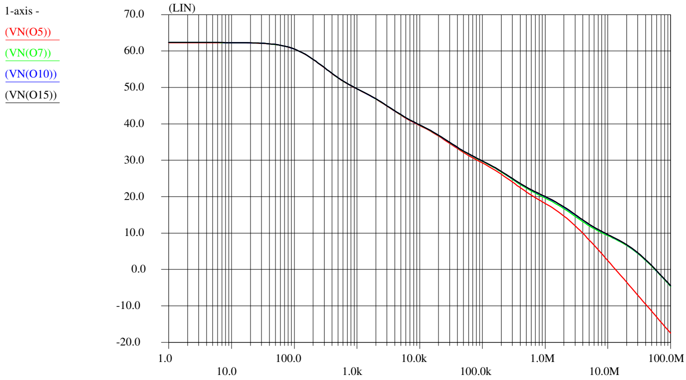
- Node voltages and currents $\mathbf{x} \in \mathbb{R}^n$
- Nonlinear vector valued $\mathbf{q}(t,\mathbf{x})$, $\mathbf{j}(t,\mathbf{x}) \in \mathbb{R}^n$
- Input $\mathbf{b}u(t) \in \mathbb{R}^n$ (sources)
- Simulation of schematic (left, n small): minutes – hours
- Simulation of layout (right, n large): minutes – ∞

Stability analysis



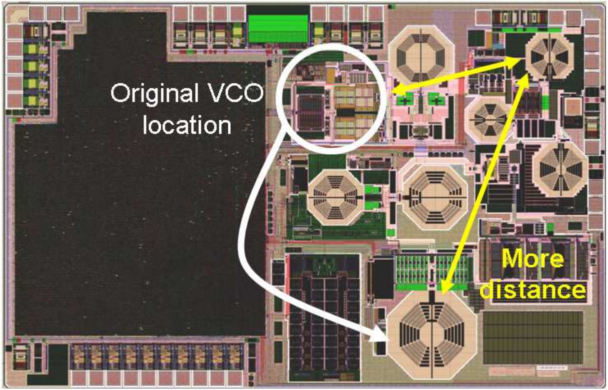
- Regulator IC: is the steady-state stable?
- Numerical challenges include
  - Matrices can be large due to parasitic elements
  - Direct methods not applicable
  - Eigenvalues at $\pm\infty$

Behavioral modeling of thermal effects



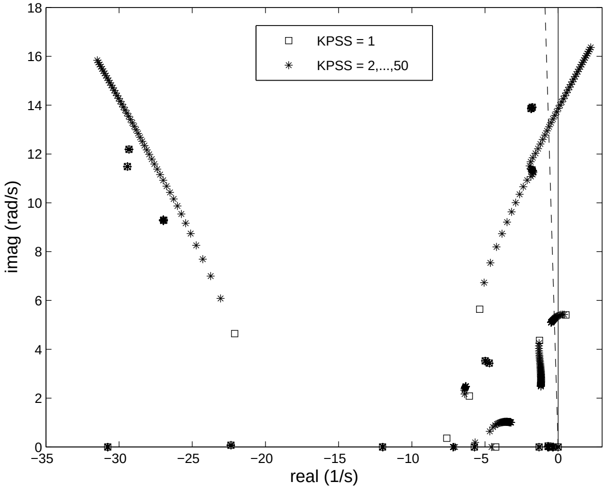
- Toplevel system simulation should cover all effects
- Computationally often not feasible
- Designers use *handmade* models to replace subsystems
- Automatic construction of behavioral models is open challenge

Oscillator coupling and pulling, phase-noise models



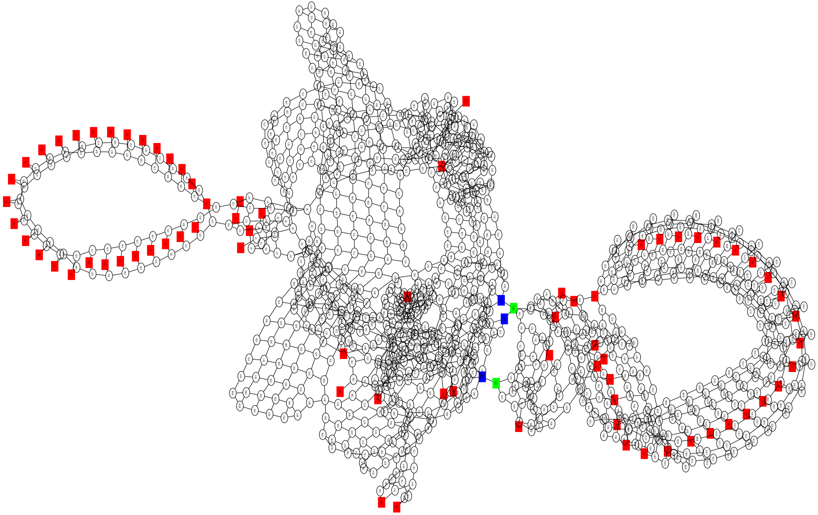
- Perturbation projection vector is eigenvector of large operator
- See e.g. Harutyunyan etal. (IEEE TCAD 2009)
- Also topic in EU project ASIVA14 (TU/e, Mentor)

Parameter sensitivity



How to compute the eigenvalues that are most sensitive to parameter changes?

Partitioning

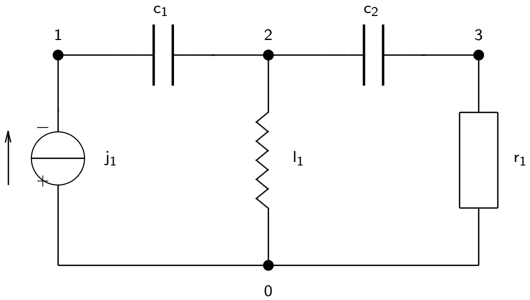


How to partition this graph?

Outline

Eigenvalue problems

Circuit equations



- Kirchhoff's Current Law: $\sum_k i_k^n = 0$
- Kirchhoff's Voltage Law: $\sum_{k \in \text{loop}} v_k = 0$
- Branch constitutive equations:
  - Resistor: $i = v/R$
  - Capacitor: $i = C\frac{dv}{dt}$
  - Inductor: $v = L\frac{di}{dt}$

Leads to system of Differential Algebraic Equations:

$$\frac{d}{dt}\mathbf{q}(t, \mathbf{x}) + \mathbf{j}(t, \mathbf{x}) = \mathbf{b}u(t)$$

Linearization

Let $\mathbf{x}_{DC}$ be steady-state solution and

$$E = \left.\frac{\partial \mathbf{q}}{\partial \mathbf{x}}\right|_{\mathbf{x}_{DC}} \quad \text{and} \quad A = -\left.\frac{\partial \mathbf{j}}{\partial \mathbf{x}}\right|_{\mathbf{x}_{DC}}$$

Linearization around steady-state gives dynamical system

$$\begin{cases} E\dot{\mathbf{x}}(t) & = & A\mathbf{x}(t) + \mathbf{b}u(t) \\ y(t) & = & \mathbf{c}^*\mathbf{x}(t), \end{cases}$$

where

$$\begin{array}{rcl} u(t), y(t) & \in & \mathbb{R},\ \text{input, output} \\ \mathbf{x}(t), \mathbf{b}, \mathbf{c} & \in & \mathbb{R}^n,\ \text{state, input-to-, -to-output} \\ E & \in & \mathbb{R}^{n\times n}\ \text{capacitance matrix} \\ A & \in & \mathbb{R}^{n\times n}\ \text{conductance matrix} \end{array}$$

Transfer function

First-order SISO dynamical system:

$$\begin{cases} E\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + \mathbf{b}u(t) \\ y(t) &= \mathbf{c}^*\mathbf{x}(t) \end{cases}$$

with transfer function

$$H(s) = \mathbf{c}^*(sE - A)^{-1}\mathbf{b}$$

Poles are $\lambda \in \mathbb{C}$ for which

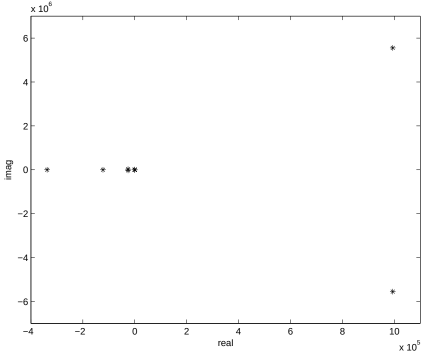
$$\lim_{s\to\lambda} |H(s)| = \infty,$$

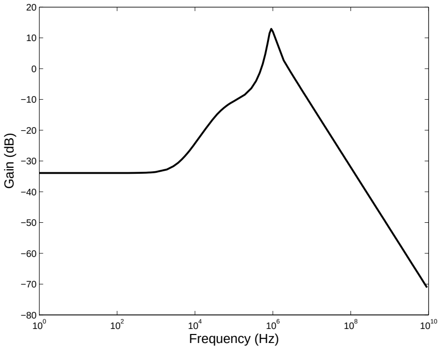
or, equivalently,

$$\det(\lambda E - A) = 0,$$

i.e. the eigenvalues of (A, E)

Eigenvalue problems in practice: Pole-zero analysis





Poles ($\Lambda(A, E)$)

Bode plot $(\omega, |H(i\omega)|)$

- poles λ with real(λ)>0: unstable solution
- dominant poles cause peaks

The generalized eigenvalue problem

Given $A, E \in \mathbb{R}^{n \times n}$, find $(\lambda, \mathbf{x}, \mathbf{y})$ that satisfy

$$\begin{aligned} A\mathbf{x} &= \lambda E\mathbf{x}, \qquad \mathbf{x} \neq 0 \\ \mathbf{y}^* A &= \lambda \mathbf{y}^* E, \qquad \mathbf{y} \neq 0 \end{aligned}$$

An eigentriplet $(\lambda, \mathbf{x}, \mathbf{y})$ consists of

$\lambda \in \mathbb{C}$	eigenvalue
$\mathbf{x} \in \mathbb{C}^n$	right eigenvector
$\mathbf{y} \in \mathbb{C}^n$	left eigenvector

- (A, E) has n eigenvalues (real / complex conjugated pairs)
- Corresponding eigenspaces need not be n-dimensional
- Bi-orthogonality: $\lambda_i \neq \lambda_j \Rightarrow \mathbf{y}_j^* E \mathbf{x}_i = 0$

Eigenvalue decompositions

Complete eigenvalue decomposition (Λ, X, Y):

$$AX = EX\Lambda, \quad Y^*A = \Lambda Y^*E \quad \text{with } Y^*EX = I, Y^*AX = \Lambda$$
$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^{n\times n}$$
$$X = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n] \in \mathbb{C}^{n\times n}$$
$$Y = [\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n] \in \mathbb{C}^{n\times n}$$

In practice only interest in $k \ll n$ eigentriplets: partial ED

$$AX_k = EX_k\Lambda_k, \quad Y_k^*A = \Lambda_k Y_k^*E \quad \text{with } Y_k^*EX_k = I, Y_k^*AX_k = \Lambda_k$$
$$\Lambda_k = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{C}^{k\times k}$$
$$X_k = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k] \in \mathbb{C}^{n\times k}$$
$$Y_k = [\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k] \in \mathbb{C}^{n\times k}$$

Computational problems for large $A\mathbf{x} = \lambda E\mathbf{x}$

Brute force approach:

1. Compute all eigenvalues (and left and right eigenvectors)
2. Select eigenvalues of interest (positive real part, dominant)

Computational complications:

- Matrices can become very large: n of $O(10^3)$ up to $O(10^6)$
- Dense methods QR/QZ too expensive ($O(n^3)$ CPU, memory)
- Spurious eigenvalues

In practice:

- Only few ($k \ll n$) specific eigenvalues of practical interest
- How to compute specifically these eigenvalues?

Similar eigenproblems arise in many other areas:

- Fluid dynamics, structural engineering, power systems

Outline

Stability analysis and spurious eigenvalues

Pole-zero stability analysis

- Generalized eigenproblem

$$Ax = \lambda Ex$$

- Wanted: eigenvalues with largest real part

$$\text{Re}(\lambda) > 0 \rightarrow \text{ unstable}$$

- A, E are large, sparse matrices
- E may be singular
- Few ($k \ll n$) specific eigenvalues are wanted
- Full space methods like QR and QZ too expensive ($O(n^3)$)

Shift-and-Invert

Generalized eigenproblem

$$A\mathbf{x} = \lambda E\mathbf{x}$$

Choose shift $\sigma \in \mathbb{C}$:

$$(A - \sigma E)\mathbf{x} = (\lambda - \sigma)E\mathbf{x}$$

and invert:

$$(A - \sigma E)^{-1}E\mathbf{x} = (\lambda - \sigma)^{-1}\mathbf{x}$$

With $S = (A - \sigma E)^{-1}E$:

$$Ax = \lambda Ex \iff Sx = \tilde{\lambda}x, \qquad \tilde{\lambda} = (\lambda - \sigma)^{-1}$$

$\lambda(A, E)$ near σ are transformed to outside of spectrum $\Lambda(S)$

The Arnoldi method [Arnoldi 1951]

Orthonormal basis $\mathbf{v}_1, \ldots, \mathbf{v}_{k+1}$ for Krylov space $\mathcal{K}^{k+1}(S, \mathbf{v}_1)$:

$$\begin{aligned} V_k &= [\mathbf{v}_1, \ldots, \mathbf{v}_k] \in \mathbb{C}^{n \times k} \\ V_k^* V_k &= I, \\ SV_k &= V_k H_k + h_{k+1,k} \mathbf{v}_{k+1} \mathbf{e}_k^T \end{aligned}$$

Require for approximate eigenpair $(\theta, V_k \mathbf{y})$

$$S(V_k \mathbf{y}) - \theta(V_k \mathbf{y}) \perp V_k \quad \text{(Ritz-Galerkin)}$$

1. Compute eigenpairs $(\theta_i, \mathbf{y}_i)$ of $H_k = V_k^* S V_k \in \mathbb{C}^{k \times k}$

$$H_k \mathbf{y}_i = \theta_i \mathbf{y}_i$$

2. Compute Ritz pairs $(\theta_i, V_k \mathbf{y}_i)$ of S and select wanted
3. Check residual norm $\|\mathbf{r}\|_2 = \|SV_k \mathbf{y}_i - \theta_i V_k \mathbf{y}_i\|_2 = |h_{k+1,k} \mathbf{y}_{i(k)}|$

Eigenvalues at infinity

- One finite, one infinite eigenvalue

$$A = A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \lambda(A, E) = \{1, \infty\}$$

- Defective, infinite eigenvalue

$$A = A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \lambda(A, E) = \{\infty\}$$

- Note $\lambda(A, E) = \infty$ becomes $\tilde{\lambda}(A^{-1}E) = 0$
- Eigenvalues at ∞ are not of interest

Numerical problem

- Start Arnoldi with $\mathbf{v}_1 = S^2\mathbf{1} \in \text{range}(S^2)$
- $P_{\mathcal{N}}$: projection on $\mathcal{N} = \ker(S)$
- $P_{\mathcal{G}}$: projection on $\mathcal{G} = \ker(S^2) \setminus \ker(S)$

j	$\|\|P_{\mathcal{N}}\mathbf{v}_j\|\|_2$	$\|\|P_{\mathcal{G}}\mathbf{v}_j\|\|_2$
1	$3.5 \cdot 10^{-11}$	$7.6 \cdot 10^{-12}$
2	$7.5 \cdot 10^{-9}$	$1.2 \cdot 10^{-10}$
3	$2.1 \cdot 10^{-7}$	$2.5 \cdot 10^{-9}$
4	$5.5 \cdot 10^{-7}$	$5.1 \cdot 10^{-8}$
5	$1.5 \cdot 10^{-4}$	$1.1 \cdot 10^{-6}$
15	$3.1 \cdot 10^{+7}$	$3.0 \cdot 10^{-4}$

One *spurious* eigenvalue $\theta = 6.4 \cdot 10^{10}$

Numerical problem

- Recall $V_\infty = \mathcal{N}(S) = \mathcal{N}(E) = \{\mathbf{x} \in \mathbb{R}^n \mid E\mathbf{x} = 0\}$
- In exact arithmetic: $\mathbf{v}_1 \in \mathcal{R} \Rightarrow \mathbf{v}_j = S\mathbf{v}_{j-1} \in \mathcal{R}$

However, in finite arithmetic

- Rounding errors ($S\mathbf{v}_j$, orth) lead to components in $\mathcal{N} + \mathcal{G}$ in v_j
- Arnoldi can find approximations θ_i to $\tilde{\lambda} = 0$:

$$(V_k^* S V_k) y_i = \theta_i y_i$$

- Back transformation $\lambda = \theta_i^{-1} + \sigma$ leads to *spurious* eigenvalues

Purification:

1. Remove/prevent spurious eigenvalue approximations
2. Improve wanted eigenpair approximations by removing components in $\mathcal{N} + \mathcal{G}$ from $\mathbf{v}_j$

Exploiting structure [Bomhof (2000), R. (2008)]

Consider block structured generalized eigenvalue problem

$$\begin{bmatrix} K & C \\ C^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \lambda \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix},$$

with $C \in \mathbb{R}^{m \times k}$, and $K, M \in \mathbb{R}^{m \times m}$ $(n = m + k)$
Corresponding ordinary eigenproblem is

$$\begin{bmatrix} S_1 & 0 \\ S_2 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \tilde{\lambda} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix}, \qquad S_1 \in \mathbb{R}^{m \times m}, \ S_2 \in \mathbb{R}^{k \times m},$$

Reduced problem

$$S_1 \mathbf{u} = \tilde{\lambda} \mathbf{u} \longleftrightarrow \begin{bmatrix} S_1 & 0 \\ S_2 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \tilde{\lambda}^{-1} S_2 \mathbf{u} \end{bmatrix} = \tilde{\lambda} \begin{bmatrix} \mathbf{u} \\ \tilde{\lambda}^{-1} S_2 \mathbf{u} \end{bmatrix}$$

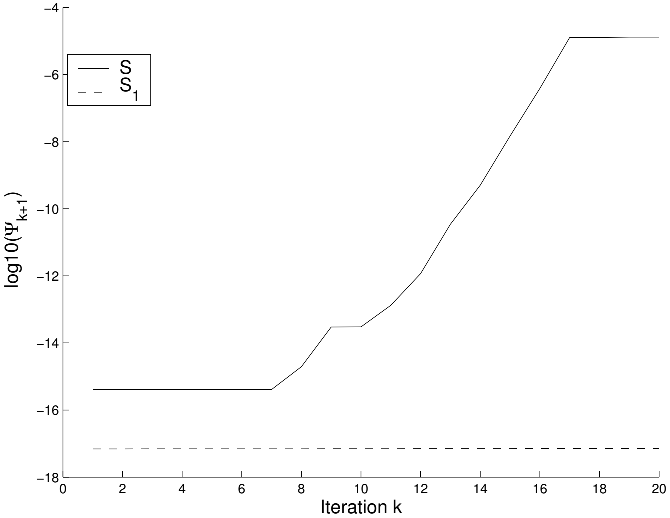
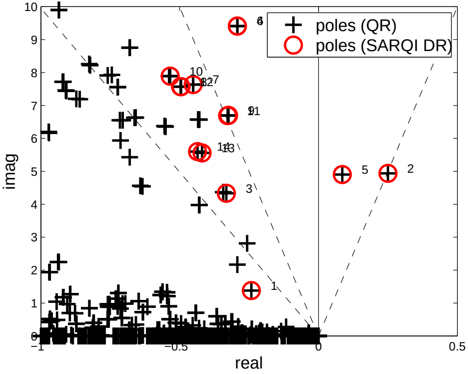


Figure: The size of $\|\Psi_{k+1}\|_2 = \|V_{k+1}\underline{H}_k - SV_k\|_2$ for Arnoldi applied to $S = (A - 60E)^{-1}E$, and Arnoldi applied to S_1.

Further improvements

- Implicit restarts [Sorensen 1992]:
  - Additional purification [Meerbergen/Spence 1995]
  - Control convergence [R. 2008/2011]
- Find missed eigenvalues:
  - Clever shifts [Cliffe/Garratt/Spence 1994, R. 2008/2011]
  - Cayley transformations [Cliffe/Garratt/Spence 1994, R. 2008]
- Very large problems ($LU = (A - \sigma B)$ not feasible):
  - Jacobi-Davidson methods [Sleijpen/Van der Vorst 1996, R. 2008]

SARQI to compute rightmost eigenvalues ($n = 40366$)



Using damping ratio to select shifts is robust [R. etal. 2010]

$$\zeta = -\frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}$$

Outline

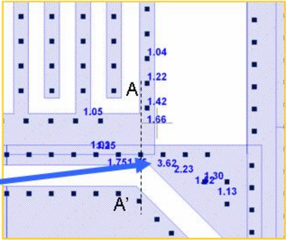
Partitioning

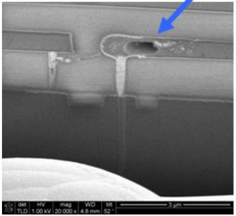
Electro Static Discharge analysis

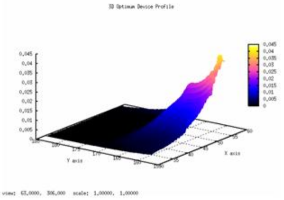
Damaged interconnect that was too small to conduct current

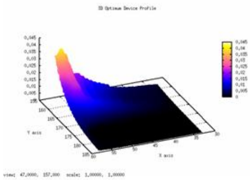
W/E	Resistor	Index	Layer	X:Y	Width	Current	Ratio
E	RRm1957	1023	$rm1	58.458:157.32	1.106	1	3.62
E	RRm1956	1022	$rm1	59.142:157.086	1.79	1	2.23
E	RRm1959	1025	$rm1	56.452:157.36	2.28	1	1.75
E	RRm1958	1024	$rm1	57.262:157.36	2.28	1	1.75
E	RRm11980	1747	$rm1	58.0515:159.1	0.6	-0.249099911	1.66
E	RRm1955	1021	$rm1	59.982:156.636	2.63	1	1.52
E	RRm11971	1739	$rm1	58.052:159.82	0.6	-0.213122023	1.42
E	RRm1954	1020	$rm1	60.352:156.81	3.08	1	1.3
E	RRm1762	896	$rm1	65.492:135.16	1.56	-0.493862312	1.27
E	RRm1960	1026	$rm1	56.002:158.04	2.28	0.710632608	1.25
E	RRm11970	1738	$rm1	58.052:160.82	0.6	-0.182341944	1.22
E	RRm1953	1019	$rm1	60.882:156.186	3.53	1	1.13

•Worst r=3.62 in M1

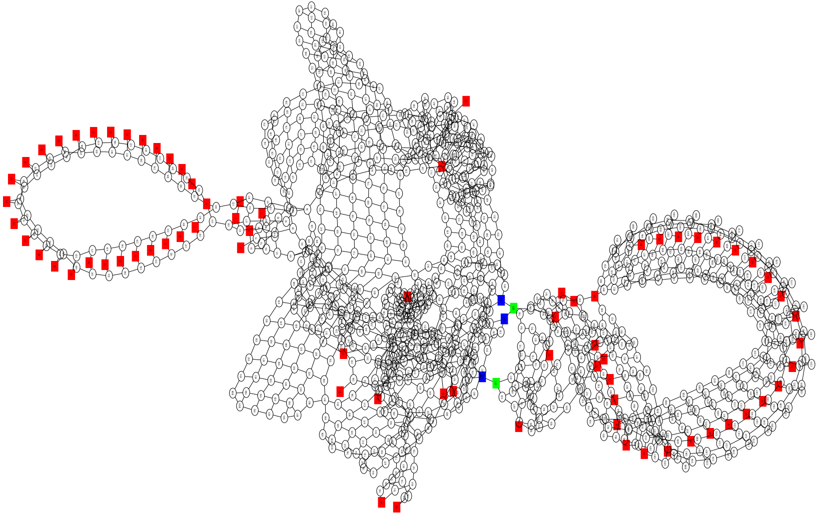








Partitioning of electrical circuits



How to partition this network?

Spectral partitioning

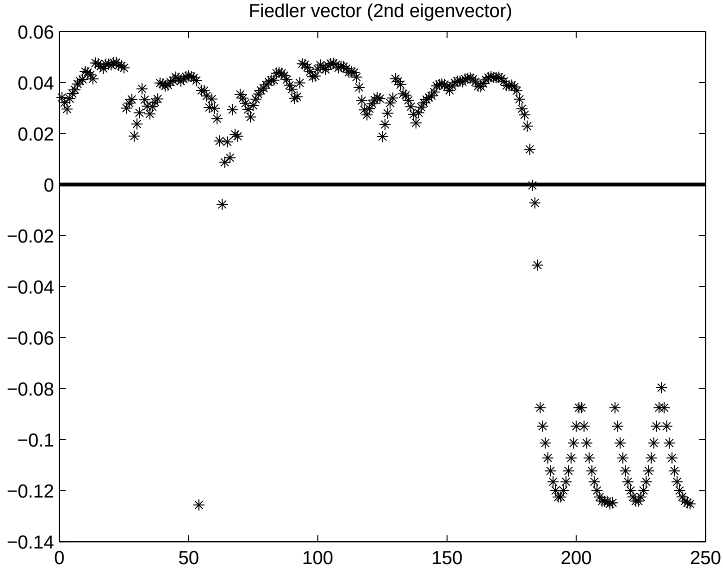
- Given undirected graph G with equally weighted edges g_{ij}
- Note $\text{diag}(G) = 0$
- Define diagonal D with $d_{ii} = \text{degree}(\text{node } i)$
- Laplacian of G is defined as $L = D - G$

Partitioning G with fewest cut edges:

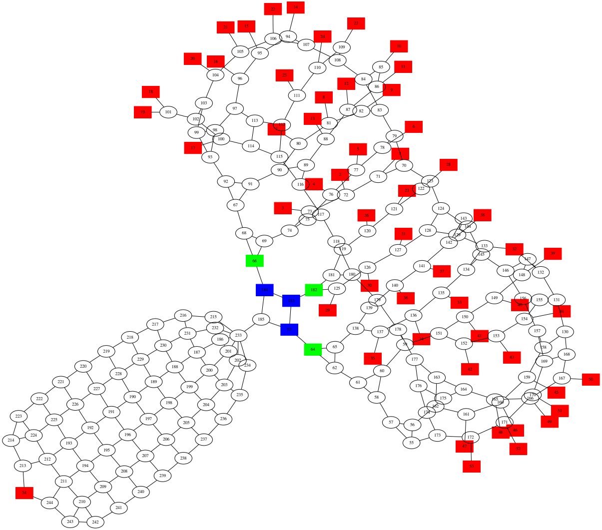
$$\min_{y_i \in \mathbb{R}^n} \sum_{i,j} (y_i - y_j)^2 g_{ij} \tag{1}$$

- Note $0 = \lambda_1(L) < \lambda_2(L) < \ldots < \lambda_n(L)$
- Eigenvector v_2 corresponding to λ_2 is called Fiedler vector
- Fiedler vector solves (1): partitioning reduces to eigenproblem!
- See [Fiedler, Pothen, D. Higham]

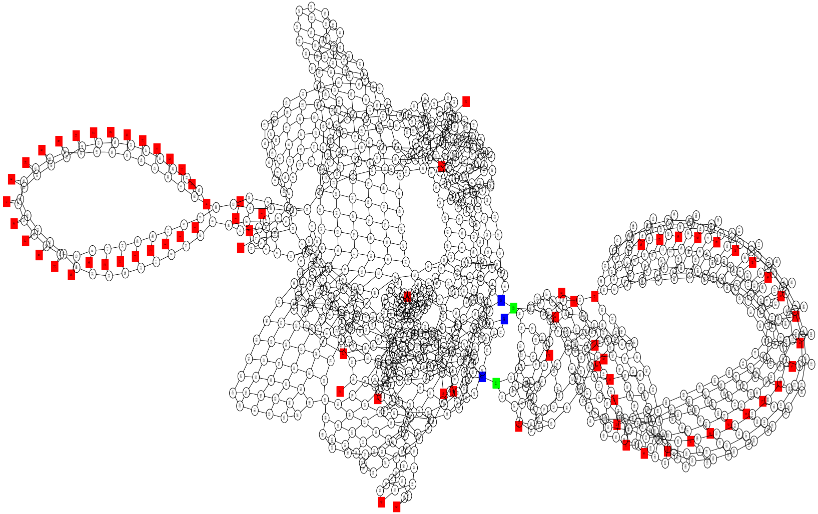
Example: biconnected component



Example: biconnected component



Example: difficult network for reduceR



76 terminals vs. 43 and 33 terminals, (3 and 2) cutnodes

Outline

Eigenanalysis for model order reduction

Transfer function $H(s) = \mathbf{c}^*(sE - A)^{-1}\mathbf{b}$

Can be expressed as

$$H(s) = \sum_{i=1}^{n} \frac{R_i}{s - \lambda_i},$$

where residues R_i are

$$R_i = (\mathbf{c}^*\mathbf{x}_i)(\mathbf{y}_i^*\mathbf{b}),$$

and $(\lambda_i, \mathbf{x}_i, \mathbf{y}_i)$ are eigentriplets $(i = 1, \ldots, n)$

$$\begin{array}{rcll} A\mathbf{x}_i &=& \lambda_i E\mathbf{x}_i, & \text{right eigenpairs} \\ \mathbf{y}_i^* A &=& \lambda_i \mathbf{y}_i^* E, & \text{left eigenpairs} \\ \mathbf{y}_i^* E\mathbf{x}_i &=& 1, & \text{normalization} \\ \mathbf{y}_j^* E\mathbf{x}_i &=& 0 \ (i \neq j), & E\text{-orthogonality} \end{array}$$

Dominant poles cause peaks in Bode-plot

$$H(s) = \mathbf{c}^*(sE - A)^{-1}\mathbf{b} = \sum_{i=1}^{n} \frac{R_i}{s - \lambda_i} \quad \text{with} \quad R_i = (\mathbf{c}^*\mathbf{x}_i)(\mathbf{y}_i^*\mathbf{b})$$

Bode-plot is graph of $(\omega, |H(i\omega)|)$

- frequency $\omega \in \mathbb{R}$
- magnitude $|H(i\omega)|$ usually in dB (note dB(x)= $20 \cdot {}^{10}\log(x)$)

Consider pole $\lambda = \alpha + \beta i$ with residue R, then

$$\begin{aligned} \lim_{\omega \to \beta} H(i\omega) &= \lim_{\omega \to \beta} \frac{R}{i\omega - (\alpha + \beta i)} + \sum_{j=1}^{n-1} \frac{R_j}{i\omega - \lambda_j} \\ &= -\frac{R}{\alpha} + H_{n-1}(i\beta) \end{aligned}$$

Pole λ with large $\left|\frac{R}{\text{Re}(\lambda)}\right|$ is dominant and causes peak

Dominant poles cause peaks in Bode-plot

Dominant Pole Algorithm [Martins (1996)]

$$H(s) = \mathbf{c}^*(sE - A)^{-1}\mathbf{b}$$

- Pole λ: $\lim_{s\to\lambda} |H(s)| = \infty$, or $\lim_{s\to\lambda} \frac{1}{H(s)} = 0$

Apply Newton's Method to $1/H(s)$:

$$\begin{aligned} s_{k+1} &= s_k + \frac{1}{H(s_k)} \frac{H^2(s_k)}{H'(s_k)} \\ &= s_k - \frac{\mathbf{c}^*(s_k E - A)^{-1}\mathbf{b}}{\mathbf{c}^*(s_k E - A)^{-1} E (s_k E - A)^{-1}\mathbf{b}} \end{aligned}$$

Note $\frac{dH}{ds} = -\mathbf{c}^*(s_k E - A)^{-1} E (s_k E - A)^{-1}\mathbf{b}$

Twosided Rayleigh quotient iteration

Note that with $\mathbf{v} \equiv \mathbf{v}_k$ and $\mathbf{w} \equiv \mathbf{w}_k$

$$
\begin{aligned}
s_{k+1} &= s_k - \frac{\mathbf{c}^*(s_k E - A)^{-1}\mathbf{b}}{\mathbf{w}^* E \mathbf{v}} \\
&= s_k \frac{\mathbf{w}^* E \mathbf{v}}{\mathbf{w}^* E \mathbf{v}} - \frac{\mathbf{c}^*(s_k E - A)^{-1}(s_k E - A)(s_k E - A)^{-1}\mathbf{b}}{\mathbf{w}^* E \mathbf{v}} \\
&= \frac{\mathbf{w}^* A \mathbf{v}}{\mathbf{w}^* E \mathbf{v}}
\end{aligned}
$$

Step	DPA	Twosided RQI
3	solve $(s_k E - A)\mathbf{v}_k = \mathbf{b}$	solve $(s_k E - A)\mathbf{v}_k = E\mathbf{v}_{k-1}$
4	solve $(s_k E - A)^*\mathbf{w}_k = \mathbf{c}$	solve $(s_k E - A)^*\mathbf{w}_k = E^*\mathbf{w}_{k-1}$

Original work on twosided RQI [Ostrowski (1958), Parlett (1974)]

Convergence behavior: DPA vs. RQI

Typically, with initial pole guess s_0,

- DPA converges to *dominant* pole closest to s_0
  - with $\angle(\mathbf{c}, \mathbf{x})$ and $\angle(\mathbf{b}, \mathbf{y})$ small
  - i.e., large $|R|$ with $R = (\mathbf{c}^*\mathbf{x})(\mathbf{y}^*\mathbf{b})$
- Quadratic rate of convergence
- See also [R., Sleijpen (2006)]

while

- RQI converges to pole closest to s_0
- Originally intended for refinement of eigenpairs
- Cubic rate of convergence
- See also [Ostrowski (1958), Parlett (1974)]

Computation of most sensitive eigenvalues

- Suppose system matrix A depends on parameter p
- Sensitivity of eigenvalue is given by

$$\frac{\partial \lambda}{\partial p} = \mathbf{y}^* \frac{\partial A}{\partial p} \mathbf{x}$$

  where $\mathbf{y}$ and $\mathbf{x}$ are left and right eigenvectors
- If $\frac{\partial A}{\partial p}$ has rank 1:

$$\frac{\partial \lambda}{\partial p} = \mathbf{y}^* \frac{\partial A}{\partial p} \mathbf{x} = (\mathbf{y}^* \mathbf{b})(\mathbf{c}^* \mathbf{x}) = (\mathbf{c}^* \mathbf{x})(\mathbf{y}^* \mathbf{b})$$

- Apply DPA to $(A, \mathbf{b}, \mathbf{c})$ to compute sensitive eigenvalues!
- See [R., Martins, IEEE TPWRS 2008] for more details

Computation of most sensitive eigenvalues

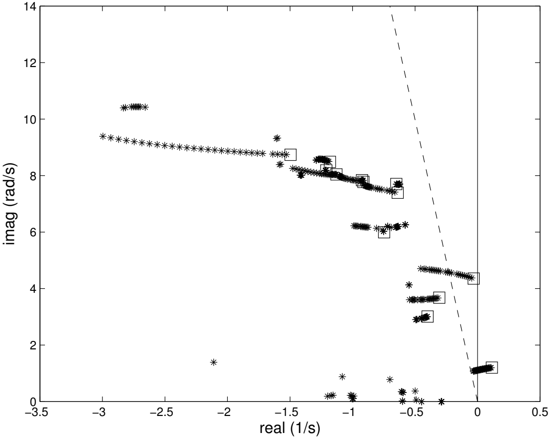


Figure: Root locus of most sensitive eigenvalues for 13k state system.

Dominant Pole Algorithm (DPA) and extensions

DPA computes dominant poles of $H(s) = \mathbf{c}^*(sE - A)^{-1}\mathbf{b}$

1. Newton scheme [M., Lima, Pinto (IEEE TPWRS 11(1) 1996]
2. Convergence analysis [R., Sleijpen (SIMAX 30(1) 2008)]
3. Subspace acceleration, selection, deflation: SADPA [R., Martins (IEEE TPWRS 21(3) 2006)]
4. Poles of MIMO systems: SAMDP [R., Martins (IEEE TPWRS 21(4) 2006)]
5. Dominant zeros [R., Martins, Pellanda (IEEE TPWRS 22(4) 2007)]
6. Poles of second-order systems: QDPA [R., Martins (SISC 30(4) 2008)]
7. Spectral zeros [Ionutiu, R., Antoulas (IEEE TCAD 27(12) 2008]
8. Sensitive poles: SPA [R., Martins (IEEE TPWRS 23(2) 2008)]
9. Computing rightmost eigenvalues: SARQI [R., Freitas, Martins 2010]
10. Time-delay systems [Meerbergen etal 2012]
11. Parameterized systems [Saadvandi etal 2014]
12. H_∞-norm of descriptor systems [Benner/Voigt 2014]

Model order reduction

Given large-scale dynamical system

$$\begin{cases} E\dot{\mathbf{x}}(t) & = & A\mathbf{x}(t) + \mathbf{b}u(t) \\ y(t) & = & \mathbf{c}^*\mathbf{x}(t) + du(t) \end{cases}$$

where $\mathbf{x}(t), \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ and $E, A \in \mathbb{R}^{n \times n}$, find

$$\begin{cases} E_k\dot{\mathbf{x}}_k(t) & = & A_k\mathbf{x}_k(t) + \mathbf{b}_k u(t) \\ y_k(t) & = & \mathbf{c}_k^*\mathbf{x}_k(t) + du(t) \end{cases}$$

where $\mathbf{x}_k(t), \mathbf{b}_k, \mathbf{c}_k \in \mathbb{R}^k$, $E_k, A_k \in \mathbb{R}^{k \times k}$ and

- $k \ll n$
- approximation error $\|y - y_k\|$ small

Antoulas (2005) and Schilders, Van der Vorst, R. (2008)

Additional constraints on reduced order model

$$\left\{\begin{array}{lcl} E\dot{\mathbf{x}}(t) & = & A\mathbf{x}(t) + \mathbf{b}u(t) \\ y(t) & = & \mathbf{c}^*\mathbf{x}(t) + du(t) \end{array}\right. \Longrightarrow \left\{\begin{array}{lcl} E_k\dot{\mathbf{x}}_k(t) & = & A_k\mathbf{x}_k(t) + \mathbf{b}_k u(t) \\ y_k(t) & = & \mathbf{c}_k^*\mathbf{x}_k(t) + du(t) \end{array}\right.$$

Size may be reduced, but what about complexity?

- Original model may have sparse system matrices, while reduced order model has dense system matrices
- Time domain simulation may become more expensive
- Reuse: ROM must be available as, e.g., netlist
- Simulators and software may introduce additional constraints

Modal approximation

General framework for modal approximation of

$$H(s) = \mathbf{c}^*(sE - A)^{-1}\mathbf{b} = \sum_{i=1}^{n} \frac{R_i}{s - \lambda_i} = \sum_{i=1}^{n} \frac{(\mathbf{c}^*\mathbf{x}_i)(\mathbf{y}_i^*\mathbf{b})}{s - \lambda_i}$$

where $\mathbf{y}_i$ and $\mathbf{x}_i$ are left and right eigenvectors of (A, E):

1. Sort (λ_i, R_i) in decreasing $|R_i|/\mathrm{Re}(\lambda_i)$ order
2. Truncate at $|R_i|/\mathrm{Re}(\lambda_i) < R_{min}$
3. Project with $Y_k = [\mathbf{y}_1, \ldots, \mathbf{y}_k]$ and $X_k = [\mathbf{x}_1, \ldots, \mathbf{x}_k]$

$$\begin{cases} \dot{\tilde{\mathbf{x}}} & = & \Lambda_k \tilde{\mathbf{x}}(t) + \tilde{\mathbf{b}} u(t) \\ y(t) & = & \tilde{\mathbf{c}}^* \tilde{\mathbf{x}}(t) \end{cases} \qquad H_k(s) = \sum_{i=1}^{k} \frac{R_i}{s - \lambda_i}$$

Use SADPA [R.,Martins (2006)] to compute dominant poles

Moment matching

Series expansion of $H(s) = \mathbf{c}^*(sE - A)^{-1}\mathbf{b}$ around s_0 is

$$H(s) = \sum_{i=0}^{\infty} m_i (s - s_0)^i$$

with moments $m_i = \mathbf{c}^* G^i (s_0 E - A)^{-1}\mathbf{b}$ and $G = (s_0 E - A)^{-1} E$

Model order reduction: Match only $2k \ll n$ moments:

1. Compute bases $V \in \mathbb{R}^{n \times k}$ and $W \in \mathbb{R}^{n \times k}$ for (Arnoldi)

$$\mathcal{K}^k((s_0 E - A)^{-1} E, \mathbf{b}) \text{ and } \mathcal{K}^k((s_0 E - A)^{-*} E^*, \mathbf{c})$$

2. Petrov-Galerkin projection gives k-th order system:

$$\left\{ \begin{array}{lcl} E\dot{\mathbf{x}} & = & A\mathbf{x}(t) \\ & + & \mathbf{b}u(t) \\ y(t) & = & \mathbf{c}^*\mathbf{x}(t) \end{array} \right. \Rightarrow \left\{ \begin{array}{lcl} (W^*EV)\dot{\tilde{\mathbf{x}}} & = & (W^*AV)\tilde{\mathbf{x}}(t) \\ & + & (W^*\mathbf{b})u(t) \\ \tilde{y}(t) & = & (\mathbf{c}^*V)\tilde{\mathbf{x}}(t) \end{array} \right.$$

Modal approximation and moment matching

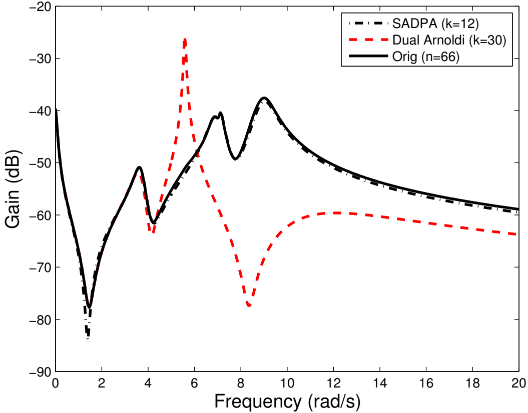


Figure: Frequency response of complete system ($n = 66$), modal approximation ($k = 12$), and dual Arnoldi model ($k = 30$).

Dominant poles: location in complex plane

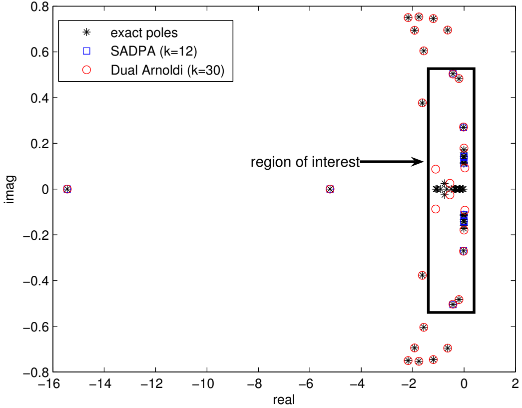


Figure: Pole spectrum of complete system ($n = 66$), modal approximation ($k = 12$), and dual Arnoldi model ($k = 30$).

Dominant poles: location in complex plane (zoom)

Dominant poles not necessarily at outside of spectrum

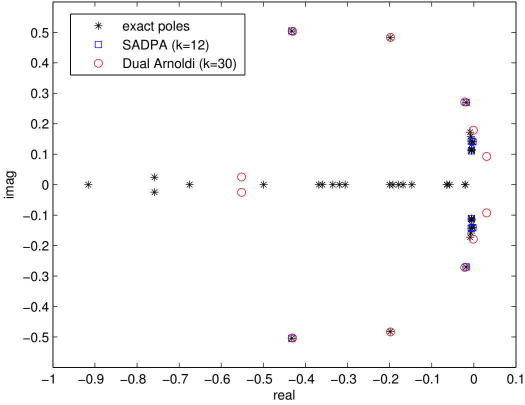


Figure: Pole spectrum (zoom) of complete system ($n = 66$), modal approximation ($k = 12$), and dual Arnoldi model ($k = 30$).

Rational Krylov methods [Ruhe (1998)]

General approach:

1. Choose m interpolation points s_i
2. Construct $V_i, W_i \in \mathbb{C}^{n \times k_i}$ such that

$$\begin{aligned}\text{colspan}(V_i) &= \mathcal{K}^{k_i}((s_i E - A)^{-1} E, (s_i E - A)^{-1} E\mathbf{b}) \\ \text{colspan}(W_i) &= \mathcal{K}^{k_i}((s_i E - A)^{-*} E^*, (s_i E - A)^{-*} E^*\mathbf{c})\end{aligned}$$

3. Project with $V = [V_1, \ldots, V_m]$ and $W = [W_1, \ldots, W_m]$

Open question:

- How to choose interpolation points s_i?
- See also PhD thesis Grimme (1997)

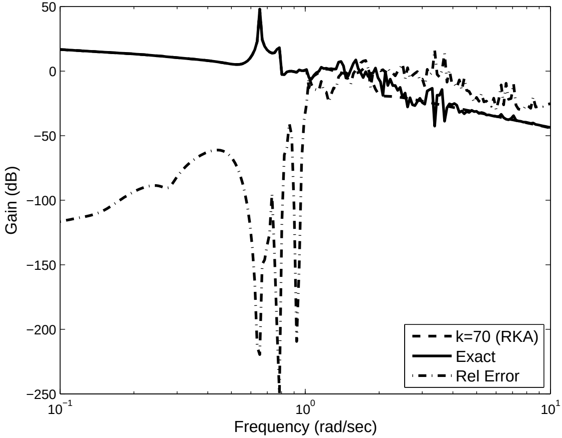


Figure: Breathing sphere ($n = 17611$). Exact transfer function (solid), 70th order SOAR [Bai/Su 2005] RKA model (dash) using interpolation points based on dominant poles, and relative error (dash-dot).

Outline

Concluding remarks

Concluding remarks

- Eigenproblems arise in many application domains
- Nature and difficulties vary
  - Stability analysis (rightmost eigenvalues)
  - MOR (dominant modes)
  - Phase noise analysis (left eigenvector for $\lambda = 1$)
  - Partitioning (Fiedler vector)
- Open challenges include
  - How to know we did not miss any eigenvalues?
  - Avoiding piling up of rounding errors (deflation)
  - Robustness and performance for inexact variants
  - Selection of shifts
  - Robustness of graph partitioning algorithms

Thank you!

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