Interpolatory Model Reduction for Flow Control

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- Interpolatory Model Reduction for DAEs
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Outline

- Flow Control Problem: Wake Stabilization by Cylinder Rotation
  - Methodology and the LQR Problem
  - Discretization and Linearization
- Model Reduction Problem For DAEs
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  - Numerical Results
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  - Bilinear Systems
  - Quadratic-in-state Systems
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Wake Stabilization by Cylinder Rotation

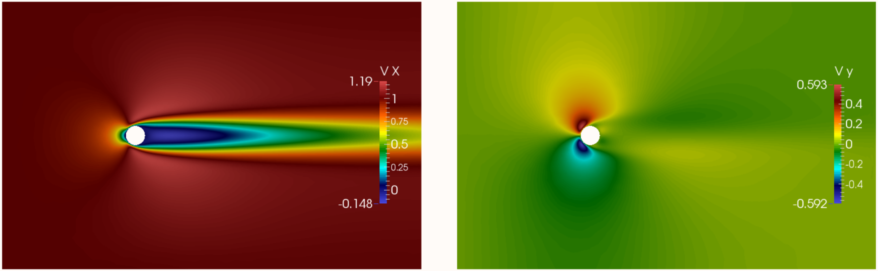


Figure : Steady-State Velocity Components at $Re_d = 60$

Objective

Stabilize the wake behind a circular cylinder using cylinder rotation.

Plan

Use linear feedback control to stabilize the steady-state solution.

Linearize about the steady-state

- An incomplete list: [Tokumaru/Dimotakis,91], [Blackburn/Henderson,99], [Dennis et al.,00], [He et al.,00], [Bergmann et al.,00], [Noack et al.,03], [Gerhard et al.,03], [Stoyanov,09], [Benner/Heiland,14], ...
- Linearize the Navier-Stokes equations about the steady-state flow:

$$\mathbf{v}(t) = \mathcal{V} + \mathbf{v}'(t) \qquad p(t) = \mathcal{P} + p'(t), \quad t > 0.$$

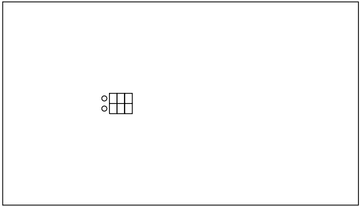
- Leads to the Oseen Equations

$$\begin{aligned} \mathbf{v}'_t &= -\mathcal{V} \cdot \nabla \mathbf{v}' - \mathbf{v}' \cdot \nabla \mathcal{V} + \tau(\mathbf{v}') - \nabla p' + Bu \\ 0 &= \nabla \cdot \mathbf{v}' \end{aligned}$$

where $\tau(\mathbf{v}) \equiv \mu \left(\nabla \mathbf{v} + \nabla \mathbf{v}^T \right)$, with boundary conditions

- Inflow: $\mathbf{v}'(t) = 0, \quad t > 0.$
- Outflow edges: $(\mathbf{v}'(t), p'(t))$ is stress-free.
- $Bu(t)$ provides tangential velocity on the cylinder.

Two-cylinder case



LQR Problem:
Find $\mathbf{u}(\cdot)$ (tangential velocities) that minimizes

$$J(\mathbf{u}(\cdot)) = \int_0^\infty \mathbf{y}^T(t)\mathbf{y}(t) + 10\|\mathbf{u}(t)\|^2 dt.$$

Controlled Outputs:

$$y_{i\star j}(t) = \int_{\Omega_i} \mathbf{v}'_j(\xi, t) d\xi$$

$i = 1, \ldots, 6$ and $j = 1, 2$.

Seek feedback solutions in the form

$$u_i(t) = -\int_\Omega h_1^i(\xi)\mathbf{v}'_1(t,\xi) + h_2^i(\xi)\mathbf{v}'_2(t,\xi) dt.$$

Computation of $h_1^i(\cdot)$, $h_2^i(\cdot)$

- Solve steady-state Navier-Stokes equations for $\mathcal{V}$.
- Discretized Oseen equations and control outputs

$$\begin{aligned} \mathbf{E}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned}$$

where

$$\mathbf{E} = \begin{bmatrix} \mathbf{E}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{21}^T \\ \mathbf{A}_{21} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{C} = [\mathbf{C}_1 \;\; \mathbf{0}]$$

- $\mathbf{E}_{11} \in \mathbb{R}^{n_1 \times n_1}$ has full rank.
- $\mathbf{A}_{11} \in \mathbb{R}^{n_1 \times n_1}$, $\mathbf{A}_{21} \in \mathbb{R}^{n_2 \times n_1}$, $\mathbf{B}_1 \in \mathbb{R}^{n_1 \times 2}$ and $\mathbf{C}_1 \in \mathbb{R}^{12 \times n_1}$.
- $\mathbf{A}_{21}$ has full rank and $\mathbf{A}_{21}\mathbf{E}_{11}^{-1}\mathbf{A}_{21}^T$ is nonsingular.

- The LQR problem becomes: Find a control $\mathbf{u}(\cdot)$ that solves

$$\min_{\mathbf{u}} \int_0^\infty \left\{ \mathbf{x}_1^T(t)\mathbf{C}_1^T\mathbf{C}_1\mathbf{x}_1(t) + 10\|\mathbf{u}\|^2(t) \right\} \, dt,$$

subject to

$$\begin{bmatrix} \mathbf{E}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1(t) \\ \dot{\mathbf{x}}_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{21}^T \\ \mathbf{A}_{21} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}(t),$$

- $\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t)$
- Computing $\mathbf{K}$ requires solving an $n = n_1 + n_2$ dimensional large-scale algebraic Riccati equation:
- Instead, reduce the dimension first.

- Apply *Interpolatory Model Reduction* to obtain

$$\widetilde{\mathbf{E}}\,\dot{\widetilde{\mathbf{x}}} = \widetilde{\mathbf{A}}\,\widetilde{\mathbf{x}} + \widetilde{\mathbf{B}}\,\mathbf{u}(t)$$
$$\widetilde{\mathbf{y}} = \widetilde{\mathbf{C}}\,\widetilde{\mathbf{x}}$$

- where $\widetilde{\mathbf{E}} \in \mathbb{R}^{r\times r}$, $\widetilde{\mathbf{A}} \in \mathbb{R}^{r\times r}$, $\widetilde{\mathbf{B}} \in \mathbb{R}^{r\times 2}$, and $\widetilde{\mathbf{C}} \in \mathbb{R}^{12\times r}$ with

$$r \ll n = n_1 + n_2$$

- Solve the reduced LQR problem

$$\widetilde{\mathbf{A}}_{11}^T \mathbf{P}\widetilde{\mathbf{E}}_{11} + \widetilde{\mathbf{E}}_{11}^T \mathbf{P}\widetilde{\mathbf{A}}_{11} - \widetilde{\mathbf{E}}_{11}^T \mathbf{P}\widetilde{\mathbf{B}}_1 \mathbf{R}^{-1}\widetilde{\mathbf{B}}_1^T \mathbf{P}\widetilde{\mathbf{E}}_{11} + \widetilde{\mathbf{C}}_1^T \widetilde{\mathbf{C}}_1 = \mathbf{0}$$
$$\widetilde{\mathbf{K}} = \mathbf{R}^{-1}\widetilde{\mathbf{B}}_1^T \mathbf{P}\widetilde{\mathbf{E}}_{11}.$$

- Then

$$\begin{aligned}\mathbf{u} &= -\tilde{\mathbf{K}}\tilde{\mathbf{x}} \\ &= -\underbrace{\tilde{\mathbf{K}}\mathbf{V}^T}_{\approx \mathbf{K}}\,\underbrace{\mathbf{V}\tilde{\mathbf{x}}}_{\approx \mathbf{x}}\end{aligned}$$

Interpolatory Model Reduction for DAEs

- Full-order model: Linearized/Discretized Model

$$\begin{aligned} \mathbf{E}\,\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t), \end{aligned}$$

- $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$, and $\mathbf{D} \in \mathbb{R}^{p\times m}$.
- Let $\mathbf{U}(s)$ and $\mathbf{Y}(s)$ denote the Laplace transforms of $\mathbf{u}(t)$ and $\mathbf{y}(t)$
- Transfer function:

$$\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{U}(s), \quad \text{where} \quad \mathbf{G}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}.$$

Model Reduction

- The goal is to construct a reduced model of the form

$$\widetilde{\mathbf{E}}\,\dot{\widetilde{\mathbf{x}}}(t) = \widetilde{\mathbf{A}}\widetilde{\mathbf{x}}(t) + \widetilde{\mathbf{B}}\mathbf{u}(t), \qquad \widetilde{\mathbf{y}}(t) = \widetilde{\mathbf{C}}\widetilde{\mathbf{x}}(t) + \widetilde{\mathbf{D}}\mathbf{u}(t),$$

  where $\widetilde{\mathbf{E}}, \widetilde{\mathbf{A}} \in \mathbb{R}^{r\times r}$, $\widetilde{\mathbf{B}} \in \mathbb{R}^{r\times m}$, $\widetilde{\mathbf{C}} \in \mathbb{R}^{p\times r}$, and $\widetilde{\mathbf{D}} \in \mathbb{R}^{p\times m}$ with $r \ll n$

- Construct $\mathbf{V} \in \mathbb{R}^{n\times r}$ and $\mathbf{W}^T \in \mathbb{R}^{n\times r}$, assume $\mathbf{x}(t) \approx \mathbf{V}\widetilde{\mathbf{x}}(t)$:

$$\widetilde{\mathbf{E}} = \mathbf{W}^T\mathbf{E}\mathbf{V}, \quad \widetilde{\mathbf{A}} = \mathbf{W}^T\mathbf{A}\mathbf{V}, \quad \widetilde{\mathbf{B}} = \mathbf{W}^T\mathbf{B}, \quad \text{and} \quad \widetilde{\mathbf{C}} = \mathbf{C}\mathbf{V}.$$

- Define $\widetilde{\mathbf{G}}(s) = \widetilde{\mathbf{C}}(s\widetilde{\mathbf{E}} - \widetilde{\mathbf{A}})^{-1}\widetilde{\mathbf{B}} + \widetilde{\mathbf{D}}$
- $\widetilde{\mathbf{G}}(s)$ has the same number of inputs and outputs but a smaller state-space dimension: Low-order rational approximation to $\mathbf{G}(s)$.
- $\widetilde{\mathbf{Y}}(s) - \mathbf{Y}(s) = \left(\widetilde{\mathbf{G}}(s) - \mathbf{G}(s)\right)\mathbf{U}(s)$

Model Reduction by Tangential Interpolation

- Pick interpolation points $\{\sigma_i\}_{i=1}^r \in \mathbb{C}$ together with the left directions $\{c_i\}_{i=1}^r \in \mathbb{C}^p$ and the right directions $\{b_i\}_{i=1}^r \in \mathbb{C}^m$:

$$c_i^T \mathbf{G}(\sigma_j) = c_i^T \widetilde{\mathbf{G}}(\sigma_j), \qquad \mathbf{G}(\sigma_j) b_j = \widetilde{\mathbf{G}}(\sigma_j) b_j, \tag{1}$$

$$\text{and} \qquad c_i^T \mathbf{G}'(\sigma_j) b_j = c_i^T \widetilde{\mathbf{G}}'(\sigma_j) b_j. \tag{2}$$

- Construct

$$\mathbf{V} = \left[(\sigma_1 \mathbf{E} - \mathbf{A})^{-1} \mathbf{B} b_1, \ \cdots, \ (\sigma_r \mathbf{E} - \mathbf{A})^{-1} \mathbf{B} b_r \right] \in \mathbb{C}^{n \times r} \text{ and}$$

$$\mathbf{W} = \left[(\sigma_1 \mathbf{E} - \mathbf{A}^T)^{-1} \mathbf{C}^T c_1 \ \cdots \ (\sigma_r \mathbf{E} - \mathbf{A}^T)^{-1} \mathbf{C}^T c_r \right] \in \mathbb{C}^{n \times r}$$

- Then the interpolation conditions (1) and (2) are satisfied.
  [Skelton *et. al.*, 87], [Grimme, 97], [Gallivan *et. al.*, 05]
- Interpolatory reduction of port-Hamiltonian systems:
  [G./Polyuga/Beattie/vanderSchaft, 12], [Beattie/G.,11] and [Chaturantabut/Beattie/G.,13]

Tangential Interpolation for DAEs

- Recall in our case

$$\mathbf{E} = \begin{bmatrix} \mathbf{E}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{21}^T \\ \mathbf{A}_{21} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{C} = [\mathbf{C}_1 \;\; \mathbf{0}]$$

- $\mathbf{E}_{11} \in \mathbb{R}^{n_1 \times n_1}$ and $\mathbf{A}_{21} \in \mathbb{R}^{n_2 \times n_1}$ have full rank and $\mathbf{A}_{21}\mathbf{E}_{11}^{-1}\mathbf{A}_{21}^T$ is nonsingular $\Longrightarrow$ Leading to an index-2 DAE.
- Let $\mathbf{G}(s)$ be additively decomposed as: $\mathbf{G}(s) = \mathbf{G}_{\mathrm{sp}}(s) + \mathbf{P}(s)$,
- We will require that $\widetilde{\mathbf{G}}(s) = \widetilde{\mathbf{G}}_{\mathrm{sp}}(s) + \widetilde{\mathbf{P}}(s)$, with $\widetilde{\mathbf{P}}(s) = \mathbf{P}(s)$,
- This will guarantee: $\mathbf{G}_{\mathrm{err}}(s) = \mathbf{G}(s) - \widetilde{\mathbf{G}}(s) = \mathbf{G}_{\mathrm{sp}}(s) - \widetilde{\mathbf{G}}_{\mathrm{sp}}(s)$.
- [Stykel,2004], [Mehrman/Stykel,2005], [Benner/Sokolov,2005], [Alì et al., 2013] [Heinkenschloss et al., 08]

- $\mathbf{G}(s) = \mathbf{G}_{sp}(s) + \mathbf{P}(s)$.
- We want $\widetilde{\mathbf{G}}(s) = \widetilde{\mathbf{G}}_{\mathrm{sp}}(s) + \widetilde{\mathbf{P}}(s)$ with $\widetilde{\mathbf{P}}(s) = \mathbf{P}(s)$,
- Problem reduces to: $\widetilde{\mathbf{G}}_{\mathrm{sp}}(s)$ interpolates $\mathbf{G}_{\mathrm{sp}}(s)$.
- $\mathbf{P}_r$ = the spectral projector onto the right deflating subspace of $(\lambda\mathbf{E} - \mathbf{A})$ corresponding to the finite eigenvalues.
- $\mathbf{P}_l$: Defined similarly for the left deflating subspace.
- $\mathbf{W}_\infty$ and $\mathbf{V}_\infty$: Span, respectively, the right and left deflating subspaces of $(\lambda\mathbf{E} - \mathbf{A})$ corresponding to the infinite eigenvalues.

Theorem ([G./Stykel/Wyatt,12])

Given are $\mathbf{G}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$*, interpolation points* $\sigma \in \mathbb{C}$ *and* $\mu \in \mathbb{C}$*; and the tangential directions* $\mathrm{b} \in \mathbb{C}^m$ *and* $\mathrm{c} \in \mathbb{C}^p$*. Define* $\mathbf{V}_f$ *and* $\mathbf{W}_f$ *such that*

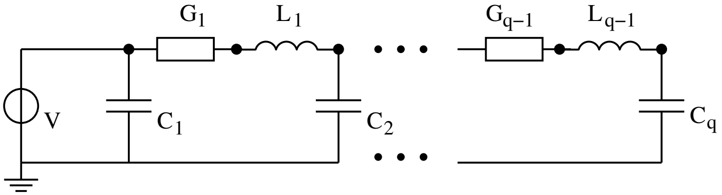
$$\mathbf{V}_f = \left[(\sigma_1\mathbf{E} - \mathbf{A})^{-1}\mathbf{P}_l\mathbf{B}\mathrm{b}_1, \; \cdots, \; (\sigma_r\mathbf{E} - \mathbf{A})^{-1}\mathbf{P}_l\mathbf{B}\mathrm{b}_r\right] \in \mathbb{C}^{n\times r} \textit{ and}$$

$$\mathbf{W}_f = \left[(\sigma_1\mathbf{E} - \mathbf{A}^T)^{-1}\mathbf{P}_r^T\mathbf{C}^T\mathrm{c}_1 \; \cdots \; (\sigma_r\mathbf{E} - \mathbf{A}^T)^{-1}\mathbf{P}_r^T\mathbf{C}^T\mathrm{c}_r\right] \in \mathbb{C}^{n\times r}$$

Define $\mathbf{W} = [\,\mathbf{W}_f, \; \mathbf{W}_\infty\,]$ *and* $\mathbf{V} = [\,\mathbf{V}_f, \; \mathbf{V}_\infty\,]$*, and construct* $\widetilde{\mathbf{G}}(s)$*. Then,*

1. $\mathbf{P}_r(s) = \mathbf{P}(s)$*, and*
2. $$\begin{aligned} \mathrm{c}_i^T\mathbf{G}(\sigma_j) &= \mathrm{c}_i^T\widetilde{\mathbf{G}}(\sigma_j) \\ \mathbf{G}(\sigma_j)\,\mathrm{b}_j &= \widetilde{\mathbf{G}}(\sigma_j)\,\mathrm{b}_j \quad \textit{for} \quad j = 1, 2, \ldots, r. \\ \mathrm{c}_i^T\mathbf{G}'(\sigma_j)\,\mathrm{b}_j &= \mathrm{c}_i^T\widetilde{\mathbf{G}}'(\sigma_j)\,\mathrm{b}_j \end{aligned}$$

A Circuit Model



$$\mathbf{E} = \begin{bmatrix} C_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_2 & 0 & 0 \\ 0 & 0 & 0 & L_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} -G_1 & G_1 & 0 & 0 & -1 \\ G_1 & -G_1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{B}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \end{bmatrix} = \mathbf{C}, \qquad \mathbf{D} = 0,$$

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} = \underbrace{\frac{sC_2G_1}{s^2C_2L_1G_1 + sC_2 + G_1}}_{\mathbf{G}_{sp}(s)} + \underbrace{sC_1}_{\mathbf{P}(s)}$$

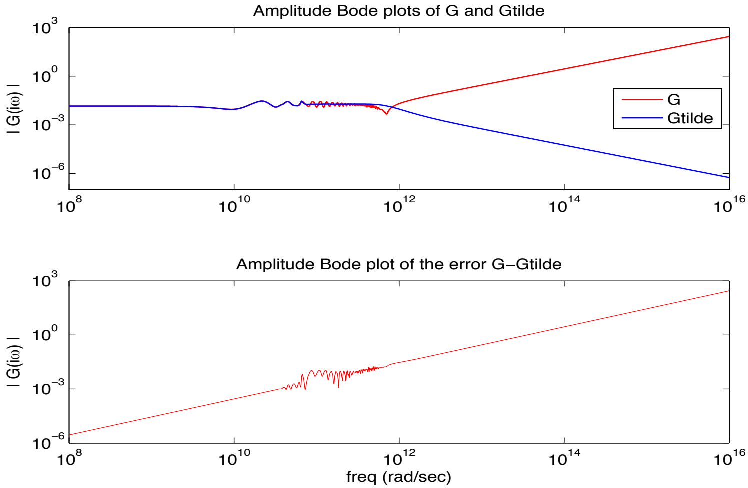
Interpolation for DAEs

- Consider the model of an RLC circuit with $n = 765$ and index-2.
- Reduce the order to $r = 20$ using complex interpolation points without the deflating subspaces:

	$\mathbf{G}(\sigma_i)$	$\mathbf{G}_r(\sigma_i)$
σ_1	$9.8479 \times 10^{-3} + \imath 3.4595 \times 10^{-3}$	$9.8479 \times 10^{-3} + \imath 3.4595 \times 10^{-3}$
σ_2	$1.1586 \times 10^{-2} + \imath 6.6549 \times 10^{-3}$	$1.1586 \times 10^{-2} + \imath 6.6549 \times 10^{-3}$
σ_3	$1.6518 \times 10^{-2} + \imath 7.9917 \times 10^{-3}$	$1.6518 \times 10^{-2} + \imath 7.9917 \times 10^{-3}$

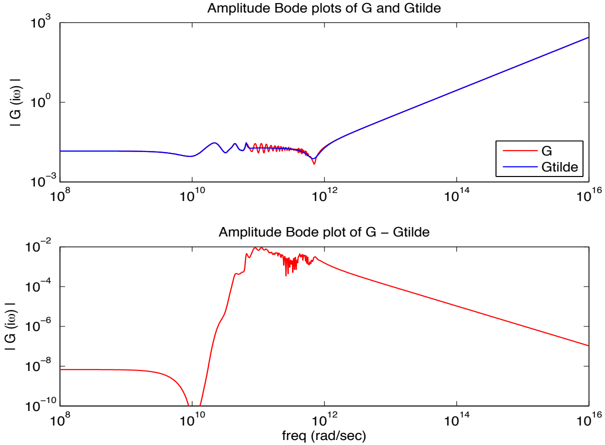
	$\mathbf{G}'(\sigma_i)$	$\mathbf{G}'_r(\sigma_i)$
σ_1	$-1.1553 \times 10^{-12} - \imath 3.7091 \times 10^{-14}$	$-1.1553 \times 10^{-12} - \imath 3.7091 \times 10^{-14}$
σ_2	$-1.1045 \times 10^{-12} + \imath 7.1250 \times 10^{-13}$	$-1.1045 \times 10^{-12} + \imath 7.1250 \times 10^{-13}$
σ_3	$-1.1846 \times 10^{-13} + \imath 1.3335 \times 10^{-12}$	$-1.1846 \times 10^{-13} + \imath 1.3335 \times 10^{-12}$

How do the Bode plots match?



- Polynomial part is completely missed.

- Re-visit the previous example and apply the projection with deflating subspaces.



- Requires computing $\mathbf{P}_l$ and $\mathbf{P}_r$. How to avoid this?

- Define $\mathbf{\Pi} = \mathbf{I} - \mathbf{A}_{12}\left(\mathbf{A}_{21}\mathbf{E}_{11}^{-1}\mathbf{A}_{12}\right)^{-1}\mathbf{A}_{21}\mathbf{E}_{11}^{-1}$
- $\mathbf{\Pi}^2 = \mathbf{\Pi},\ \mathbf{\Pi}\mathbf{E}_{11} = \mathbf{E}_{11}\mathbf{\Pi}^T,\ \mathrm{Null}(\mathbf{\Pi}) = \mathrm{Range}(\mathbf{A}_{12})$.
- Can be equivalently to reduced to ([Heinkenschloss *et al.*,08])

$$\begin{aligned} \mathbf{\Pi}\mathbf{E}_{11}\mathbf{\Pi}^T\dot{\mathbf{v}}_1(t) &= \mathbf{\Pi}\mathbf{A}_{11}\mathbf{\Pi}^T\mathbf{v}_1(t) + \mathbf{\Pi}\mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{v}_1(t) + \mathbf{D}_1\,\mathbf{u}(t) + \mathbf{D}_2\,\dot{\mathbf{u}}(t) \end{aligned}$$

- We need $(\sigma_i\,\mathbf{\Pi}\mathbf{E}_{11}\mathbf{\Pi}^T - \mathbf{\Pi}\mathbf{A}_{11}\mathbf{\Pi}^T)^{-1}\mathbf{\Pi}\mathbf{B}\mathrm{b}_i$
- Define $\mathbf{\Pi}\mathbf{E}_{11}\mathbf{\Pi}^T = \mathcal{E}, \quad \mathbf{\Pi}\mathbf{A}_{11}\mathbf{\Pi}^T = \mathcal{A}$, and, $\mathcal{B} = \mathbf{\Pi}\mathbf{B}$
- Inverse defined on a restricted subspace:

$$(\sigma\,\mathcal{E} - \mathcal{A})^I(\sigma\,\mathcal{E} - \mathcal{A}) = \mathbf{\Pi}^T,\ \ (\sigma\,\mathcal{E} - \mathcal{A})(\sigma\,\mathcal{E} - \mathcal{A})^I = \mathbf{\Pi}.$$

- The vector $\mathbf{v}_i = (\sigma\mathcal{E} - \mathcal{A})^I\mathcal{B}\mathrm{b}_i$ solves

$$\begin{bmatrix} \sigma\,\mathbf{E}_{11} + \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{v}_i \\ \mathbf{z}_i \end{bmatrix} = \begin{bmatrix} \mathbf{B}\mathrm{b}_i \\ \mathbf{0} \end{bmatrix}$$

Interpolation without $\mathbf{P}_l$ and $\mathbf{P}_r$ computations

Theorem (G./Stykel/Wyatt, 2013)

Given $\{\sigma_i\} \in \mathbb{C}$, $\{\mathrm{b}_i\} \in \mathbb{C}^m$ *and* $\{\mathrm{c}_i\} \in \mathbb{C}^p$, *let* $\mathbf{v}_i$ *and* $\mathbf{w}_i$ *solve*

$$\begin{bmatrix} \sigma_i \mathbf{E}_{11} - \mathbf{A}_{11} & \mathbf{A}_{21}^T \\ \mathbf{A}_{21} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{v}_i \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \mathrm{b}_i \\ \mathbf{0} \end{bmatrix},$$

$$\begin{bmatrix} \sigma_i \mathbf{E}_{11}^T - \mathbf{A}_{11}^T & \mathbf{A}_{21}^T \\ \mathbf{A}_{21} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{w}_i \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} \mathbf{C}^T \mathrm{c}_i \\ \mathbf{0} \end{bmatrix}.$$

for $i = 1, \ldots, r$. *Construct*

$$\mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_r], \quad \textit{and} \quad \mathbf{W} = [\mathbf{w}_1, \ldots, \mathbf{w}_r].$$

Then $\widetilde{\mathbf{G}}(s) = \mathbf{C}\mathbf{V}(s\mathbf{W}^T\mathbf{E}_{11}\mathbf{V} - \mathbf{W}^T\mathbf{A}_{11}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{B}_1 + \mathbf{D}_1\mathbf{u}(t) + \mathbf{D}_2\dot{\mathbf{u}}(t)$ *satisfies the required interpolation conditions and matches the polynomial part.*

Interpolation points for $\mathcal{H}_2$ optimal approximation

- $\|\mathbf{G}\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} \|\mathbf{G}(\imath\omega)\|_F^2\, d\omega\right)^{\frac{1}{2}}$

Problem

Given $\mathbf{G}(s)$*, find* $\widetilde{\mathbf{G}}(s)$ *of order* r *which solves:* $\min_{\mathrm{degree}(\mathbf{G}_r)=r} \|\mathbf{G}-\mathbf{G}_r\|_{\mathcal{H}_2}$.

- $\|\mathbf{G}\|_{\mathcal{H}_2} = \sup_{\mathbf{u}\neq 0} \frac{\|\mathbf{y}\|_\infty}{\|\mathbf{u}\|_2}$ for MISO and SIMO systems
- In general, $\|\mathbf{y}-\mathbf{y}_r\|_\infty \leq \|\mathbf{G}-\widetilde{\mathbf{G}}\|_{\mathcal{H}_2}\|\mathbf{u}\|_2$.
- Solution for the ODE case: [Meier /Luenberger,67], [G./Antoulas/Beattie,08]
  $\Longrightarrow$Iterative Rational Krylov Algorithm: [G./Antoulas/Beattie,08]
- Solution for the DAE case: [G./Stykel/Wyatt,13]

$\mathcal{H}_2$ optimality for DAE approximation

Theorem ([G./Stykel/Wyatt,13])

For $\mathbf{G}(s) = \mathbf{G}_{sp}(s) + \mathbf{P}(s)$*, let* $\widetilde{\mathbf{G}}(s) = \widetilde{\mathbf{G}}_{\mathrm{sp}}(s) + \widetilde{\mathbf{P}}(s)$ *minimize the* $\mathcal{H}_2$ *error* $\left\|\mathbf{G} - \widetilde{\mathbf{G}}\right\|_{\mathcal{H}_2}$*. Then,* $\widetilde{\mathbf{P}}(s) = \mathbf{P}(s)$*, and, hence* $\widetilde{\mathbf{G}}_{\mathrm{sp}}(s)$ *minimizes the* $\mathcal{H}_2$ *error* $\left\|\mathbf{G}_{sp} - \widetilde{\mathbf{G}}_{\mathrm{sp}}\right\|_{\mathcal{H}_2}$*. Moreover, let* $\widetilde{\mathbf{G}}_{\mathrm{sp}}(s) = \widetilde{\mathbf{C}}_{\mathrm{sp}}(s\widetilde{\mathbf{E}}_{\mathrm{sp}} - \widetilde{\mathbf{A}}_{\mathrm{sp}})^{-1}\widetilde{\mathbf{B}}_{\mathrm{sp}}$*. Suppose that that the reduced-order pencil* $\lambda\widetilde{\mathbf{E}}_{\mathrm{sp}} - \widetilde{\mathbf{A}}_{\mathrm{sp}}$ *has distinct eigenvalues* $\{\widetilde{\lambda}_i\}_{i=1}^r$*, i.e.,* $\widetilde{\mathbf{G}}_{\mathrm{sp}}(s) = \sum_{i=1}^r \frac{1}{s-\widetilde{\lambda}_i}\tilde{\mathbf{c}}_i\tilde{\mathbf{b}}_i^T$*. Then, for* $i = 1, \cdots, r$*,*

$$\mathbf{G}(-\widetilde{\lambda}_i)\tilde{\mathbf{b}}_i = \widetilde{\mathbf{G}}(-\widetilde{\lambda}_i)\tilde{\mathbf{b}}_i, \quad \tilde{\mathbf{c}}_i^T\mathbf{G}(-\widetilde{\lambda}_i) = \tilde{\mathbf{c}}_i^T\widetilde{\mathbf{G}}(-\widetilde{\lambda}_i),$$

$$\textit{and} \quad \tilde{\mathbf{c}}_i^T\mathbf{G}'(-\widetilde{\lambda}_i)\tilde{\mathbf{b}}_i = \tilde{\mathbf{c}}_i^T\widetilde{\mathbf{G}}'(-\widetilde{\lambda}_i)\tilde{\mathbf{b}}_i.$$

- Iterate on the interpolation points and directions until convergence

Iterative Rational Krylov Algorithm (IRKA):

Algorithm (G./Antoulas/Beattie [2008])

1. *Choose* $\{\sigma_1, \ldots, \sigma_r\}$, $\{\hat{\mathrm{b}}_1, \ldots, \hat{\mathrm{b}}_r\}$ *and* $\{\hat{\mathrm{c}}_1, \ldots, \hat{\mathrm{c}}_r\}$
2. $\mathbf{V} = \left[(\sigma_1 \mathbf{E} - \mathbf{A})^{-1}\mathbf{B}\hat{\mathrm{b}}_1 \ \cdots \ (\sigma_r \mathbf{E} - \mathbf{A})^{-1}\mathbf{B}\hat{\mathrm{b}}_r \right]$
   $\mathbf{W} = \left[(\sigma_1 \mathbf{E} - \mathbf{A}^T)^{-1}\mathbf{C}^T\hat{\mathrm{c}}_1 \ \cdots \ (\sigma_r \mathbf{E} - \mathbf{A}^T)^{-1}\mathbf{C}^T\hat{\mathrm{c}}_r \right].$
3. *while (not converged)*
   1. $\widetilde{\mathbf{A}} = \mathbf{W}^T\mathbf{A}\mathbf{V}$, $\widetilde{\mathbf{E}} = \mathbf{W}^T\mathbf{E}\mathbf{V}$, $\mathbf{B} = \mathbf{W}^T\mathbf{B}$, *and* $\mathbf{C} = \mathbf{C}\mathbf{V}$
   2. *Compute* $\widetilde{\mathbf{G}}(s) = \sum_{i=1}^{r} \frac{\mathbf{c}_i \mathbf{b}_i^T}{s - \lambda_i}$
   3. $\sigma_i \longleftarrow -\lambda_i$, $\hat{\mathrm{b}}_i \longleftarrow \mathbf{b}_i$ *and* $\hat{\mathrm{c}}_i \longleftarrow \mathbf{c}_i$.
   4. $\mathbf{V} = \left[(\sigma_1 \mathbf{E} - \mathbf{A})^{-1}\mathbf{B}\hat{\mathrm{b}}_1 \ \cdots \ (\sigma_r \mathbf{E} - \mathbf{A})^{-1}\mathbf{B}\hat{\mathrm{b}}_r \right]$
   5. $\mathbf{W} = \left[(\sigma_1 \mathbf{E} - \mathbf{A}^T)^{-1}\mathbf{C}^T\hat{\mathrm{c}}_1 \ \cdots \ (\sigma_r \mathbf{E} - \mathbf{A}^T)^{-1}\mathbf{C}^T\hat{\mathrm{c}}_r \right].$
4. $\widetilde{\mathbf{A}} = \mathbf{W}_r^T\mathbf{A}\mathbf{V}_r$, $\widetilde{\mathbf{E}} = \mathbf{W}_r^T\mathbf{E}\mathbf{V}_r$, $\widetilde{\mathbf{B}} = \mathbf{W}_r^T\mathbf{B}$, $\widetilde{\mathbf{C}} = \mathbf{C}\mathbf{V}_r$, $\widetilde{\mathbf{D}} = \mathbf{D}$.

Discretization of Navier-Stokes/Oseen equations

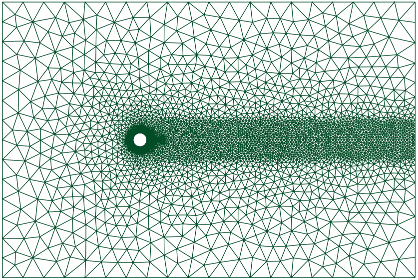
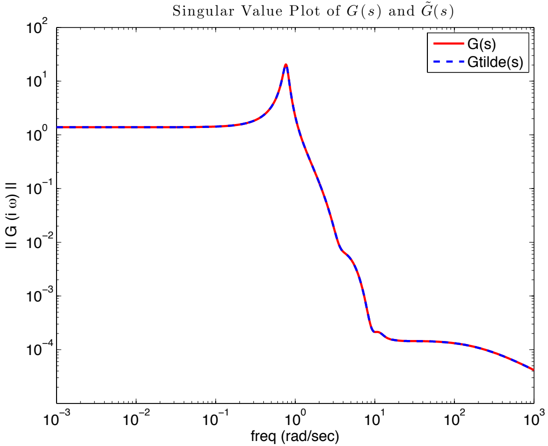


Figure : Discretization by Taylor-Hood Finite Elements

- Leads to 21,390 velocity degrees of freedom ($\mathbf{x}_1$),
- and 2,777 pressure degrees of freedom ($\mathbf{x}_2$).
- Solved at $Re_d = 60$ ($\mu = 1/Re$)

Numerical Results: One-cylinder Case

- Recall $n_1 = 21390$ and $n_2 = 2777$
- We reduce the order to $r = 60$ using interpolatory projection.



- Relative $\mathcal{L}_\infty$ error = 1.5406×10^{-5}

Functional Gains

- For this Re_d, the full-model has two unstable poles.
- These unstable poles are captured very accurately.

$$\lambda_{\text{unstable}}(\mathbf{G}(s)): \quad 5.248019596820730 \times 10^{-2} \pm \imath\, 7.672028760928972 \times 10^{-1}$$

$$\lambda_{\text{unstable}}(\widetilde{\mathbf{G}}(s)): \quad 5.248030491505502 \times 10^{-2} \pm \imath\, 7.672029050490372 \times 10^{-1}$$

- Solve the reduced LQR problem and compute the functional gains:

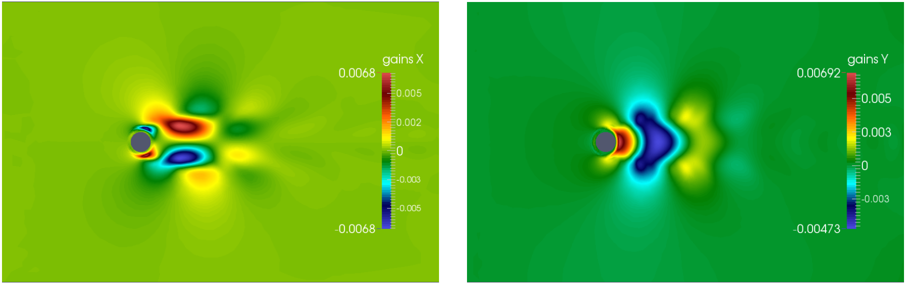
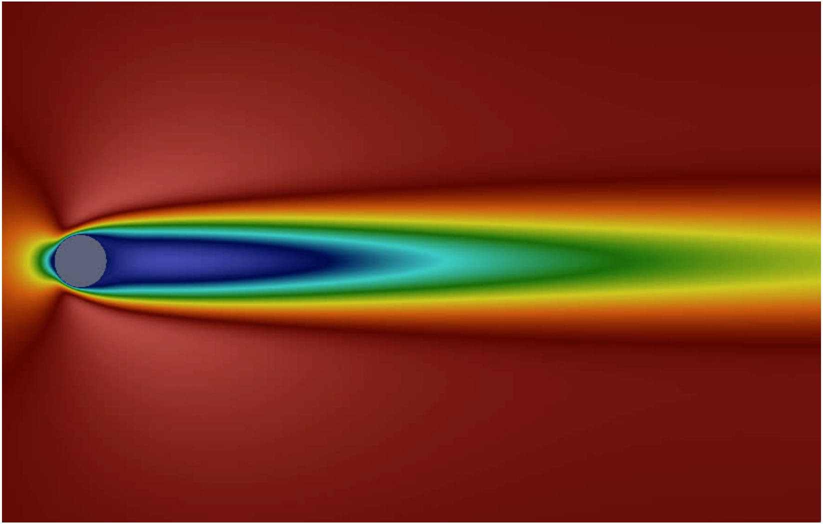
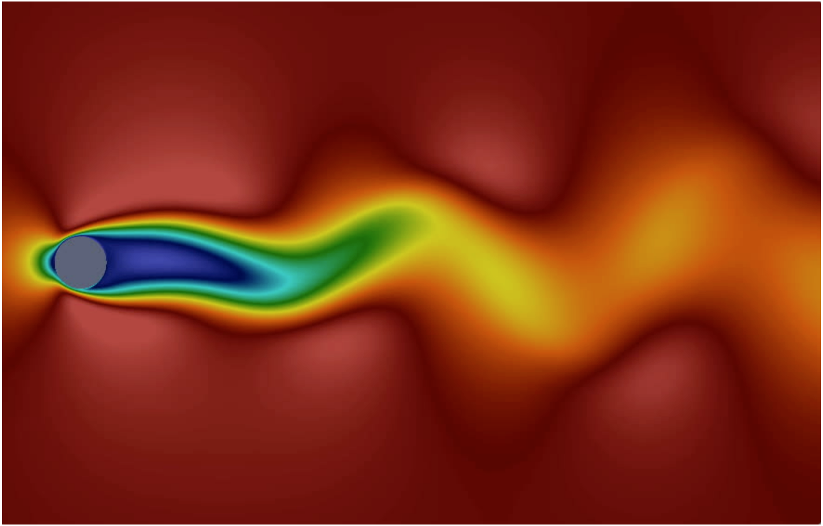


Figure : Horizontal (left) and Vertical (right) Components

Open Loop Simulation



Closed Loop: From $t = 20$



Two-cylinder case

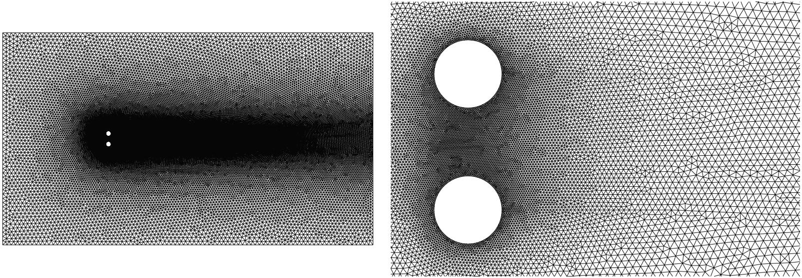
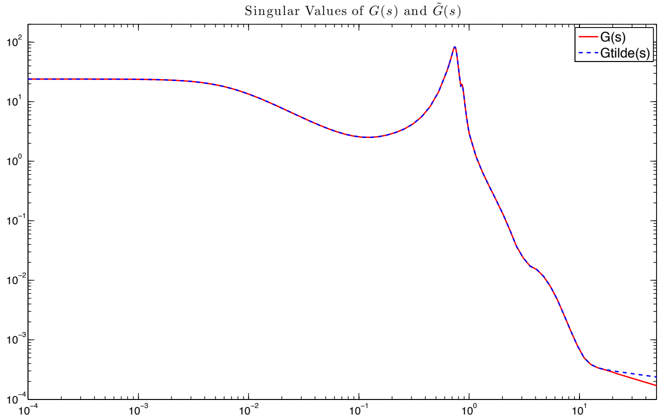


Figure : Discretization by Taylor-Hood Finite Elements

- Leads to 132476 velocity degrees of freedom ($\mathbf{x}_1$),
- and 16691 pressure degrees of freedom ($\mathbf{x}_2$).
- Solved at $Re_d = 60$ ($\mu = 1/Re$)

Model Reduction for the Two-cylinder Case

- Recall $n_1 = 132476$ and $n_2 = 16691$, $n = n_1 + n_2 = 149167$.
- We reduce the order to $r = 150$ using interpolatory projection.



- Relative $\mathcal{L}_\infty$ error = 6.3980×10^{-6}

Functional Gain

- Unstable poles are, once again, captured very accurately.

$\lambda_{\text{unstable}}(\mathbf{G}(s))$: $3.97391256163880 1 \times 10^{-2} \pm \imath\, 7.498560362688469 \times 10^{-1}$

$\lambda_{\text{unstable}}(\widetilde{\mathbf{G}}(s))$: $3.97391252608265 7 \times 10^{-2} \pm \imath\, 7.498560367601876 \times 10^{-1}$

- Solve the reduced LQR problem and compute the functional gains:

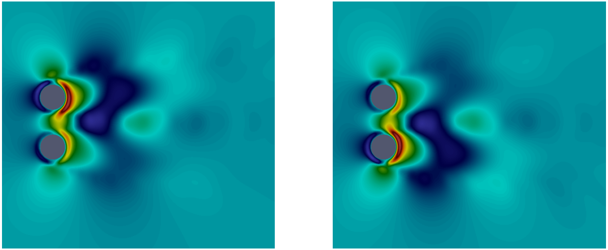
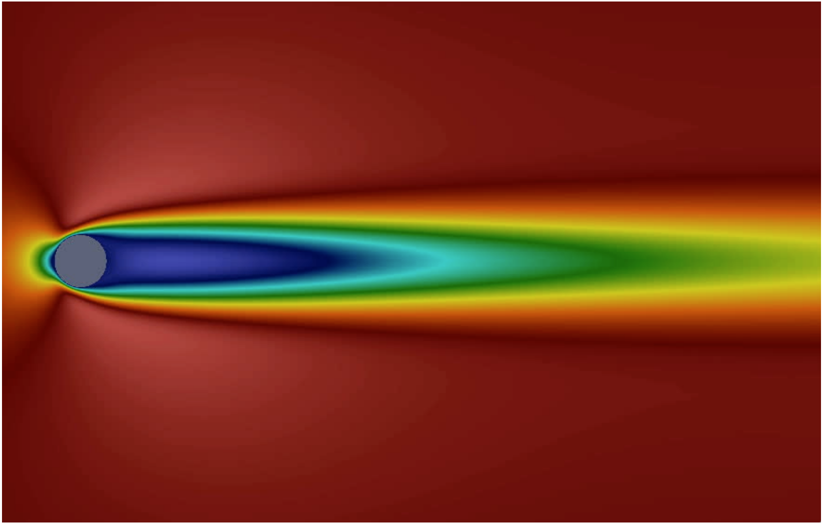
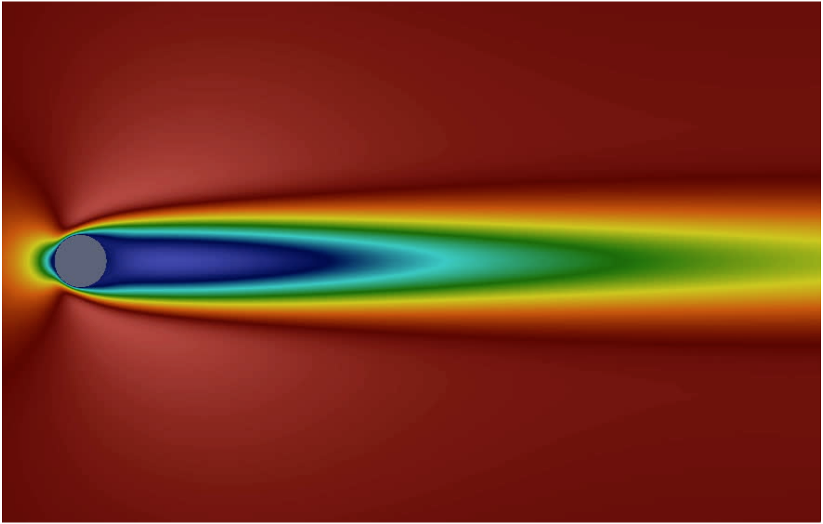


Figure : Horizontal (left) and Vertical (right) Components

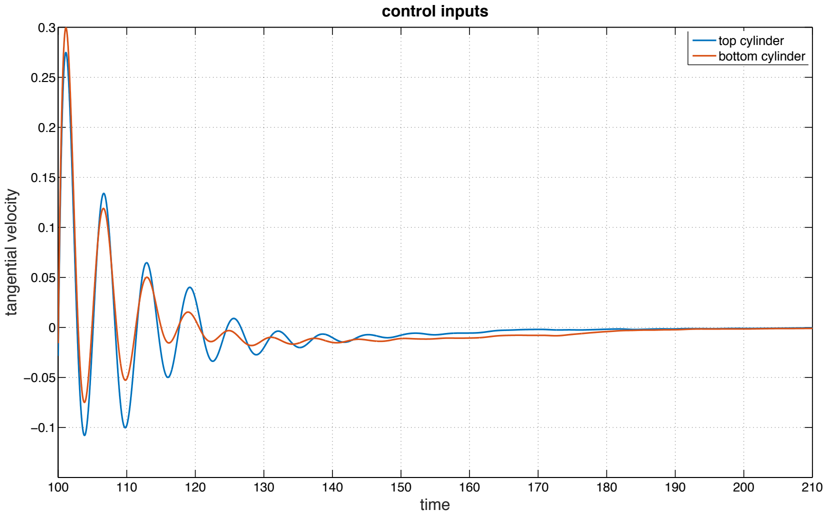
Open Loop Simulation



Closed Loop: Controlled from $t = 100$



Control Inputs



Two-cylinder case

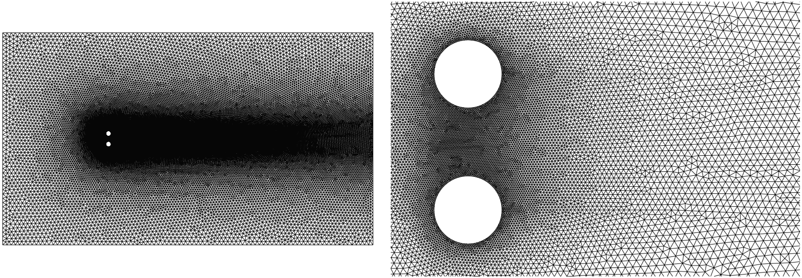
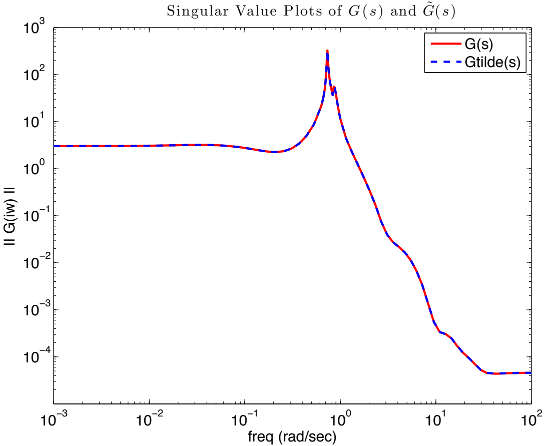


Figure : Discretization by Taylor-Hood Finite Elements

- Leads to 299338 velocity degrees of freedom ($\mathbf{x}_1$),
- and 37714 pressure degrees of freedom ($\mathbf{x}_2$).
- Solved at $Re_d = 100$ ($\mu = 1/Re$)

Model Reduction for the Two-cylinder Case

- Recall $n_1 = 299338$ and $n_2 = 37714$, $n = n_1 + n_2 = 337052$.
- We reduce the order to $r = 170$ using interpolatory projection.



- Relative $\mathcal{L}_\infty$ error = 1.5154×10^{-5}

Unstable Poles

- For $Re_d = 100$, the full-model has seven unstable poles.
- The unstable poles of the reduced model $\widetilde{\mathbf{G}}(s)$:

$$
\begin{array}{l}
1.245178576584041 \times 10^{-1} \pm \imath\, 7.507209792650027 \times 10^{-1} \\
3.195053261722973 \times 10^{-2} \pm \imath\, 8.505319185007424 \times 10^{-1} \\
8.325502142822423 \times 10^{-3} \pm \imath\, 7.314950149341377 \times 10^{-1} \\
2.580915637572443 \times 10^{-2}
\end{array}
$$

- Accurate to 5 significant digits
- Unstable poles are, once again, captured accurately.
- We follow similarly and solve the reduced LQR problem.

Convergence of Gain with Model Size

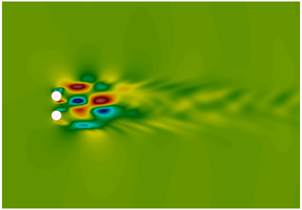


Figure : Gain h_1^1 for $r = 120$

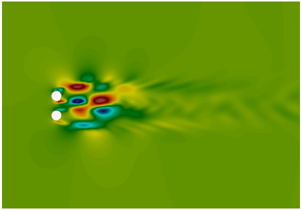


Figure : Gain h_1^1 for $r = 140$

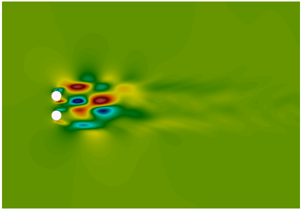
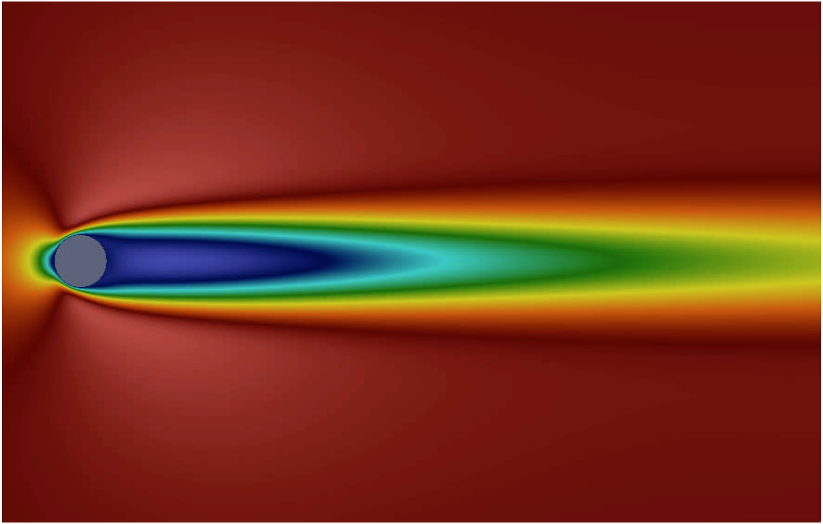
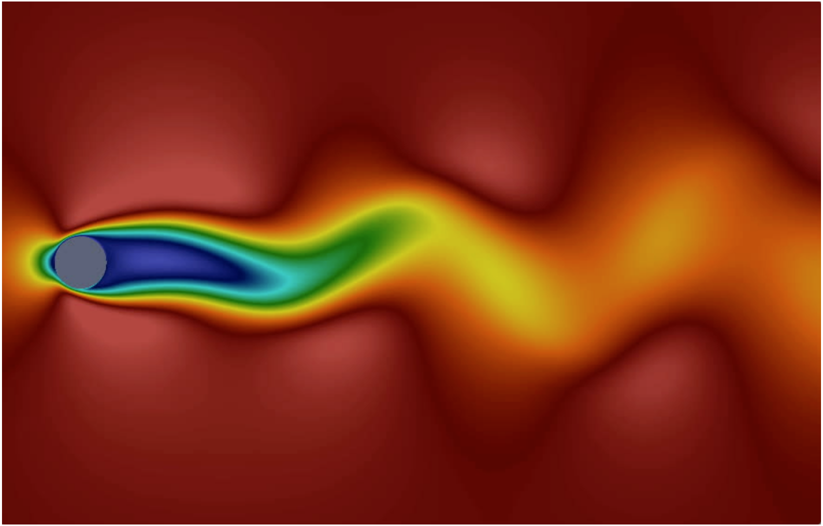


Figure : Gain h_1^1 for $r = 170$

Open Loop Simulation: disturbance for $t \in (0, 2\pi)$



Closed loop from $t = 10$



Nonlinear dynamical systems

- $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}\,(\mathbf{x}(t), \mathbf{u}(t))\,, \qquad \mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t)$
- The most common and a rather effective approach: Proper Orthogonal Decomposition (POD)
- Pick your favorite input $\mathbf{u}(t)$, run the system from $t = 0$ to $t_N = T$ and construct a snapshot matrix:

$$\mathbf{X} = [\mathbf{x}(t_1),\ \mathbf{x}(t_2),\ ,\ldots,\mathbf{x}(t_N)] \in \mathbb{R}^{n\times N}$$

- Compute the SVD of $\mathbf{X}$: $\mathbf{X} = \mathbf{U}\Sigma\mathbf{Z}^T$
- Choose $\mathbf{V}$ as the leading r columns of $\mathbf{U}$.
- $\dot{\mathbf{x}}_r(t) = \mathbf{V}^T\mathbf{A}\mathbf{V}\mathbf{x}_r(t) + \mathbf{V}^T\mathbf{f}\,(\mathbf{V}\mathbf{x}_r(t), \mathbf{u}(t))\,, \qquad \mathbf{y}_r(t) = \mathbf{C}\mathbf{V}\,\mathbf{x}_r(t)$

- Input-dependent reduced-order model.
- The reduced-model is usually only as good as the information in $\mathbf{X}$.
- For linear dynamics, $\mathbf{u}(t)$ did not enter into the model reduction step.
- Can we mimic the linear case for some special cases?
- How to extend the idea of transfer function to the nonlinear setting?

Bilinear Systems

- $$\zeta : \begin{cases} \dot{\boldsymbol{x}}(t) = & \boldsymbol{A}\boldsymbol{x}(t) + \mathbf{N}\boldsymbol{x}(t)u(t) + \boldsymbol{b}u(t) \\ y(t) = & \boldsymbol{c}^T\boldsymbol{x}(t) \end{cases},$$

  where $\mathbf{A}, \mathbf{N} \in \mathbb{R}^{n\times n}$, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$ and $\boldsymbol{x}(t) \in \mathbb{R}^n$.

- The output $y(t)$ has the Volterra series representation

$$y(t) = \sum_{k=1}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} h_k(t_1, \ldots, t_k) u(t - t_1 - t_2 - \cdots - t_k) \cdots u(t - t_k)\, \mathrm{d}t_k \cdots \mathrm{d}t_1,$$

  where $h_k(t_1, \ldots, t_k) = \mathbf{c}^T e^{\mathbf{A}t_k} \mathbf{N} e^{\mathbf{A}t_{k-1}} \mathbf{N} \cdots \mathbf{N} e^{\mathbf{A}t_1} \mathbf{b}$.

- $$\begin{aligned} \mathcal{L}[h_k(t_1, \ldots, t_k)] &= H_k(s_1, s_2, \ldots, s_k) \\ &= \mathbf{c}^T (s_k\mathbf{I} - \mathbf{A})^{-1} \mathbf{N} (s_{k-1}\mathbf{I} - \mathbf{A})^{-1} \mathbf{N} \cdots \mathbf{N} (s_1\mathbf{I} - \mathbf{A})^{-1} \mathbf{b}. \end{aligned}$$

Model Reduction in the Petrov-Galerkin Framework

- Given

$$\zeta : \begin{cases} \dot{\boldsymbol{x}}(t) = & \boldsymbol{A}\boldsymbol{x}(t) + \mathbf{N}\boldsymbol{x}(t)u(t) + \boldsymbol{b}u(t) \\ y(t) = & \boldsymbol{c}^T\boldsymbol{x}(t) \end{cases}$$

  of dimension n.

- For $r \ll n$, find

$$\zeta_r : \begin{cases} \dot{\boldsymbol{x}}_r(t) = & \tilde{\boldsymbol{A}}\boldsymbol{x}_r(t) + \tilde{\mathbf{N}}\boldsymbol{x}_r(t)u(t) + \tilde{\boldsymbol{b}}u(t) \\ y_r(t) = & \tilde{\boldsymbol{c}}^T\boldsymbol{x}_r(t) \end{cases}$$

  such that $y_r(t) \approx y(t)$

- Define ζ_r via the projected equations:

$$\zeta_r : \begin{cases} \dot{\boldsymbol{x}}_r(t) = & \boldsymbol{W}^T\mathbf{A}\mathbf{V}\boldsymbol{x}_r(t) + \boldsymbol{W}^T\mathbf{N}\mathbf{V}\boldsymbol{x}_r(t)u(t) + \boldsymbol{W}^T\mathbf{b}u(t) \\ y_r(t) = & \mathbf{c}^T\mathbf{V}\boldsymbol{x}_r(t) \end{cases}$$

What to interpolate

- Construct **V** and **W** so that

$$H_k(\sigma_1, \sigma_{1,2}, \ldots, \sigma_{1,\ldots,k}) = \widetilde{H}_k(\sigma_1, \sigma_{1,2}, \ldots, \sigma_{1,\ldots,k})$$

  for $k = 1, \ldots, N$. [Phillips, 2002], [Bai and Skoogh, 2006], [Breiten and Damm, 2009].

  $\Rightarrow$ The leading N subsystems of $\widetilde{H}(s)$ interpolates those of $H(s)$.
- Optimal $\mathcal{H}_2$ reduction for bilinear systems: [Benner/Breiten,11]: B-IRKA
  - Input-independent optimal model reduction for a nonlinear system.
  - Significantly more accurate approximations than the subsystem interpolation methods and better performance than bilinear *balanced truncation*.
- Interpolate the infinite-Volterra series, not just the subsystems: [Flagg/G.,15]:
  - Solve bilinear Sylvester equations
  - B-IRKA interpolates the infinite-Volterra series.

Quadratic-in-State Nonlinearity

- Consider the 1D Burgers equation over $[0,1] \times [0,t_f]$.

$$v_t(x,t) + v(x,t) \cdot v_x(x,t) = \nu \cdot v_{xx}(x,t),$$
$$v(0,t) = u(t), v_x(1,t) = 0, v(x,0) = v_0(x) = 0$$

- A finite difference discretization yields

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{H}\left(\mathbf{x}(t) \otimes \mathbf{x}(t)\right) + \mathbf{N}\mathbf{x}(t)u(t) + \mathbf{b}u(t)$$
$$y(t) = \mathbf{c}^T\mathbf{x}(t)$$

where

$$\mathbf{A}, \mathbf{N} \in \mathbb{R}^{n \times n}, \quad \mathbf{H} \in \mathbb{R}^{n \times n^2}, \quad \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$$

- In our tests, we took $\nu = 0.02$ and $n = 1500$.

Reduced Order Model (ROM)

- Construct

$$\dot{\tilde{\mathbf{x}}}(t) = \widetilde{\mathbf{A}}\tilde{\mathbf{x}}(t) + \widetilde{\mathbf{H}}\left(\tilde{\mathbf{x}}(t) \otimes \tilde{\mathbf{x}}(t)\right) + \widetilde{\mathbf{N}}\tilde{\mathbf{x}}(t)u(t) + \widetilde{\mathbf{b}}u(t)$$
$$\tilde{y}(t) = \widetilde{\mathbf{c}}^T\tilde{\mathbf{x}}(t)$$

via projection

$$\widetilde{\mathbf{A}} = \mathbf{V}^T\mathbf{A}\mathbf{V}, \quad \widetilde{\mathbf{H}} = \mathbf{V}^T\mathbf{H}(\mathbf{V} \otimes \mathbf{V}), \quad \widetilde{\mathbf{N}} = \mathbf{V}^T\mathbf{N}\mathbf{V}$$
$$\widetilde{\mathbf{b}} = \mathbf{V}^T\mathbf{b}, \quad \widetilde{\mathbf{c}} = \mathbf{V}^T\mathbf{c}$$

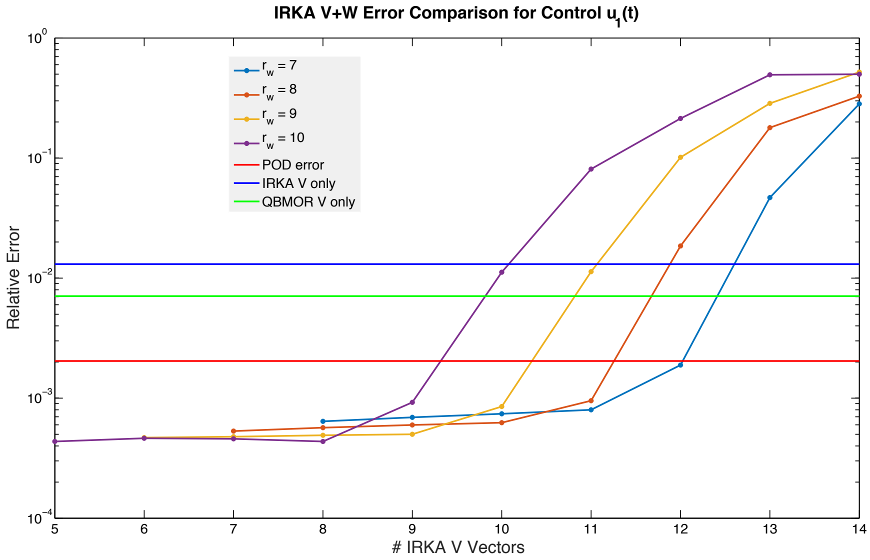
- Subsystem interpolation: [Gu,11], [Benner/Breiten,15]
- Here, we will use optimal interpolation subspaces from the linearized model.
- [Beattie/G.,11] and [Chaturantabut/Beattie/G.,13] for reducing nonlinear port-Hamiltonian systems.

Test Description

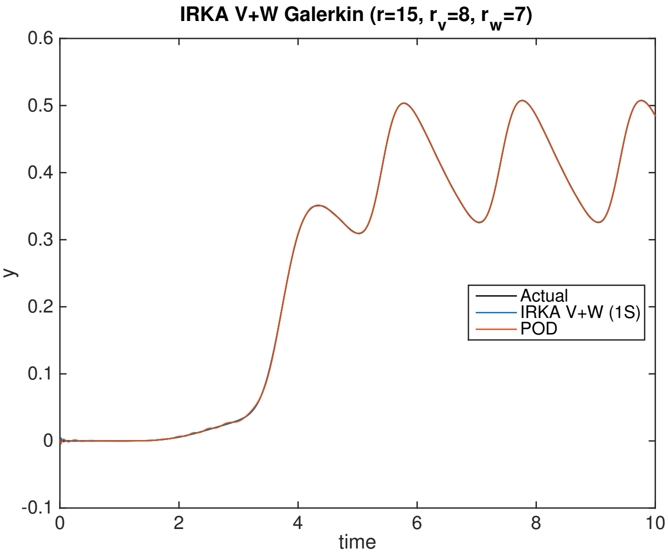
We test the technique against several input functions and various values of r_W and r_V.

- First, we generate ROMs using POD and one-sided IRKA.
- Next we picked $r_w = 7, \ldots, 10$.
- For each r_w, we calculated a ROM for each of $r_V = (r - r_w), \ldots, (r - 1)$.
- For each ROM, the output error was calculated.

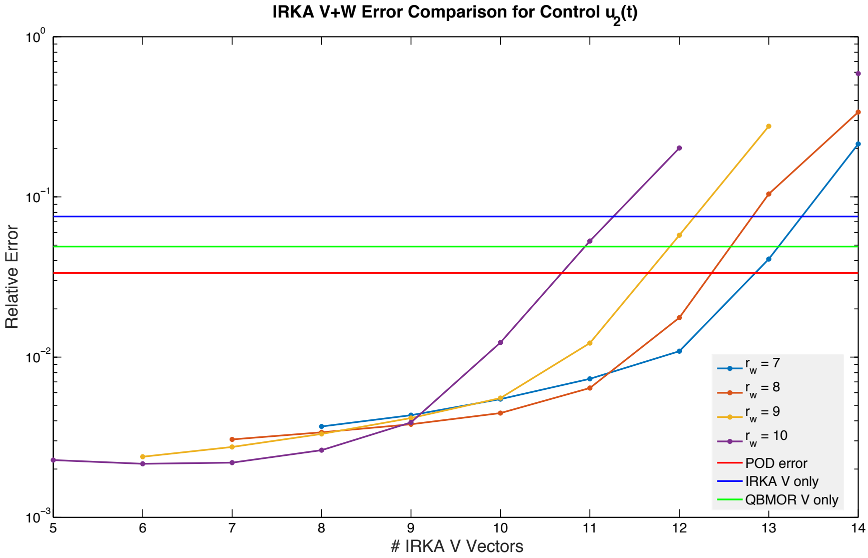
Error plots for $u_1(t) = \cos(\pi t)$



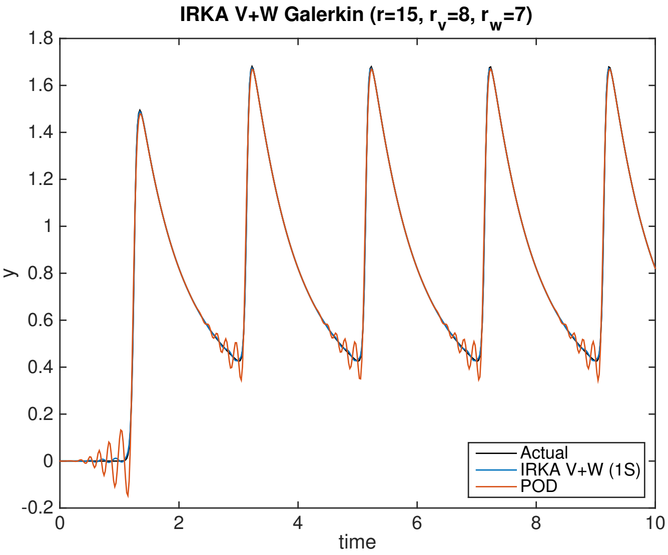
Output plot for $u_1(t)$ with $r_W = 7$



Error plots for $u_2(t) = 2\sin(\pi t)$

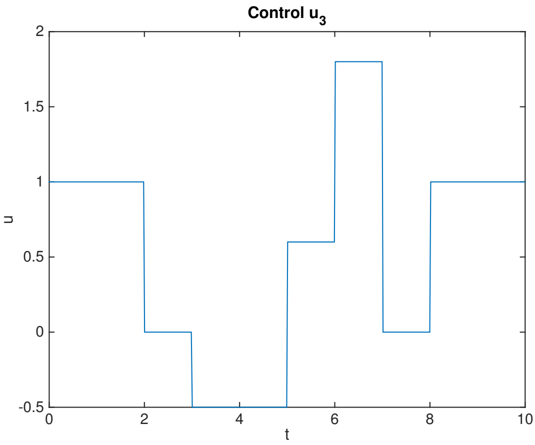


Output plots for $u_2(t) = 2\sin(\pi t)$ with $r_W = 7$

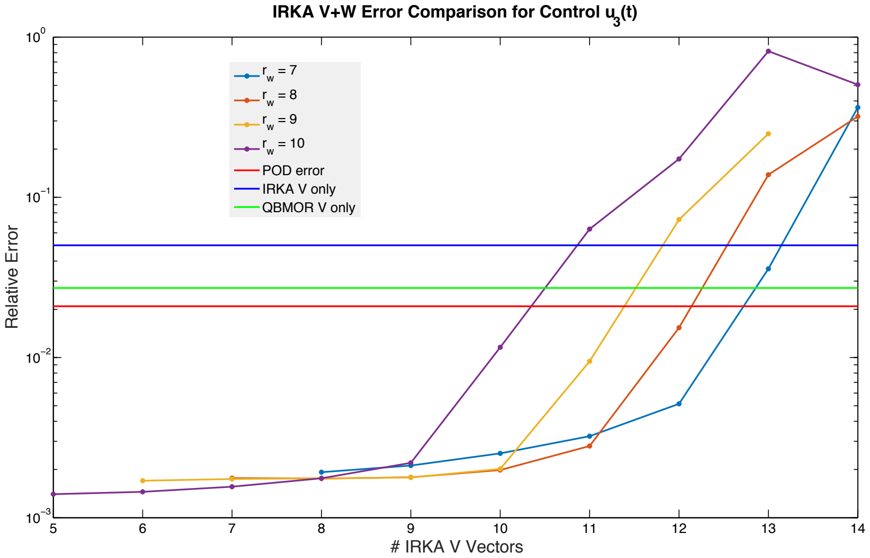


Plot of control function $u_3(t)$

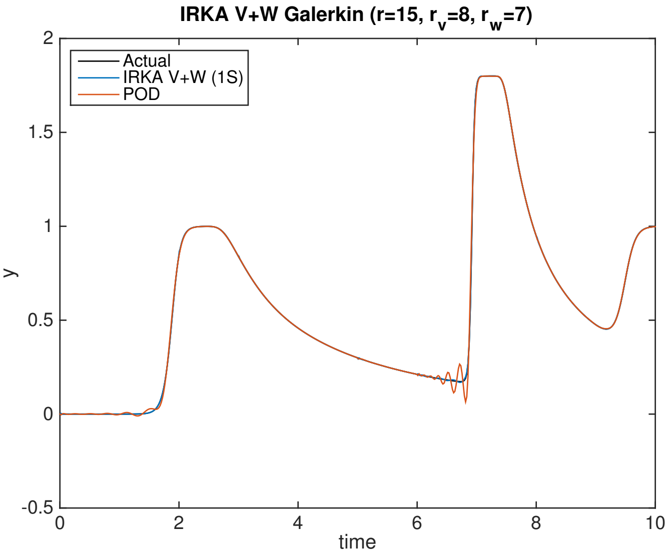
Step Function



Error plots for $u_3(t)$



Output plots for $u_3(t)$



Conclusions and Future Work

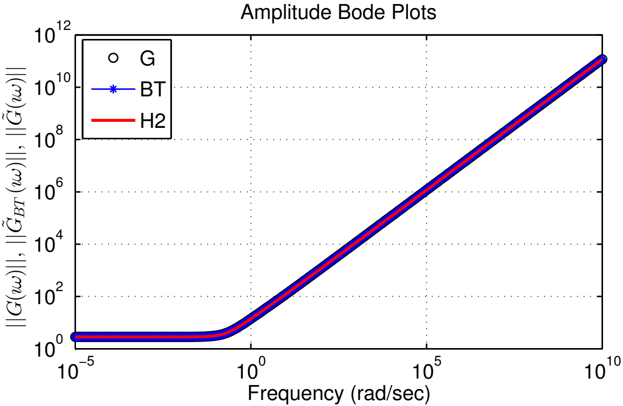
- Interpolatory model reduction for DAEs combined with LQR design for flow control
- Computationally efficient framework
- Unstable poles captured accurately
- Incorporating optimal linear subspaces into reducing nonlinear models
- Establish the connection to rational Krylov methods for eigenvalue problems.
- Test the performance for higher Reynolds numbers.
- Choice of **C**

Index-2 example: Oseen equations

- Data from [Heinkenschloss *et al.*,08]
- Discretizied the Oseen equations: describing the flow of a viscous and incompressible fluid in a domain $\Omega \in \mathbb{R}^2$ representing a channel with a backward facing step.
- $\mathbf{E}_{11}, \mathbf{A}_{11} \in \mathbb{R}^{5520\times5520}$, $\mathbf{A}_{12}, \mathbf{A}_{21}^T \in \mathbb{R}^{5520\times761}$, $\mathbf{B}_1 \in \mathbb{R}^{5520\times6}$, $\mathbf{B}_2 \in \mathbb{R}^{761\times6}$, $\mathbf{C}_1 \in \mathbb{R}^{2\times5520}$, $\mathbf{C}_2 \in \mathbb{R}^{2\times761}$, and $\mathbf{D} = 0$.
- Reduced to order $r = 20$ using interpolatory $\mathcal{H}_2$ method for index-2 DAEs.
- Also compared with balanced truncation

- Relative $\mathcal{H}_\infty$-error: $\frac{\|\mathbf{G}_{sp}-\widetilde{\mathbf{G}}_{sp}\|_{\mathcal{H}_\infty}}{\|\mathbf{G}_{sp}\|_{\mathcal{H}_\infty}}$.

$$\mathrm{IRKA-DAE} : 8.9663 \times 10^{-6} \qquad \mathrm{BT} : 3.3284 \times 10^{-6}$$



- Compare the full and reduced model for the input selections $\mathbf{u}_i(t) = \sin(6it)$ for $i = 1, \ldots, 6$

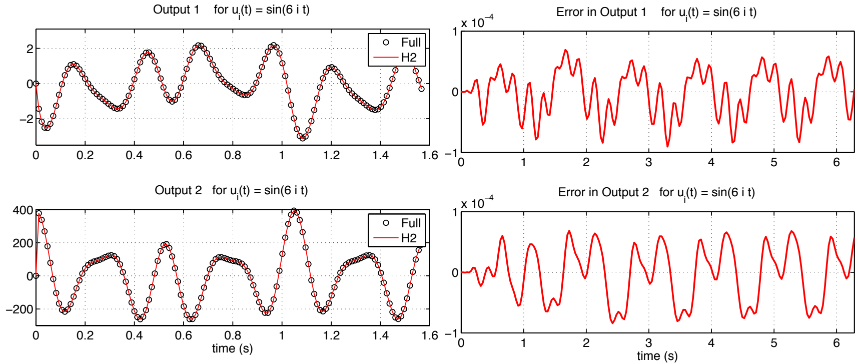


Figure : Oseen equation: (left) time domain response for $\mathbf{u}_i(t) = \sin(6it)$; (right) error in time domain response for $\mathbf{u}_i(t) = \sin(6it)$.