Note on a family of monotone quantum relative entropies

Andreas Deuchert

Department of Mathematics, University of Tübingen

YRS @ Santiago de Chile
24. July 2015

Joint work with Christian Hainzl and Robert Seiringer

Definition: Generalized relative entropy

Let $\varphi : [0,1] \to \mathbb{R}$ be a convex function with $\varphi \in \mathcal{C}([0,1]) \cap \mathcal{C}^1((0,1))$. For hermitian matrices A, B with $0 \leq A, B \leq 1$ define

$$\mathcal{H}_\varphi(A,B) = \operatorname{Tr}\left[\varphi(A) - \varphi(B) - \varphi'(B)(A-B)\right].$$

Examples:

- Boltzmann entropy: $\varphi_1(x) = x\ln(x)$,
- Fermionic quasi-free states: $\varphi_2(x) = x\ln(x) + (1-x)\ln(1-x)$,
- Bosonic quasi-free states: $\varphi_3(x) = x\ln(x) - (1+x)\ln(1+x)$.

Aim: Find a useful definition of $\mathcal{H}_\varphi$ for bounded non-compact A and B.

Generalized relative entropy for bounded operators

Definition 1 ([LS]): Assume φ' operator monotone and let $\{P_n\}_{n=1}^{\infty}$ be an increasing sequence of finite-dimensional projections on a separable Hilbert space h with $P_n \to 1$ strongly. For $A, B \in \mathcal{L}(h)$ with $0 \leq A, B \leq 1$ define

$$\mathcal{H}_\varphi(A, B) := \lim_{n\to\infty} \mathcal{H}_\varphi(P_n A P_n, P_n B P_n).$$

Theorem 1 ([LS]): Assume φ as above with φ' operator monotone on $(0, 1)$. Then the sequence $\mathcal{H}_\varphi(P_n A P_n, P_n B P_n)$ possesses a limit in $\mathbb{R}_+ \cup \{+\infty\}$. This limit does not depend on the chosen sequence $\{P_n\}_{n=1}^{\infty}$ and hence $\mathcal{H}_\varphi(A, B)$ is well defined.

[LS] M. Lewin, J. Sabin, *A Family of Monotone Quantum Relative Entropies*, Lett. Math. Phys. **104** (2014), 691-705.

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$$\lim_{n\to\infty} \mathcal{H}_\varphi(P_nAP_n, P_nBP_n) = \mathrm{Tr}\left[\varphi(A) - \varphi(B) - \varphi'(B)(A-B)\right]$$

hold?

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hold?

The answer is "in principle yes".

Limit of the relative entropy

Theorem 3 ([DHS]): Let $\varphi \in C([0,1])$ with φ' operator monotone on $(0,1)$ and take $\{P_n\}_{n=1}^{\infty}$ as before. Then

$$\lim_{n\to\infty} \mathcal{H}_\varphi(P_nAP_n, P_nBP_n) = \mathrm{Tr}\left[\varphi(A) - \varphi(B) - \frac{\mathrm{d}}{\mathrm{d}\alpha}\varphi\left(\alpha A + (1-\alpha)B\right)\Big|_{\alpha=0}\right].$$

[DHS] A. D., C. Hainzl, R. Seiringer, *Note on a Family of Monotone Quantum Relative Entropies*, arXiv:1502.07205 [math-ph] (2015).

Recovering the original formula

Theorem 4 ([DHS]): Let $\varphi \in C([0,1])$ with φ' operator monotone on $(0,1)$. Assume in addition that $(A-B)$, $\varphi(A)-\varphi(B)$ and $\varphi'(B)(A-B)$ are trace-class. Then

$$\mathrm{Tr}\left[\varphi(A)-\varphi(B)-\frac{\mathrm{d}}{\mathrm{d}\alpha}\varphi(\alpha A+(1-\alpha)B)\Big|_{\alpha=0}\right] = \mathrm{Tr}\left[\varphi(A)-\varphi(B)-\varphi'(B)(A-B)\right]$$

holds.

A crucial ingredient of the proof

There exists a unique Borel probability measure μ on $[-1,1]$ and a $b \geq 0$ such that

$$\mathrm{Tr}\Big[\varphi(A) - \varphi(B) - \frac{\mathrm{d}}{\mathrm{d}\alpha}\varphi(\alpha A + (1-\alpha)B)\Big|_{\alpha=0}\Big] = 2b\int_{-1}^{1}\int_{0}^{\infty}\mathrm{Tr}\Big[R_{\lambda,t}(B)QR_{\lambda,t}(A)QR_{\lambda,t}(B)\Big]\mathrm{d}t\mathrm{d}\mu(\lambda),$$

with

$$Q = A - B \quad \text{and} \quad R_{\lambda,t}(A) = \frac{1}{1+\lambda(1-2A)+t}.$$

Conclusion

- We have computed the limit $\lim_{n\to\infty} \mathcal{H}_\varphi(P_nAP_n, P_nBP_n)$ for general bounded operators A and B,
- and shown that it equals $\mathrm{Tr}\left[\varphi(A) - \varphi(B) - \varphi'(B)(A-B)\right]$ for nice enough A and B.

- This makes the generalized relative entropy of Lewin and Sabin easier to use
- and suggest that one should define the generalized relative entropy directly by $\mathcal{H}_\varphi(A,B) = \mathrm{Tr}\Big[\varphi(A) - \varphi(B) - \frac{\mathrm{d}}{\mathrm{d}\alpha}\varphi(\alpha A + (1-\alpha)B)\big|_{\alpha=0}\Big]$.

Thank you for your attention!