The Gibbs state of the weakly interacting Bose gas, full counting statistics, and effective field theory

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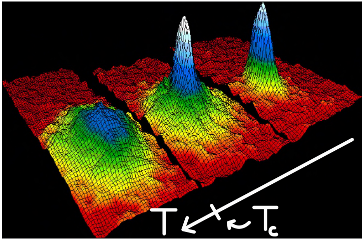
A gas of quantum mechanical particles

Background: Bose–Einstein condensation (BEC)



"Bose–Einstein condensation (BEC) is one of the most intriguing phenomena predicted by quantum statistical mechanics."
Wolfgang Ketterle, Nobel Prize in Physics in 2001.

Interacting Bose gases display wealth of interesting phenomena related to the BEC phase transition, that are notoriously difficult to study mathematically because:

- Curse of dimensionality $\Rightarrow$ Analysis extremely challenging (in experiments $N \approx 10^3 - 10^6$),
- Small denominator problem $\Rightarrow$ Perturbation theory fails in interacting many-body quantum systems.

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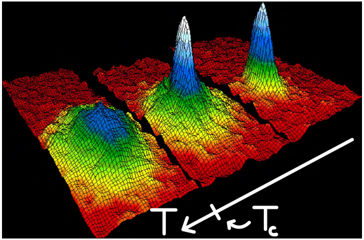

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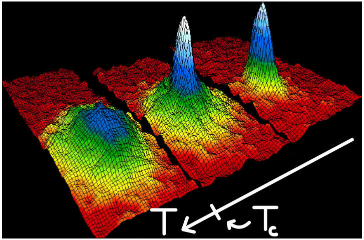

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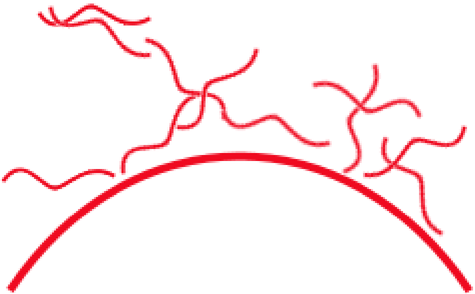
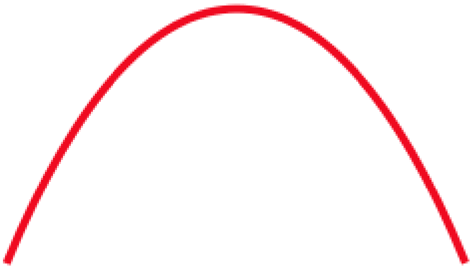
Above the critical point

Below the critical point



At zero temperature



Starting point: the microscopic model

We are interested in a system of bosons captured in the unit torus with fluctuating particle number. The Hilbert space of the system is therefore the bosonic Fock space

$$\mathscr{F}(L^2([0,1]^3) = \bigoplus_{n=0}^{\infty} L^2_{\mathrm{sym}}([0,1]^{3n}).$$

Here $L^2_{\mathrm{sym}}([0,1]^{3n})$ denotes the set of all L^2-functions that satisfy

$$\Psi(x_1, ..., x_i, ..., x_j, ..., x_n) = \Psi(x_1, ..., x_j, ..., x_i, ..., x_n) \quad \forall\ i < j.$$

The Hamiltonian of the system reads

$$\mathcal{H}_N = 0 \bigoplus_{n=1}^{\infty} \left[\sum_{i=1}^{n} -\Delta_j + \frac{1}{N} \sum_{1 \leq i < j \leq n} v(x_i - x_j) \right].$$

Here $v \in L^1$ is nonnegative and N denotes the expected particle number.

Creation and annihilation operators

By a_p^* and a_p we denote the creation and annihilation operators on $\mathscr{F}$ that create and annihilate a particle in the function $\varphi_p(x) = e^{\mathrm{i}p\cdot x}$ with $p \in 2\pi\mathbb{Z}^3$, respectively. That is,

$$(a_p^* \Psi_n)(x_1, ..., x_n, x_{n+1}) = \mathrm{Sym}[\varphi_p(x_{n+1})\Psi_N(x_1, ..., x_n)].$$

They satisfy the canonical commutation relations (CCR)

$$[a_p, a_q^*] = \delta_{p,q}, \qquad [a_p, a_q] = 0 = [a_p^*, a_q^*].$$

The Hamiltonian can be written as

$$\mathcal{H}_N = \sum_{p \in 2\pi\mathbb{Z}^3} p^2 a_p^* a_p + \frac{1}{2N} \sum_{p,u,v \in 2\pi\mathbb{Z}^3} \hat{v}(p) a_{u+p}^* a_{v-p}^* a_u a_v$$

with the Fourier coefficients $\hat{v} \geq 0$ of v.

Free energy and Gibbs state

Equilibrium properties of the system are encoded in the Gibbs state

$$G_{\beta,N} = \frac{\exp(-\beta(\mathcal{H}_N - \mu\mathcal{N}))}{\operatorname{Tr}\exp(-\beta(\mathcal{H}_N - \mu\mathcal{N}))}$$

at inverse temperature $\beta > 0$. The chemical potential μ is chosen s.t.

$$\operatorname{Tr}[\mathcal{N} G_{\beta,N}] = N$$

holds, where $\mathcal{N} = \sum_{p \in 2\pi\mathbb{Z}^3} a_p^* a_p$ denotes the number operator.

Bose–Einstein condensation (BEC)

We are interested in the parameter regime $N \to \infty$, $\beta = \kappa \beta_c$ with

$$\beta_c = \frac{1}{4\pi} \left(\frac{N}{\zeta(3/2)} \right)^{-2/3}$$

and $\kappa \in (0, \infty)$.

The expected number of particles displays the aymptotic behavior

$$\frac{N_0(\beta, N)}{N} = \frac{\mathrm{Tr}[a_0^* a_0 G_{\beta,N}]}{N} \simeq \left[1 - \frac{1}{\kappa^{3/2}} \right]_+ .$$

in this limit. That is, we see a BEC phase transition.

Literature

Low lying excitation spectrum

- Seiringer, Commun. Math. Phys. (2011)
- Lewin, Nam, Serfaty, Solovej, Commun. Pure Appl. Math. (2015)
- Nam, Seiringer, Arch. Rational Mech. Anal. (2015)

BEC and dependence of critical temperature for BEC on interaction

- D., Seiringer, Yngvason, Commun. Math. Phys. (2019)
- D., Seiringer, Arch. Rational Mech. Anal. (2020)
- D., Seiringer, J. Funct. Anal. (2021)

Derivation of nonlinear Gibbs measures at critical point

- Lewin, Nam, Rougerie, Invent. Math. (2021)
- Fröhlich, Knowles, Schlein, Sohinger, JAMS (2022)

Theorem 1 (Convergence of Gibbs state)

The Gibbs state $G_{\beta,N}$ is close (in trace norm) to the **non quasi-free** state

$$\Gamma = \int_{\mathbb{C}} |z\rangle\langle z| \otimes G^{\mathrm{Bog}}(z) g^{\mathrm{BEC}}(z)\,\mathrm{d}z,$$

on $\mathscr{F} \simeq \mathscr{F}_0 \otimes \mathscr{F}_+$ with the coherent state $|z\rangle = \exp(za_0^* + \overline{z}a_0)|\Omega\rangle$.

Here $G^{\mathrm{Bog}}(z)$ is the Gibbs state of the Bogoliubov Hamiltonian

$$\begin{aligned}\mathcal{H}^{\mathrm{Bog}}(z) &= \sum_{p\in 2\pi\mathbb{Z}^3\setminus\{0\}} p^2 a_p^* a_p \\ &+ \frac{N_0(\beta,N)}{2N} \sum_{p\in 2\pi\mathbb{Z}^3\setminus\{0\}} \hat{v}(p) \left(2a_p^* a_p + (z/|z|)^2 a_p^* a_{-p}^* + (\overline{z}/|z|)^2 a_p a_{-p}\right)\end{aligned}$$

and the condensate is described by

$$g^{\mathrm{BEC}}(z) \propto \exp(-\beta(\hat{v}(0)/(2N)|z|^4 - \mu^{\mathrm{BEC}}|z|^2)).$$

The Hamiltonian

The Hamiltonian can be written as

$$\begin{aligned}\mathcal{H}_N = &\sum_{p\in\Lambda_+^*} p^2 a_p^* a_p + \frac{\hat{v}(0)}{2N} \sum_{u,v\in\Lambda^*} a_u^* a_v^* a_u a_v \\ &+ \frac{1}{2N} \sum_{p\in\Lambda_+^*} \hat{v}(p) \left\{ 2a_p^* a_0^* a_p a_0 + a_0^* a_0^* a_p a_{-p} + a_p^* a_{-p}^* a_0 a_0 \right\} \\ &+ \frac{1}{N} \sum_{p,k,p+k\in\Lambda_+^*} \hat{v}(p) \left\{ a_{k+p}^* a_{-p}^* a_k a_0 + h.c. \right\} \\ &+ \frac{1}{2N} \sum_{u,v,p,u+p,v-p\in\Lambda_+^*} \hat{v}(p) a_{u+p}^* a_{v-p}^* a_u a_v .\end{aligned}$$

Theorem 2 (Two-particle density matrix)

Matrix elements of two-particle density matrix of Gibbs state converge to that of Γ. Selected implications: The variance of the number of particles in the condensate satisfies

$$\mathrm{Tr}[a_0^* a_0 a_0^* a_0 G_{\beta,N}] - (\mathrm{Tr}[a_0^* a_0 G_{\beta,N}])^2 = \frac{N}{\beta \hat{v}(0)}(1 + o(1)).$$

Moreover,

$$\mathrm{Tr}[a_0^* a_0^* a_p a_{-p} G_{\beta,N}] = N_0(\beta, N)\alpha_{\beta,N}(p)(1 + o(1)),$$
$$\mathrm{Tr}[a_p^* a_{-p}^* a_p a_{-p} G_{\beta,N}] = \alpha_{\beta,N}^2(p)(1 + o(1)),$$

where $N_0(\beta, N)$ is the expected number of particles in the condensate and $\alpha_{\beta,N}$ is the pairing function of a Bogoliubov Gibbs state.

Theorem 3 (Distribution of condensate particle number)

We consider the random variable $\mathbf{N}_0$ defined by

$$\mathbf{P}(\mathbf{N}_0 \in [a, b]) = \mathrm{Tr}[\mathbb{1}(a_0^* a_0 \in [a, b]) G_{\beta,N}].$$

The centered and normalized version

$$\widetilde{\mathbf{N}}_0 = \frac{\mathbf{N}_0 - \mathbf{E}(\mathbf{N}_0)}{\sqrt{\mathbf{Var}(\mathbf{N}_0)}}$$

convergences, as $N \to \infty$, to a standard normal distribution in the condensed phase. In other parameter regimes we find three other limiting distributions (mixture of Gaussian and exponential, exponential, and geometric distributions).

A new Bogoliubov field

Let $\{(\Psi_p, \Psi_{-p})\}_{p \in A}$ be a family of independent, centered complex Gaussian random vectors with covariance and pseudo-covariance matrices

$$\mathbf{E}\begin{pmatrix} |\Psi_p|^2 & \Psi_p \overline{\Psi}_{-p} \\ \overline{\Psi}_p \Psi_{-p} & |\Psi_{-p}|^2 \end{pmatrix} = \frac{p^2 + g(\kappa)\widehat{v}(p)}{p^2(p^2 + 2g(\kappa)\widehat{v}(p))} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\mathbf{E}\begin{pmatrix} \Psi_p^2 & \Psi_p \Psi_{-p} \\ \Psi_p \Psi_{-p} & \Psi_{-p}^2 \end{pmatrix} = -\frac{g(\kappa)\widehat{v}(p)}{p^2(p^2 + 2g(\kappa)\widehat{v}(p))} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The Bogoliubov field is defined as the random Fourier series

$$\Psi(x) = \sum_{p \neq 0} \Psi_p e^{\mathrm{i} p \cdot x}.$$

Theorem 5 (Counting statistics on macroscopic scale)

Let $a_x = \sum_{p\in\Lambda^*} e^{\mathrm{i}p\cdot x} a_p$ denote the operator annihilating a particle at $x \in \Lambda$. The random variable N_D counting the number of particles in D is defined by

$$\mathbf{P}(N_D \in B) = \mathrm{Tr}\left[\mathbb{1}\left(\int_D a_x^* a_x \mathrm{d}x \in B\right) G_{\beta,N}\right].$$

Let X be a standard Gaussian random variable, independent of the Bogoliubov field Ψ. As $N \to \infty$, we have

$$\sqrt{\frac{\beta}{N}}(N_D - N|D|) \xrightarrow{d} \frac{|D|}{\sqrt{\widehat{v}(0)}} X + 2\sqrt{g(\kappa)}\mathrm{Re}\langle \Psi, \mathbf{1}_D\rangle.$$

The general result holds for joint distributions.

Theorem 6 (Counting statistics on microscopic scale)

One can associate a point process $\Theta_{\beta,N}$ to the Gibbs state $G_{\beta,N}$. On the microscopic length scale $N^{-1/3}$ we expect to have locally a finite number of particles.

We show that $\Theta_{\beta,N}$ converges weakly, as $N \to \infty$, to the boson process, which is a permanental point process that does not depend on the interaction potential v.

The boson process is a Poisson point process with a random rate (Cox process) given by the local L^2 mass of a Gaussian random field.

Theorem 7 (A new abstract correlation inequality)

Let A be self-adjoint, let B be symmetric and assume that

$$\|B\psi\| \leq a\|A\psi\| + b\|\psi\|$$

with $0 < a < 1$ and $b \geq 0$. Assume additionally that $\exp(-(1-a)A)$ is trace-class and that the state

$$\Gamma_t = \frac{\exp(A + tB)}{\mathrm{Tr}[\exp(A+tB)]} \quad \text{satisfies} \quad \sup_{t\in[-1,1]} |\,\mathrm{Tr}[B\Gamma_t]| \leq \eta$$

for some constant $\eta > 0$. Then we have

$$\mathrm{Tr}[B^2\Gamma_0] \leq \eta\exp(\eta) + \frac{1}{4}\,\mathrm{Tr}([[B,A],B])\Gamma_0).$$

This greatly simplifies and generalizes a correlation inequality in Lewin, Nam, Rougerie, Invent. Math. (2021). We also have a higher order correlation inequality!

Theorem 8 (An infnite-dimensional version of Stahl's theorem)

Let A be self-adoint, let B be symmetric and assume that

$$\|B\psi\| \leq a\|A\psi\| + b\|\psi\|$$

holds $0 < a < 1$ and $b \geq 1$. Assume additionally that $\exp(-(1-a)A)$ is trace-class and define ($t \in [-1, 1]$)

$$Z(t) = \mathrm{Tr}[\exp(-(A + tB))].$$

Then there exists a nonnegative Borel measure μ such that

$$Z(t) = \int_{-\infty}^{\infty} e^{-ts} \, \mathrm{d}\mu(s)$$

holds.

References (Stahl's theorem f.k.a. BMV conjecture):

- Bessis, Moussa, Villani, J. Math. Phys. (1975)
- Stahl, Acta Math. (2013)

Summary

- Approximation of Gibbs state in trace norm.
- Computation of 2-pdm and condensate distributions.
- New Bogoliubov random field describing the non-condensed particles.
- Limiting distributions for macroscopic and microscopic counting statistics.
- New abstract correlation inequalities.
- Infinite-dimensional version of Stahl's theorem.