Dynamics and equilibrium states of infinite systems of lattics bosons

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December 11, 2025
Joint work with Jonas Lampart and Marius Lemm

Mathematical setup

We consider bosons moving in $\mathbb{Z}^d$.

- Bosonic Fock space:

$$\mathscr{F}(L^2(\mathbb{Z}^d)) = \bigoplus_{n=0}^{\infty} L^2_{\text{sym}}(\mathbb{Z}^{dn}).$$

- Creation and annihilation operators a_x^* and a_x of a particle at $x \in \mathbb{Z}^d$ satisfy CCR:

$$[a_x, a_y^*] = \delta_{xy}, \quad [a_x, a_y] = 0 = [a_x^*, a_y^*].$$

- Field operators: For $f \in \ell^2(\mathbb{Z}^d)$ we define

$$\Phi(f) = a(f) + a^*(f), \quad a^*(f) = \sum_{x \in \mathbb{Z}^d} a_x^* f(x).$$

Dynamics of finite many-particle

We are interested in dynamics generated by the Hamiltonian

$$H = -\sum_{\substack{x,y\in\mathbb{Z}^d\\ d(x,y)=1}} (a_x^* a_y + a_y^* a_x) + \sum_{x,y\in\mathbb{Z}^d} v(x,y) a_x^* a_y^* a_x a_y.$$

Existence of the Schrödinger dynamics $\psi(t) = e^{-\mathrm{i}Ht}\psi_0$ for finitely many particles follows from the self-adjointness of H.

In case of infinitely many degrees of freedom we need to consider the Heisenberg dynamics

$$A(t) = e^{\mathrm{i}Ht} A e^{-\mathrm{i}Ht}$$

on a suitabel C^*-algebra of observables encoding some sort of locality. States on this C^*-algebra replace wave functions.

[1] M. PA. Fisher, P. B. Weichman, G. Grinstein, D. S. Fisher, Phys. Rev. B 40(1):546 (1989)

[2] M. Aizenman, E. H. Lieb, R. Seiringer, J. P. Solovej, J. Yngvason, Phys. Rev. A, 70:023612 (2004)

Quantum spin systems vs. bosonic lattics systems

Quantum spin systems and fermionic lattice systems:

- finite-dimensional Hilbert space on each lattice site,
- bounded operators (matrices) describe local observables,
- the closure of the algebra of local observables in operator norm (quasi-local observables) is a natural C^*-algebra of observables,
- this C^*-algebra is invariant under physically interesting dynamics (Lieb–Robinson bounds).

Bosonic lattics systems:

- infinite-dimensional Hilbert space on each lattice site,
- unbounded operators describe local observables,
- to obtain a C^*-algebra one needs to consider bounded functions of local observables,
- infinite energies lead to nonlocal dynamics.

C^* algebraic formalism for bosons

- Weyl algebra [1]: not invariant under interacting dynamics in $\mathbb{R}^d$ (even with finitely many degrees of freedom)
- Greens functions [2,3]: approach gives limited information about dynamics in $\mathbb{R}^d$
- Resolvent algebra [4,5]: for finitely many degrees of freedom invariant under dynamics in $\mathbb{R}^d$
- Extension of resolvent algebra [6,7]: invariant under dynamics in $\mathbb{R}^d$ with infinitely many degrees of freedom (not quasi-local)

[1] M. Fannes, A. Verbeure, CMP 35, 257–264 (1974)

[2] Y. M. Park, CMP 94:1–33 (1984)

[3] Y. M. Park, J. Stat. Phys. 40:259–302 (1985)

[4] D. Kastler, CMP 1, 14-48 (1965)

[5] D. Buchholz, H. Grundling, JFA 254, 2725–2779 (2008)

[6] D. Buchholz, CMP 362(3):949–981 (2018)

[7] D. Buchholz, CMP 375:1159–1199 (2020)

Resolvent algebra following Buchholz

- The resolvent algebra is the C^*-algebra generated by $\mathbb{1}$ and

$$R(f,z) = (\Phi(f) - z)^{-1} \quad \text{with} \quad f \in \ell^2_{\mathrm{fin}}(\mathbb{Z}^d), \mathrm{Im}(z) \neq 0.$$

- The gauge invariant resolvent algebra $\mathcal{R}^{\mathrm{inv}}$ is the subalgebra of all operators A that satisfy $e^{\mathrm{i}Nt} A e^{-\mathrm{i}Nt} = A$ for all $t \in \mathbb{R}$.
- The Buchholz algebra $\mathcal{B}$ consists of all $B \in \mathcal{L}(\mathscr{F})$ s.t. for all $n \in \mathbb{N}$ there is a $A \in \mathcal{R}^{\mathrm{inv}}$ with

$$B \mathbb{1}_{N \leq n} = A \mathbb{1}_{N \leq n}.$$

- Example: The operator $\sin(N_x)$, $x \in \mathbb{Z}^d$ is in $\mathcal{B}$ but not in $\mathcal{R}^{\mathrm{inv}}$.

Proposition (D., Lampart, Lemm 2025)

For all $t \in \mathbb{R}$ the map $A \mapsto \tau_t(A) = e^{\mathrm{i}Ht} A e^{-\mathrm{i}Ht}$ defines a $*$-automorphism on $\mathcal{B}$.

Lack of continuity due to lack of lokality

- For $d = 1$ consider

$$\mathrm{i}\partial_t \psi_t(x) = -\Delta \psi_t(x) \quad \text{with} \quad \psi_0(x) = \delta_{x,0}.$$

  Then $\psi_t(x) = I_{|x|}(2\mathrm{i}t)$ with the modified Bessel function $I_{|x|}(2\mathrm{i}t)$.
- For fixed $x \in \mathbb{Z}$ the set of times t such that $\psi_t(x) = 0$ holds is countable.
- For $x \in \mathbb{Z}$ and a function $f : \mathbb{N}_0 \to \mathbb{R}$ such that $\lim_{n\to\infty} f(n)$ exists, we define the state

$$\gamma(f(N_x)) = \lim_{m\to\infty} \frac{1}{m!} \langle \Omega, a_0^m f(N_x)(a_0^*)^m \Omega \rangle$$

  with the vacuum vector Ω.
- We have (t such that $\psi_t(x) \neq 0$)

$$\gamma\left(e^{\mathrm{i}\,\mathrm{d}\Gamma(-\Delta)t} \frac{1}{1+N_x} e^{-\mathrm{i}\,\mathrm{d}\Gamma(-\Delta)t} \right) = 0 \neq 1 = \gamma\left(\frac{1}{1+N_x} \right).$$

Lieb–Robinson bounds (simple 5 page proof!)

Theorem (D., Lampart, Lemm 2025)

Assume that $p > 2d + 2$ and $M > 0$, $R > 0$. $\forall \epsilon > 0\ \exists\ m_0 \in \mathbb{N}$ s.t. $\forall m \geq m_0$, $X \subset \mathbb{Z}^d$ with $\operatorname{diam}(X) \leq R$, states γ on $\mathcal{R}^{\mathrm{inv}}$ satisfying

$$\sup_{x \in \mathbb{Z}^d} \gamma((1 + N_x)^p) \leq M,$$

we have for all $A, B \in \mathcal{R}^{\mathrm{inv}}$, $\operatorname{supp}(A) \subset X$

$$\sup_{|t| \leq T} \left| \gamma\left(\left(\tau_t(A) - \tau_t^{X[m]}(A) \right) B \right) \right| \leq \epsilon \|A\| \ \|B\|.$$

[1] T. Kuwahara, T. V. Vu, K. Saito, Nat. Commun. 15(1):2520 (2024)

[2] T. Kuwahara, M. Lemm, arXiv:2405.04672 (2024)

[3] J. Faupin, M. Lemm, I. M. Sigal, CMP, 394(3):1011–1037 (2022)

[4] C. Yin, A. Lucas, PRX, 12(2):021039 (2022)

[5] N. Schuch, S. K. Harrison, T. J. Osborne, J. Eisert, PRA, 84(3):032309 (2011)

[6] B. Nachtergaele, H. Raz, B. Schlein, R. Sims, CMP 286(3) (2009)

Consequence I: continuous dynamics in GNS representation

Theorem (D., Lampart, Lemm 2025)

Let γ be a state on $\mathcal{B}$, which satisfies $\gamma(\tau_t(A)) = \gamma(e^{\mathrm{i}Ht} A e^{-\mathrm{i}Ht}) = \gamma(A)$ for all $A \in \mathcal{R}^{\mathrm{inv}}$ and

$$\sup_{x \in \mathbb{Z}^d} \gamma((1 + N_x)^p) < \infty.$$

Let $(\mathcal{H}_\gamma, \pi_\gamma, \Omega_\gamma)$ be the GNS representation of $\mathcal{R}^{\mathrm{inv}}$.

Then there exists a unique strongly continuous unitary group $U_\gamma(t)$ on H_γ so that for all $A, B \in \mathcal{R}^{\mathrm{inv}}$

$$\langle U_\gamma(t) \pi_\gamma(A) \Omega_\gamma, \pi_\gamma(B) \Omega_\gamma \rangle = \gamma(\tau_t(A) B).$$

[1] Y. M. Park, CMP 94:1–33 (1984)

[2] Y. M. Park, J. Stat. Phys. 40:259–302 (1985)

Consequence II: existence of KMS states

Theorem (D., Lampart, Lemm 2025)

Assume that the state

$$\gamma_\Lambda = Z_\Lambda^{-1}\,\mathrm{Tr}[Ae^{-\beta(H_\Lambda-\mu_\Lambda N_\Lambda)}\otimes|\Omega_{\Lambda^c}\rangle\langle\Omega_{\Lambda^c}|]$$

satisfies

$$\sup_{\Lambda\Subset\mathbb{Z}^d}\sup_{x\in\Lambda}\gamma_\Lambda((1+N_x)^p)<\infty$$

with some $p>2d+2$. Then any weak-$*$ accumulation point of $\{\gamma_\Lambda\}_\Lambda$ satifies the KMS condition

$$\gamma(\tau_t(A)B)=F(t),\quad \gamma(B\tau_t(A))=F(t-\mathrm{i}\beta)$$

for all $A,B\in\mathcal{R}^{\mathrm{inv}}$ and a function F that is analytic in the strip $-\beta<\mathrm{Im}(z)<0$.

[1] Y. M. Park, CMP 94:1–33 (1984); J. Stat. Phys. 40:259–302 (1985)

[2] J. Dereziński, V. Jakšić, C-A Pillet, Rev. Math. Phy. 15(05):447–489 (2003)

Summary

- Simple proof of invariance of Buchholz algebra under interacting dynamics for bosonic lattice systems
- Simplified proof of Lieb–Robinson bounds
- Continuous dynamics in GNS representation of resolvent algebra for invariant states satisfying local moment bounds
- Existence of KMS states